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Some criteria for discreteness of spectrum of half-linear fourth order Sturm-Liouville problem

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Abstract. We prove a necessary and sufficient conditions for discreteness of the set of all eigenvalues of half–linear eigenvalue problem with locally integrable weights. Our conditions appear to be equivalent to the compact embedding of certain weighted Sobolev and Lebesgue spaces.

Key words: Hardy inequality, weighted spaces, Sturm–Liouville problem, oscillatory theory

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1 Introduction

Let $p > 1$ be a real number. We study the eigenvalue problem

$$\begin{cases} (\rho(t)\varphi(u''(t)))'' - \lambda\sigma(t)\varphi(u(t)) = 0, & t > 0, \\ u'(0) = (\rho(t)\varphi(u''(t)))'|_{t=0} = 0, & u(\infty) = u'(\infty) = 0 \end{cases} \tag{1.1}$$

where $\varphi(s) = |s|^{p-2}s$ for $s \neq 0$ and $\varphi(0) = 0$. For the functions $\rho = \rho(t)$ and $\sigma = \sigma(t)$ we assume continuity and positivity on $[0, \infty)$, with $t^{p'}\rho^{1-p'} \in L^1(0, \infty)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. We emphasize that we *do not assume* $\sigma \in L^1(0, \infty)$ in general!

By a *solution* of (1.1) we understand a function $u \in C^2(0, \infty)$ such that $\rho\varphi(u'') \in C^2(0, \infty)$, the equation in (1.1) holds at every point, the boundary conditions are satisfied and the Dirichlet integral $\int_0^\infty \rho(t)|u''(t)|^p dt$ is finite.

The parameter λ is called an *eigenvalue* of (1.1) if this problem has a nontrivial (i.e. nonzero) solution. This solution is then called an *eigenfunction* of (1.1) associated with λ.

We say that the *D-property* for (1.1) is satisfied if "the set of all eigenvalues of (1.1) forms an increasing sequence $\{\lambda_n\}_{n=1}^\infty$ such that $\lambda_1 > 0$ and $\lim_{n\to+\infty} \lambda_n = +\infty$; the set of all normalized eigenfunctions associated with a given eigenvalue is finite (multiplicity of the eigenvalue of nonlinear problem is finite); every eigenfunction has finite number of nodes."

Let $a, t \in [0, \infty)$ be such that $a \le t$ and denote

$$A_1(a;t) := \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^\infty (\tau - t)^{p'}\rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1};$$
$$A_2(a;t) := \left(\int_a^t (t-\tau)^p\sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^\infty \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1}.$$

The main result of this paper is the following theorem.

Theorem 1.1. *The D-property for (1.1) is satisfied if and only if the following two conditions hold:*

$$\limsup_{t\to\infty} A_1(0;t) = \limsup_{t\to\infty} A_2(0;t) = 0. \tag{1.2}$$

Remark 1.2. The conditions in (1.2) are equivalent to the *compact embedding*

$$W^{2,p}_\infty(\rho) \hookrightarrow\hookrightarrow L^p(\sigma), \tag{1.3}$$

where $L^p(\sigma)$ is the weighted Lebesgue space of all functions $u = u(t)$ defined on $(0, \infty)$, for which

$$\|u\|_{p;\sigma} \stackrel{def}{=} \left(\int_0^\infty \sigma(t)|u(t)|^p\,\mathrm{d}t\right)^{\frac{1}{p}} < \infty;$$

$W^{2,p}_\infty(\rho)$ is the weighted Sobolev space of all functions $u \in C^1[0, \infty)$, u' is absolutely continuous, $u'(0) = u(\infty) = u'(\infty) = 0$ and

$$\|u\|_{2,p;\rho} := \left(\int_0^\infty \rho(t)|u''(t)|^p\mathrm{d}t\right)^{\frac{1}{p}} < \infty. \tag{1.4}$$

Note that $L^p(\sigma)$ and $W^{2,p}_\infty(\rho)$ equipped with the norms $\|\cdot\|_{p;\sigma}$ and $\|\cdot\|_{2,p;\rho}$, respectively, are uniformly convex Banach spaces.

