1 Volumes of convex polyhedrons

To compute the volume \( V(P) \) for a convex polyhedron \( P \) with vertices \( p = \{p_1, \ldots, p_n\} \), we first introduce a new vertex

\[
p(f_j) = \frac{1}{\sigma(j)} \sum_{k=1}^{\sigma(j)} p_{jk},
\]

for every non-triangular face \( f_j \) with vertices \( p_{j1}, p_{j2}, \ldots, p_{j\sigma(j)} \).

Connecting \( p(f_j) \) with all vertices of \( f_j \) results in a triangulation of the polyhedron. Then the volume of the polyhedron can be computed as [Allgower and Schmidt 1986]

\[
V(P) = \frac{1}{6} \sum_{t_i \in T} \det \left( p^1(t_i), p^2(t_i), p^3(t_i) \right),
\]

where \( T \) is the set of faces for the triangulated polyhedron, and \( p^1(t_i), p^2(t_i), p^3(t_i) \) are the vertices of triangle \( t_i \) in positive orientation. In our optimization, the positive orientation is determined from the initial polyhedron shape, by choosing a consistent ordering of triangle vertices such that Equation (1) produces a positive value.

2 Surface sampling for convex polyhedrons

Our optimization requires sample points \( \{q_i\} \) on the surface of a polyhedron \( P \), represented as \( q_i = b_i \), where \( b_i \in \mathbb{R}^n \) are pre-computed convex combination coefficients with respect to the polyhedron vertex positions. To generate the samples and compute the coefficient vectors \( \{b_i\} \), we first triangulate the polyhedron by introducing new vertices on non-triangular faces (see Section 1). We then compute three types of sample points from the triangulated polyhedron \( T \):

1. Vertices of \( T \): such a sample point \( q_i \) is either a vertex of the original polyhedron \( P \), or an interior point on a face of \( P \). In the former case, vector \( b_i \) has exactly one non-zero element of value 1. In the latter case, there are \( \sigma(j) \) non-zero elements in \( b_i \), each with value \( 1/\sigma(j) \), where \( \sigma(j) \) is the number of vertices of the original polyhedron face that contains \( q_i \) (see Equation (1)).

2. Interior points on an edge \( e_i \) of \( T \): such a point can be represented as a convex combination of the two vertex sample points that belong to \( e_i \). In our implementation, we generate \( K \) internal sample points for each edge. Let \( q_{i1}, q_{i2} \) be the coefficient vectors for the two end vertex samples for \( e_i \), then the \( K \) interior samples on \( e_i \) are computed as:

\[
q_{i|e_i|} = \frac{j}{K+1} q_{i1} + \frac{K-j+1}{K+1} q_{i2}, \quad j = 1, \ldots, K.
\]

3. Interior points on a triangle \( t_i \) of \( T \): such a point can be represented as a convex combination of the three vertex sample points that belongs to \( t_i \). Let \( q_{i1}, q_{i2}, q_{i3} \) be the coefficient vectors for the vertex samples, then according to the parameter \( K \) the sample points are computed as:

\[
q_{a,b,c}(t_i) = \frac{a}{K+1} q_{i1} + \frac{b}{K+1} q_{i2} + \frac{c}{K+1} q_{i3}, \quad a, b, c \in \mathbb{N} \text{ and } a + b + c = K + 1.
\]

We determine the value of \( K \) from a user-specified parameter \( N_s \) for the preferred number of samples. \( K \) is chosen as the smallest number such that the total number of sample points is at least \( N_s \).

