1 Volumes of convex polyhedrons

To compute the volume $V(P)$ for a convex polyhedron $P$ with vertices $P = \{p_1, \ldots, p_n\}$, we first introduce a new vertex

$$p(f_j) = \frac{1}{\sigma(j)} \sum_{k=1}^{\sigma(j)} p_{jk}$$

for every non-triangular face $f_j$ with vertices $p_{j1}, p_{j2}, \ldots, p_{j\sigma(j)}$. Connecting $p(f_j)$ with all vertices of $f_j$ results in a triangulation of the polyhedron. Then the volume of the polyhedron can be computed as [Allgower and Schmidt 1986]

$$V(P) = \frac{1}{6} \sum_{t_i \in T} \det (p^1(t_i), p^2(t_i), p^3(t_i)), \quad (1)$$

where $T$ is the set of faces for the triangulated polyhedron, and $p^1(t_i), p^2(t_i), p^3(t_i)$ are the vertices of triangle $t_i$ in positive orientation. In our optimization, the positive orientation is determined from the initial polyhedron shape, by choosing a consistent ordering of triangle vertices such that Equation (1) produces a positive value.

2 Surface sampling for convex polyhedrons

Our optimization requires sample points $\{q_i\}$ on the surface of a polyhedron $P$, represented as $q_i = \mathbf{b}_i$, where $\mathbf{b}_i \in \mathbb{R}^n$ are pre-computed convex combination coefficients with respect to the polyhedron vertex positions. To generate the samples and compute the coefficient vectors $\{\mathbf{b}_i\}$, we first triangulate the polyhedron by introducing new vertices on non-triangular faces (see Section 1). We then compute three types of sample points from the triangulated polyhedron $T$:

1. Vertices of $T$: such a sample point $q_i$ is either a vertex of the original polyhedron $P$, or an interior point on a face of $P$. In the former case, vertex $\mathbf{b}_i$ has exactly one non-zero element of value 1. In the latter case, there are $\sigma(j)$ non-zero elements in $\mathbf{b}_i$, each with value $1/\sigma(j)$, where $\sigma(j)$ is the number of vertices of the original polyhedron face that contains $q_i$ (see Equation (1)).

2. Interior points on an edge $e_i$ of $T$: such a point can be represented as a convex combination of the two vertex sample points that belong to $e_i$. In our implementation, we generate $K$ internal sample points for each edge. Let $q_{i1}, q_{i2}$ be the coefficient vectors for the two end vertex samples for $e_i$, then the $K$ interior samples on $e_i$ are computed as:

$$q_{i_1}(e_i) = \frac{j}{K+1} q_{i1} + \frac{K-j+1}{K+1} q_{i2}, \quad j = 1, \ldots, K.$$

3. Interior points on a triangle $t_i$ of $T$: such a point can be represented as a convex combination of the three vertex sample points that belongs to $t_i$. Let $q_{i1}, q_{i2}, q_{i3}$ be the coefficient vectors for the vertex samples, then according to the parameter $K$ the samples points are computed as:

$$q_{i,a,b,c}(t_i) = \frac{a}{K+1} q_{i1} + \frac{b}{K+1} q_{i2} + \frac{c}{K+1} q_{i3},$$

where $a, b, c \in \mathbb{N}$ and $a + b + c = K + 1$.

We determine the value of $K$ from a user-specified parameter $N_s$ for the preferred number of samples. $K$ is chosen as the smallest number such that the total number of sample points is at least $N_s$.