Remark 1.3. A key part of the 'D Property' is that all eigenfunctions have finitely many nodes. This is substantially more difficult to establish in the fourth order case considered here than in the more usual second-order case. For $\rho \equiv \sigma \equiv 1$, in $L^2(0,1)$, Pinkus [13] proved, as a key step in establishing various n-widths for approximations of functions in $L^p(0,1)$ by functions in $W^{r,p}(0,1)$, that for the corresponding problem of order $2r$ the n-th eigenfunction has n sign changes, at least for $n > r$. However a key step in his approach is the observation that the higher eigenfunctions are obtained by gluing together multiple copies of dilations of the lowest index eigenfunction, which does not work when ρ and σ are not constant. A further reason to be surprised by the 'finitely many nodes' property here is that when $p = 2$ and $\rho \equiv \sigma \equiv 1$, then the linear eigenvalue problem in $L^2(0,\infty)$ has spectrum bounded below, essential spectrum in $[0,\infty)$, yet the solutions of the differential equation all have infinitely many nodes, whatever value λ may take. This is in contrast to the second order case with $p = 2$ where it is well known that if the spectrum is bounded below then the n-th eigenfunction has $n-1$ zeros (see, e.g., Dunford and Schwartz [1, Chapter XIII]).

Remark 1.4. We conjecture that similar results can be proved also for other boundary conditions typical for the ordinary differential equations of the fourth order. However, some of the technical estimates might be different from those above. To avoid lengthening our current manuscript, we do not discuss this issue here in detail and postpone it for possible future research.

A function $u \in W^{2,p}_{\infty}(\rho)$ is called a *weak solution* of (1.1) if the integral identity

$$\int_0^\infty \rho(t)\varphi(u''(t))v''(t)\mathrm{d}t = \lambda \int_0^\infty \sigma(t)\varphi(u(t))v(t)\mathrm{d}t \tag{1.5}$$

holds for all $v \in W^{2,p}_{\infty}(\rho)$ (with both integrals being finite).

It is clear that every solution of (1.1) is also a weak solution. The converse is true as well. Indeed, take arbitrary $v \in C_0^\infty(0,\infty)$ (smooth functions with compact support in $(0,\infty)$) as a test function in (1.5) and integrate by parts. We get that there are constants $A, B \in \mathbb{R}$ such that

$$\rho(t)\varphi(u''(t)) = \lambda \int_0^t (t-s)\sigma(s)\varphi(u(s))\,\mathrm{d}s + A + Bt \tag{1.6}$$

for a.e. in $(0,\infty)$. Hence, continuity of $s \longmapsto \sigma(s)\varphi(u(s))$ in $[0,\infty)$ implies that $\rho\varphi(u'') \in C^2[0,\infty)$ and (1.6) (and thus also the equation in (1.1)) holds at every point $t \in (0,\infty)$. Now, testing (1.5) with $v \in W^{2,p}_{\infty}(\rho)$, $v(0) \neq 0$, $v \equiv 0$ in the left neighborhood of ∞, and integrating by parts we arrive at $(\rho(t)\varphi(u''(t)))'\,|_{t=0} = 0$. Since we have $u'(0) = u(\infty) = u'(\infty) = 0$ by $u \in W^{2,p}_{\infty}(\rho)$, a weak solution u is a solution in the sense of our definition at the same time.

Remark 1.5. Further we will consider problem (1.1) with positive λ, since, for nonpositive λ the problem has only trivial solutions.

There is a vast literature which deals with the boundedness from below and the discreteness of the spectrum of the linear Sturm–Liouville problem with singular and/or degenerate coefficients. However, there are not too many works dedicated to the same topic for nonlinear homogeneous problems. The main reason is the fact that the machinery of linear functional analysis cannot be applied. Let us mention the pioneering work [11] where new methods of nonlinear analysis had to be employed. These results were generalized in papers [2], [3] and [4] where an interesting connection between the discreteness of the spectrum of nonlinear Sturm–Liouville problem and the embeddings of weighted Sobolev and Lebesgue spaces was also revealed.

Nonlinear homogeneous Sturm–Liouville problems of the fourth order were studied in paper [7]. The authors address similar issues as for the second order problem. The *purpose of* our *paper* is to deal with a rather general *nonlinear Sturm–Liouville problem* of the *fourth order* with *degenerate* and/or *singular* coefficients. We prove *necessary* and *sufficient conditions* (1.2) for the *discreteness* of the set of all *eigenvalues* and *isolatedness* of the set of all *normalized eigenfunctions.* We relate our conditions to a *compact embedding* between suitable *weighted* Sobolev and Lebesgue spaces (1.3).

This paper is organized as follows. In Section 2 we present sufficient conditions which guarantee that solutions of (1.1) have either an infinite or else a finite number of nodes in $(0,\infty)$. Section 3 brings sufficient conditions for discreteness of the set of normalized eigenfunctions. The proof of the main result is elaborated in Section 4.

2 Oscillation and nonoscillation results

A solution $u = u(t)$ of problem (1.1) is called *nonoscillatory*, if there exists $T \in (0,\infty)$ such that $u(t) \neq 0$ for all $t \in (T, \infty)$. Otherwise, the solution is called *oscillatory.*

Oscillation results. In this section we first discuss oscillatory solutions of (1.1).