3 Computation of centroids

To compute the centroid \( C \) of the final model, we consider the final model as the combination of a hollow polyhedron made from uniform thin-sheet materials, and a 3D volume shell with uniform density. Then

\[
C = \left( \frac{C_1 V_1 - C_3 V_3}{V_1 - V_3} \right) \rho_1 + \frac{C_2 A_2 \rho_2}{V_1 - V_3} + A_2 \rho_2,
\]

where \( C_1, C_3 \) are the solid centroids of the target shape and the polyhedron, respectively; \( C_2 \) is the surface centroid of the polyhedron; \( V_1, V_3 \) are the internal volumes of the target surface and the polyhedron, respectively; \( A_2 \) is the polyhedron surface area; \( \rho_1 \) and \( \rho_2 \) are parameters for the volume density of the 3D printed part and the area density of the laser-cut material, respectively. Here \( V_1, V_3 \) can be computed using Equation (1). Using the same notation as Equation (1), the solid centroid of a polyhedron shape can be computed as

\[
C(P) = \sum_{t_i \in T} \frac{\det \left( p^1(t_i), p^2(t_i), p^3(t_i) \right)}{4 \cdot \sum_{t_i \in T} \det \left( p^1(t_i), p^2(t_i), p^3(t_i) \right)} \left( p^1(t_i) + p^2(t_i) + p^3(t_i) \right)
\]

while the surface area of a polyhedron is

\[
A_P = \frac{1}{2} \sum_{t_i \in T} \left\| p^2(t_i) - p^3(t_i) \right\| \times \left\| p^3(t_i) - p^1(t_i) \right\|,
\]

and its surface centroid is

\[
C_A(P) = \frac{\sum_{t_i \in T} \left\| p^2(t_i) - p^3(t_i) \right\| \times \left\| p^3(t_i) - p^1(t_i) \right\| \sum_{k=1}^{3} p^k(t_i)}{\sum_{t_i \in T} \left\| p^2(t_i) - p^3(t_i) \right\| \times \left\| p^3(t_i) - p^1(t_i) \right\|}.
\]

4 Constraints for optimizing multiple polyhedrons

The two faces \( (f^k_i, f^l_i) \) chosen for the connection between two polyhedrons must satisfy the following conditions:

1. \( f^k_i \) and \( f^l_i \) are parallel, with their outward normals pointing towards each other;

2. there exists a cylinder with radius \( r \) and with its axis parallel to the normals of \( f^k_i, f^l_i \), such that its two ends touch the two faces \( (f^k_i, f^l_i) \) and lie within the interior of each face, and the whole cylinder lie inside the target shape.
For the first condition, we require
\[ n_i^k + n_j^l = 0, \]
where \( n_i^k \) and \( n_j^l \) are the outward normal variables for the two faces.

For the second condition, we introduce auxiliary variables \( c_i^k, c_j^l \in \mathbb{R}^3 \) for the centers of the circles, where the cylinder touches the two faces. \( c_i^k \) and \( c_j^l \) are required to lie on the two faces, respectively. The line segment between these two points must be orthogonal to the two faces, thus requiring
\[ c_i^k + t_i^k n_i^k = c_j^l, \]
with auxiliary variable \( t_i^k > 0 \). Moreover, each face must be kept inside a disc with radius \( r \) and center \( c_i^k \) (or \( c_j^l \), respectively). Taking face \( f_i^k \) as an example, we require
\[ (c_i^k - p_{j1}) \cdot \frac{n_i^k \times (p_{j2} - p_{j1})}{\|n_i^k \times (p_{j2} - p_{j1})\|} \geq r, \]
where \( p_{j1}, p_{j2} \) are two adjacent vertices in \( f_i^k \) in an appropriate order. A similar constraint is defined for face \( f_j^l \). Finally, we compute a set of sample points \( \{q\} \) on the cylinder, and enforce a constraint
\[ D(q) \geq d_{\text{min}}, \]
where \( D \) is the signed distance function from the surface of the whole object. Each sample \( q \) is computed as
\[ q = a c_i^k + (1 - a) n_j^l + r (e_{1,i}^k \cos b + e_{2,i}^k \cos b), \]
where parameters \( a \in [0, 1] \) and \( b \in [0, 2\pi] \) are pre-determined, \( e_{1,i}^k, e_{2,i}^k \) are auxiliary variables that form an orthonormal frame with \( n_i^k \), previously used for enforcing the bounding rectangle constraints.

References