3 Computation of centroids

To compute the centroid $C$ of the final model, we consider the final model as the combination of a hollow polyhedron made from uniform thin-sheet materials, and a 3D volume shell with uniform density. Then

$$C = \frac{(C_1 V_1 - C_3 V_3) \rho_1 + C_2 A_2 \rho_2}{(V_1 - V_3) \rho_1 + A_2 \rho_2},$$

where $C_1, C_3$ are the solid centroids of the target shape and the polyhedron, respectively; $C_2$ is the surface centroid of the polyhedron; $V_1, V_3$ are the internal volumes of the target surface and the polyhedron, respectively; $A_2$ is the polyhedral surface area; $\rho_1$ and $\rho_2$ are parameters for the volume density of the 3D printed part and the area density of the laser-cut material, respectively. Here $V_1, V_3$ can be computed using Equation (1). Using the same notation as Equation (1), the solid centroid of a polyhedron shape can be computed as

$$C(P) = \frac{\sum_{t_i \in T} \det (p^1(t_i), p^2(t_i), p^3(t_i)) (p^1(t_i) + p^2(t_i) + p^3(t_i))}{4 \cdot \sum_{t_i \in T} \det (p^1(t_i), p^2(t_i), p^3(t_i))} , \quad (2)$$

while the surface area of a polyhedron is

$$A_P = \frac{1}{2} \sum_{t_i \in T} \|p^2(t_i) - p^1(t_i)\| \times \|p^3(t_i) - p^1(t_i)\|, \quad (3)$$

and its surface centroid is

$$C_A(P) = \frac{\sum_{t_i \in T} \|p^2(t_i) - p^1(t_i)\| \times \|p^3(t_i) - p^1(t_i)\| \sum_{j=1}^3 p^j(t_i)}{\sum_{t_i \in T} \|p^2(t_i) - p^1(t_i)\| \times \|p^3(t_i) - p^1(t_i)\|} , \quad (4)$$

$C_1, C_3$ are computed using formula (2), while $A_s$ and $C_2$ are computed using formulas (3) and (4), respectively.

4 Constraints for optimizing multiple polyhedrons

The two faces $(f_1, f_1')$ chosen for the connection between two polyhedrons must satisfy the following conditions:

1. $f_1, f_1'$ are parallel, with their outward normals pointing towards each other;

2. there exists a cylinder with radius $r$ and with its axis parallel to the normals of $f_1, f_1'$, such that its two ends touch the two faces $(f_1, f_1')$ and lie within the interior of each face, and the whole cylinder lie inside the target shape.
For the first condition, we require
\[ n_i^k + n_j^l = 0, \]
where \( n_i^k \) and \( n_j^l \) are the outward normal variables for the two faces.

For the second condition, we introduce auxiliary variables \( c_i^k, c_j^l \in \mathbb{R}^3 \) for the centers of the circles, where the cylinder touches the two faces. \( c_i^k \) and \( c_j^l \) are required to lie on the two faces, respectively. The line segment between these two points must be orthogonal to the two faces, thus requiring
\[ c_i^k + t_i^k n_i^k = c_j^l, \]
with auxiliary variable \( t_i^k > 0 \). Moreover, each face must be kept inside a disc with radius \( r \) and center \( c_i^k \) (or \( c_j^l \), respectively). Taking face \( f_i^k \) as an example, we require
\[ (c_i^k - p_{j_1}) \cdot \frac{n_i^k \times (p_{j_2} - p_{j_1})}{\|n_i^k \times (p_{j_2} - p_{j_1})\|} \geq r, \]
where \( p_{j_1}, p_{j_2} \) are two adjacent vertices in \( f_i^k \) in an appropriate order. A similar constraint is defined for face \( f_j^l \). Finally, we compute a set of sample points \( \{q\} \) on the cylinder, and enforce a constraint
\[ D(q) \geq d_{\text{min}}, \]
where \( D \) is the signed distance function from the surface of the whole object. Each sample \( q \) is computed as
\[ q = ac_i^k + (1 - a)n_j^l + r(e_1^{k,i} \cos b + e_2^{k,i} \cos b), \]
where parameters \( a \in [0, 1] \) and \( b \in [0, 2\pi] \) are pre-determined, \( e_1^{k,i}, e_2^{k,i} \) are auxiliary variables that form an orthonormal frame with \( n_i^k \), previously used for enforcing the bounding rectangle constraints.

References