Theorem 2.1. *Let λ be an eigenvalue of (1.1). If*

$$\limsup_{t\to\infty} A_1(0;t) > \frac{1}{\lambda} \qquad \text{or} \qquad \limsup_{t\to\infty} A_2(0;t) > \frac{1}{\lambda} \tag{2.1}$$

then any eigenfunction associated with λ is oscillatory.

Proof. We prove the theorem by contradiction. Let assumptions of the theorem hold, but suppose problem (1.1) has a nonoscillatory solution u. Then there exists T such that u and u'' do not change the sign in (T,∞) and $u''(T) = 0$. Indeed, it is enough to prove that u'' can have only finite number of zeros in $(0,\infty)$. In fact, if this is not the case, we apply the Lagrange mean value theorem to $\rho(t)\varphi(u''(t))$ between its zero points and derive that $\big(\rho(t)\varphi(u''(t))\big)'$ has infinitely many zero points. Repeating this argument we get that $\big(\rho(t)\varphi(u''(t))\big)''$ has also infinitely many zero points and then from the equation it would follow that u is oscillatory, a contradiction.

Further, without loss of generality we assume that the function u is positive in (T,∞). Then it can be shown by using the boundary conditions at infinity that u' is also positive in (T,∞). Moreover, using

$$\rho(t)\varphi(u''(t)) = \lambda \int_T^t (\rho(\tau)\varphi(u''(\tau)))'\,\mathrm{d}\tau > 0 \qquad \text{for all} \quad t \in (T,\infty),$$

we get the existence of $T_1 \in (T,\infty)$ such that $(\rho(t)\varphi(u''(t)))'\,|_{t=T_1} > 0$.

Successively integrating both sides of the equation in (1.1) over the interval $(T_1,\, t)$ we have

$$\rho(t)\varphi(u''(t)) = \lambda \int_{T_1}^t (t-\tau)\sigma(\tau)\varphi(u(\tau))\mathrm{d}\tau + A(t-T_1) + B, \tag{2.2}$$

where $A = (\rho(t)\varphi(u''(t)))'|_{t=T_1}$ and $B = \rho(T_1)\varphi(u''(T_1))$. From this we find

$$u''(t) \;=\; \rho^{1-p'}(t)\varphi^{-1}\left(\lambda \int_{T_1}^t (t-\tau)\sigma(\tau)\varphi(u(\tau))\mathrm{d}\tau + A(t-T_1) + B\right).$$

Since $u(\infty) = u'(\infty) = 0$, we get

$$
\begin{aligned}
u(t) &= -\int_t^\infty u'(s)\,\mathrm{d}s \\
&= \int_t^\infty \left[\int_\tau^\infty u''(s)\,\mathrm{d}s\right] \mathrm{d}\tau \\
&= \int_t^\infty (s-t)u''(s)\,\mathrm{d}s \\
&= \int_t^\infty (s-t)\rho^{1-p'}(s)\varphi^{-1}\left(\lambda \int_{T_1}^s (s-\tau)\sigma(\tau)\varphi(u(\tau))\mathrm{d}\tau + A(s-T_1) + B\right) \mathrm{d}s,
\end{aligned}
$$

i.e.,

$$
u(t) = \int_t^\infty (s-t)\rho^{1-p'}(s)\varphi^{-1}\left(\lambda \int_{T_1}^s (s-\tau)\sigma(\tau)\varphi(u(\tau))\mathrm{d}\tau + A(s-T_1) + B\right) \mathrm{d}s \tag{2.3}
$$

for $t \in (T_1, \infty)$. Taking into account the positivity of A and B we obtain from (2.3) that the function u is decreasing in (T_1, ∞).

Using the monotonicity of u, we estimate the right hand side of (2.3) for $t \in (T, \infty)$:

$$
\begin{aligned}
u(t) &= \int_t^\infty (s-t)\rho^{1-p'}(s)\Big(\lambda \int_{T_1}^s (s-\tau)\sigma(\tau)\varphi(u(\tau))\mathrm{d}\tau \\
&\quad + A(s-T_1) + B\Big)^{\frac{1}{p-1}} \mathrm{d}s \\
&> \int_t^\infty (s-t)\rho^{1-p'}(s)\left(\lambda \int_{T_1}^s (s-\tau)\sigma(\tau)\varphi(u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s \\
&\geq \int_t^\infty (s-t)\rho^{1-p'}(s)\left(\lambda \int_{T_1}^t (s-\tau)\sigma(\tau)\varphi(u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \mathrm{d}s \\
&\geq \int_t^\infty (s-t)^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\left(\lambda \int_{T_1}^t \sigma(\tau)\varphi(u(\tau))\mathrm{d}\tau\right)^{\frac{1}{p-1}} \\
&\geq \int_t^\infty (s-t)^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\left(\lambda \int_{T_1}^t \sigma(\tau)\mathrm{d}\tau\right)^{\frac{1}{p-1}} u(t),
\end{aligned}
$$

i.e.

$$
\limsup_{t\to\infty}\left(\int_t^\infty (s-t)^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\right)^{p-1}\left(\int_{T_1}^t \sigma(\tau)\mathrm{d}\tau\right) \leq \frac{1}{\lambda}.
$$

Consequently, due to the assumptions on the weights, we obtain

$$
\begin{aligned}
& \limsup_{t\to\infty} \left(\int_0^t \sigma(\tau)\,\mathrm{d}\tau \right) \left(\int_t^\infty (\tau - t)^{p'} \rho(\tau)^{1-p'}\,\mathrm{d}\tau \right)^{p-1} \\
= & \limsup_{t\to\infty} \left[\int_0^{T_1} \sigma(\tau)\,\mathrm{d}\tau + \int_{T_1}^t \sigma(\tau)\,\mathrm{d}\tau \right] \left(\int_t^\infty (\tau - t)^{p'} \rho(\tau)^{1-p'}\,\mathrm{d}\tau \right)^{p-1} \\
= & \limsup_{t\to\infty} \left[\int_0^{T_1} \sigma(\tau)\,\mathrm{d}\tau \left(\int_t^\infty (\tau - t)^{p'} \rho(\tau)^{1-p'}\,\mathrm{d}\tau \right)^{p-1} \right. \\
& \qquad \left. + \int_{T_1}^t \sigma(\tau)\,\mathrm{d}\tau \left(\int_t^\infty (\tau - t)^{p'} \rho(\tau)^{1-p'}\,\mathrm{d}\tau \right)^{p-1} \right] \\
= & \limsup_{t\to\infty} \left(\int_{T_1}^t \sigma(\tau)\,\mathrm{d}\tau \right) \left(\int_t^\infty (\tau - t)^{p'} \rho(\tau)^{1-p'}\,\mathrm{d}\tau \right)^{p-1} \le \frac{1}{\lambda},
\end{aligned}
$$

i.e.

$$
\limsup_{t\to\infty} A_1(0;t) \le \frac{1}{\lambda}.
$$

This is a contradiction with (2.1).

To obtain similar estimate for A_2 we proceed as follows:

If we denote

$$
v(t) = \rho(t)\varphi(u''(t))
$$

then $u''(t) = \rho^{1-p'}(t)\varphi^{-1}(v(t))$ and from (2.2) we get

$$
\begin{aligned}
v(t) &= \lambda \int_{T_1}^t (t-\tau)\sigma(\tau)\varphi(u(\tau))\mathrm{d}\tau + A(t-T_1) + B \\
&= \lambda \int_{T_1}^t (t-\tau)\sigma(\tau)\varphi\left(\int_\tau^\infty (s-\tau)u''(s)\mathrm{d}s \right)\mathrm{d}\tau + A(t-T_1) + B \\
&= \lambda \int_{T_1}^t (t-\tau)\sigma(\tau)\varphi\left(\int_\tau^\infty (s-\tau)\rho^{1-p'}(s)\varphi^{-1}(v(s))\mathrm{d}s \right)\mathrm{d}\tau + A(t-T_1) + B.
\end{aligned}
$$

From (2.2) we obtain also that v is positive and monotone increasing in (T_1,∞), which we use to estimate v as follows:

$$
\begin{aligned}
v(t) &\ge \lambda \int_{T_1}^t (t-\tau)\sigma(\tau)\varphi\left(\int_\tau^\infty (s-\tau)\rho^{1-p'}(s)\varphi^{-1}(v(s))\mathrm{d}s \right)\mathrm{d}\tau \\
&\ge \lambda \int_{T_1}^t (t-\tau)\sigma(\tau)\varphi\left(\int_t^\infty (s-\tau)\rho^{1-p'}(s)\varphi^{-1}(v(s))\mathrm{d}s \right)\mathrm{d}\tau \\
&\ge \lambda \int_{T_1}^t (t-\tau)^p\sigma(\tau)\mathrm{d}\tau\,\varphi\left(\int_t^\infty \rho^{1-p'}(s)\varphi^{-1}(v(s))\mathrm{d}s \right) \\
&\ge \lambda \int_{T_1}^t (t-\tau)^p\sigma(\tau)\mathrm{d}\tau \left(\int_t^\infty \rho^{1-p'}(s)\mathrm{d}s \right)^{p-1} v(t).
\end{aligned}
$$

This implies that

$$\int_{T_1}^{t}(t-\tau)^p\sigma(\tau)\mathrm{d}\tau\left(\int_t^\infty \rho^{1-p'}(s)\mathrm{d}s\right)^{p-1}\leq\frac{1}{\lambda}$$

for all $t\in(T_1,\infty)$. From this we obtain that

$$\begin{aligned}
& \limsup_{t\to\infty}\int_0^t(t-\tau)^p\sigma(\tau)\mathrm{d}\tau\left(\int_t^\infty \rho^{1-p'}(s)\mathrm{d}s\right)^{p-1} \\
= \ & \limsup_{t\to\infty}\left[\int_0^{T_1}(t-\tau)^p\sigma(\tau)\mathrm{d}\tau+\int_{T_1}^{t}(t-\tau)^p\sigma(\tau)\mathrm{d}\tau\right]\left(\int_t^\infty \rho^{1-p'}(s)\mathrm{d}s\right)^{p-1} \\
= \ & \limsup_{t\to\infty}\left[\int_0^{T_1}(t-\tau)^p\sigma(\tau)\mathrm{d}\tau\left(\int_t^\infty \rho^{1-p'}(s)\mathrm{d}s\right)^{p-1}\right. \\
& \left.+\int_{T_1}^{t}(t-\tau)^p\sigma(\tau)\mathrm{d}\tau\left(\int_t^\infty \rho^{1-p'}(s)\mathrm{d}s\right)^{p-1}\right] \\
\leq \ & \limsup_{t\to\infty}\left[\int_0^{T_1}\sigma(\tau)\mathrm{d}\tau\left(\int_t^\infty s^{p'}\rho^{1-p'}(s)\mathrm{d}s\right)^{p-1}+\frac{1}{\lambda}\right] \\
\leq \ & \frac{1}{\lambda}.
\end{aligned}$$

Consequently, we get that

$$\limsup_{t\to\infty}A_2(0;t)\leq\frac{1}{\lambda}$$

contradicting again (2.1). Theorem 2.1 is proved. □

Nonoscillation results. Further, we suppose that

$$\sup_{t>0}A_1(0;t)<\infty \qquad \text{and} \qquad \sup_{t>0}A_2(0;t)<\infty.$$

These conditions are equivalent to the *continuous embedding*

$$W^{2,p}_{\infty}(\rho)\hookrightarrow L^p(\sigma). \tag{2.4}$$

Lemma 2.2. *Let $0\leq a<b\leq\infty$, then inequality*

$$\int_a^b\left(\int_x^b(t-x)w(t)\,\mathrm{d}t\right)^p\sigma(x)\,\mathrm{d}x\leq C\int_a^b w^p(x)\rho(x)\,\mathrm{d}x \tag{2.5}$$

or its equivalent form

$$\int_a^b\left(\int_a^x(x-t)w(t)\,\mathrm{d}t\right)^{p'}\rho^{1-p'}(x)\,\mathrm{d}x\leq C^{p'-1}\int_a^b w^{p'}(x)\sigma^{1-p'}(x)\,\mathrm{d}x \tag{2.6}$$

holds for all measurable $w(x) \geq 0$ on $(a,\, b)$ if and only if

$$\hat{A}_1(a,b) \;:=\; \sup_{(a,b)} \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau \right) \left(\int_t^b (\tau - t)^{p'} \rho^{1-p'}(\tau)\,\mathrm{d}\tau \right)^{p-1} < \infty, \tag{2.7}$$

$$\hat{A}_2(a,b) \;:=\; \sup_{(a,b)} \left(\int_a^t (t-\tau)^p \sigma(\tau)\,\mathrm{d}\tau \right) \left(\int_t^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau \right)^{p-1} < \infty. \tag{2.8}$$

Moreover, the best constant $C = C(a,b)$ in (2.5) satisfies

$$\hat{A}(a,b) \leq C(a,b) \leq 2^{(p-1)p+1} p^p (p')^p \hat{A}(a,b), \tag{2.9}$$

where

$$\hat{A}(a,b) := \max\{\hat{A}_1(a,b), \hat{A}_2(a,b)\}. \tag{2.10}$$

Proof. The necessity and sufficiency of conditions (2.7), (2.8) for satisfying the inequality and the lower estimate $\hat{A}(a,b) \leq C(a,b)$ (even more general cases) can be found in [9, Theorem 4]. Further we prove the upper estimate

$$C(a,b) \leq 2^{(p-1)p+1} p^p (p')^p \hat{A}(a,b).$$

Using Fubini's theorem and the inequality $(a+b)^{p-1} \leq 2^{p-1}(a^{p-1}+b^{p-1})$, we estimate the left hand side of the inequality in the form

$$\begin{aligned}
I &= \int_a^b \sigma(x) \left(\int_x^b (t-x) w(t)\,\mathrm{d}t \right)^p \mathrm{d}x \\
&= p \int_a^b \sigma(x) \left[\int_x^b (t-x) w(t) \left(\int_t^b (s-x) w(s)\,\mathrm{d}s \right)^{p-1} \mathrm{d}t \right] \mathrm{d}x \\
&= p \int_a^b w(t) \left[\int_a^t (t-x) \sigma(x) \left(\int_t^b (s-x) w(s)\,\mathrm{d}s \right)^{p-1} \mathrm{d}x \right] \mathrm{d}t \\
&= p \int_a^b w(t) \left[\int_a^t (t-x) \sigma(x) \left(\int_t^b (s-t) w(s)\,\mathrm{d}s + (t-x) \int_t^b w(s)\,\mathrm{d}s \right)^{p-1} \mathrm{d}x \right] \mathrm{d}t \\
&\leq 2^{p-1} p \left[\int_a^b w(t) \left(\int_t^b (s-t) w(s)\,\mathrm{d}s \right)^{p-1} \left(\int_a^t (t-x) \sigma(x)\,\mathrm{d}x \right) \mathrm{d}t \right. \\
&\quad \left. + \int_a^b w(t) \left(\int_t^b w(s)\,\mathrm{d}s \right)^{p-1} \left(\int_a^t (t-x)^p \sigma(x)\,\mathrm{d}x \right) \mathrm{d}t \right] \\
&=: 2^{p-1} p \left[I_1 + I_2 \right].
\end{aligned}$$

Now we estimate I_1 and I_2, separately.

$$\begin{aligned}
I_1 &:= \int_a^b w(t) \left(\int_t^b (s-t) w(s)\,\mathrm{d}s \right)^{p-1} \left(\int_a^t (t-x) \sigma(x)\,\mathrm{d}x \right) \mathrm{d}t \\
&\leq \left(\int_a^b w^p(t) \rho(t)\,\mathrm{d}t \right)^{1/p} \left(\int_a^b \rho^{1-p'}(t) \left(\int_a^t (t-x) \sigma(x)\,\mathrm{d}x \right)^{p'} \left(\int_t^b (s-t) w(s)\,\mathrm{d}s \right)^p \mathrm{d}t \right)^{1/p'}.
\end{aligned}$$

The second term on the right hand side can be written as

$$
\begin{aligned}
&\left(\int_a^b \rho^{1-p'}(t)\left(\int_a^t (t-x)\sigma(x)\,\mathrm{d}x\right)^{p'}\left(-\int_t^b \mathrm{d}\left(\int_\tau^b (s-\tau)w(s)\,\mathrm{d}s\right)^p\right)\mathrm{d}t\right)^{1/p'}\\
= &\left(-\int_a^b\left[\int_a^\tau \rho^{1-p'}(t)\left(\int_a^t (t-x)\sigma(x)\,\mathrm{d}x\right)^{p'}\mathrm{d}t\right]\mathrm{d}\left(\int_\tau^b (s-\tau)w(s)\,\mathrm{d}s\right)^p\right)^{1/p'}.
\end{aligned}
$$

By Minkowski's integral inequality we have

$$
\begin{aligned}
\int_a^\tau \rho^{1-p'}(t)\left(\int_a^t (t-x)\sigma(x)\,\mathrm{d}x\right)^{p'}\mathrm{d}t &= \left(\left[\int_a^\tau \rho^{1-p'}(t)\left(\int_a^t (t-x)\sigma(x)\,\mathrm{d}x\right)^{p'}\mathrm{d}t\right]^{1/p'}\right)^{p'}\\
&\le \left(\int_a^\tau \sigma(x)\left(\int_x^\tau (t-x)^{p'}\rho^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'}\mathrm{d}x\right)^{p'}\\
&\le \left(\int_a^\tau \sigma(x)\left(\int_x^b (t-x)^{p'}\rho^{1-p'}(t)\,\mathrm{d}t\right)^{1/p'}\mathrm{d}x\right)^{p'}\\
&\le A_1(a,b)^{\frac{p'}{p}}\left(\int_a^\tau \sigma(x)\left(\int_a^x \sigma(t)\,\mathrm{d}t\right)^{-1/p}\mathrm{d}x\right)^{p'}\\
&= A_1(a,b)^{\frac{p'}{p}}\left(p'\int_a^\tau \mathrm{d}\left(\int_a^x \sigma(t)\,\mathrm{d}t\right)^{1/p'}\right)^{p'}\\
&= (p')^{p'}A_1(a,b)^{\frac{p'}{p}}\left(\int_a^\tau \sigma(t)\,\mathrm{d}t\right).
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_1 &\le p'A_1(a,b)^{\frac{1}{p}}\|w\|_{p,\rho}\left(-\int_a^b\left(\int_a^\tau \sigma(t)\,\mathrm{d}t\right)\mathrm{d}\left(\int_\tau^b (s-\tau)w(s)\,\mathrm{d}s\right)^p\right)^{1/p'}\\
&= p'A_1(a,b)^{\frac{1}{p}}\|w\|_{p,\rho}\left(\int_a^b \sigma(t)\left(\int_t^b (s-t)w(s)\,\mathrm{d}s\right)^p\mathrm{d}t\right)^{1/p'}\\
&= p'A_1(a,b)^{\frac{1}{p}}\|w\|_{p,\rho}I^{1/p'}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= \int_a^b w(t)\left(\int_t^b w(s)\,\mathrm{d}s\right)^{p-1}\left(\int_a^t (t-x)^p\sigma(x)\,\mathrm{d}x\right)\,\mathrm{d}t\\
&\le \left(\int_a^b w^p(t)\rho(t)\,\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_a^b \rho^{1-p'}(t)\left(\int_t^b w(s)\,\mathrm{d}s\right)^p\left(\int_a^t (t-x)^p\sigma(x)\,\mathrm{d}x\right)^{p'}\,\mathrm{d}t\right)^{\frac{1}{p'}}\\
&\le \hat{C}^{p-1}\left(\int_a^b w^p(t)\rho(t)\,\mathrm{d}t\right).
\end{aligned}
$$

To get the last estimate we used the classical Hardy inequality [12, page 40] where the constant is estimated by

$$
\hat{C} \le p^{1/p}(p')^{1/p'}\sup_{t>a}\left(\int_a^t\left(\int_a^\tau (\tau-x)^p\sigma(x)\,\mathrm{d}x\right)^{p'}\rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{1/p}\left(\int_t^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{1/p'}.
$$

Here, estimating the integral

$$
\begin{aligned}
\int_a^t\left(\int_a^\tau (\tau-x)^p\sigma(x)\,\mathrm{d}x\right)^{p'}\rho^{1-p'}(\tau)\,\mathrm{d}\tau &\le A_2(a,b)^{p'}\int_a^t \rho^{1-p'}(\tau)\left(\int_\tau^b \rho^{1-p'}(s)\,\mathrm{d}s\right)^{-p}\,\mathrm{d}\tau\\
&= \frac{A_2(a,b)^{p'}}{p-1}\left(\int_\tau^b \rho^{1-p'}(s)\,\mathrm{d}s\right)^{1-p}\Bigg|_a^t\\
&\le \frac{A_2(a,b)^{p'}}{p-1}\left(\int_t^b \rho^{1-p'}(s)\,\mathrm{d}s\right)^{1-p}
\end{aligned}
$$

we get

$$
\hat{C} \le p^{1/p}(p')^{1/p'}(p-1)^{-1/p}A_2(a,b)^{\frac{p'}{p}} = p'A_2(a,b)^{\frac{p'}{p}}.
$$

Therefore,

$$
I_2 \le \hat{C}^{p-1}\left(\int_a^b w^p(t)\rho(t)\,\mathrm{d}t\right) \le (p')^{p-1}A_2(a,b)\|w\|_{p,\rho}^p.
$$

Thus,

$$
\begin{aligned}
I &\le 2^{p-1}p\,(I_1+I_2) \le 2^{p-1}p\left(p'A_1(a,b)^{\frac{1}{p}}\|w\|_{p,\rho}I^{1/p'} + (p')^{p-1}A_2(a,b)\|w\|_{p,\rho}^p\right)\\
&= 2^{p-1}pp'A_1(a,b)^{\frac{1}{p}}\|w\|_{p,\rho}I^{1/p'} + 2^{p-1}p(p')^{p-1}A_2(a,b)\|w\|_{p,\rho}^p.
\end{aligned}
$$

Using Young's inequality $ab \le \frac{a^p}{p}+\frac{b^{p'}}{p'}$, with $a = 2^{p-1}pp'A_1(a,b)^{\frac{1}{p}}\|w\|_{p,\rho}$ and $b = I^{1/p'}$, we obtain

$$
I \le \frac{2^{(p-1)p}p^p(p')^p}{p}A_1(a,b)\|w\|_{p,\rho}^p + \frac{I}{p'} + 2^{p-1}p(p')^{p-1}A_2(a,b)\|w\|_{p,\rho}^p,
$$

which implies

$$
\begin{aligned}
I &\leq p\left(\frac{2^{(p-1)p}p^p(p')^p}{p}A_1(a,b)+2^{p-1}p(p')^{p-1}A_2(a,b)\right)\|w\|_{p,\rho}^p \\
&\leq 2^{(p-1)p+1}p^p(p')^pA(a,b)\|w\|_{p,\rho}^p.
\end{aligned}
$$

The proof is complete. □

The following property of solutions of (1.1) will be used in the proof of the next result.

Proposition 2.3. *Every solution u of problem (1.1) satisfies*

$$
\lim_{t\to\infty}\left[\rho(t)\varphi(u''(t))u'(t)-(\rho(t)\varphi(u''(t)))'u(t)\right]=0.
$$

Proof. We prove the proposition by estimating separately each term of the expression

$$
\rho(t)\varphi(u''(t))u'(t)-(\rho(t)\varphi(u''(t)))'u(t).
$$

Using the Lagrange mean value theorem with respect to the boundary conditions $u'(0)=u'(\infty)=0$ we obtain the existence of $\xi\in(0,\infty)$ such that $u''(\xi)=0$ and then

$$
\begin{aligned}
|\rho(t)\varphi(u''(t))| &= \left|\int_\xi^t(\rho(s)\varphi(u''(s)))'\,\mathrm{d}s\right| \\
&= \left|\int_\xi^t\left(\int_0^s(\rho(\tau)\varphi(u''(\tau)))''\,\mathrm{d}\tau\right)\mathrm{d}s\right| \\
&= \lambda\left|\int_\xi^t\left(\int_0^s\sigma(\tau)\varphi(u(\tau))\,\mathrm{d}\tau\right)\mathrm{d}s\right| \\
&\leq \lambda\int_0^t\left(\int_0^s\sigma(\tau)|u(\tau)|^{p-1}\,\mathrm{d}\tau\right)\mathrm{d}s \\
&= \lambda\int_0^t(t-\tau)\sigma(\tau)|u(\tau)|^{p-1}\,\mathrm{d}\tau \\
&\leq \lambda\left(\int_0^t(t-\tau)^p\sigma(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^t|u(\tau)|^p\sigma(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p'}}.
\end{aligned}
$$

Similarly, it can be proved also that

$$
|(\rho(t)\varphi(u''(t)))'| \leq \lambda\left(\int_0^t\sigma(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}}\left(\int_0^t|u(\tau)|^p\sigma(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p'}}.
$$

Using the conditions $u(\infty)=u'(\infty)=0$ we get also the following estimates

$$
\begin{aligned}
|u(t)| &= \left|\int_t^\infty u'(s)\,\mathrm{d}s\right| = \left|\int_t^\infty(\tau-t)u''(\tau)\,\mathrm{d}\tau\right| \\
&\leq \left(\int_t^\infty(\tau-t)^{p'}\rho(\tau)^{1-p'}\,\mathrm{d}\tau\right)^{\frac{1}{p'}}\left(\int_t^\infty|u''(\tau)|^p\rho(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}}
\end{aligned}
$$

and

$$
\begin{aligned}
|u'(t)| &= \left|\int_t^\infty u''(s)\,\mathrm{d}s\right| \\
&\le \left(\int_t^\infty \rho(\tau)^{1-p'}\,\mathrm{d}\tau\right)^{\frac{1}{p'}} \left(\int_t^\infty |u''(\tau)|^p\rho(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}}.
\end{aligned}
$$

All the foregoing estimates and (2.4) (i.e., if u is a solution of (1.1) then (2.4) implies that $u \in L^p(\sigma)$) imply that

$$
\begin{aligned}
& |\rho(t)\varphi(u''(t))u'(t) - (\rho(t)\varphi(u''(t)))'u(t)| \\
\le\ & |\rho(t)\varphi(u''(t))u'(t)| + |(\rho(t)\varphi(u''(t)))'u(t)| \\
\le\ & \lambda(A_1(0;t) + A_2(0;t)) \left(\int_0^t |u(\tau)|^p\sigma(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p'}} \left(\int_t^\infty |u''(\tau)|^p\rho(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}} \\
\le\ & \lambda\big(\sup_{t>0} A_1(0;t) + \sup_{t>0} A_2(0;t)\big)\|u\|_{p,\sigma}^{\frac{p}{p'}} \left(\int_t^\infty |u''(\tau)|^p\rho(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}},
\end{aligned}
$$

that is

$$
|\rho(t)\varphi(u''(t))u'(t) - (\rho(t)\varphi(u''(t)))'u(t)| \le C\left(\int_t^\infty |u''(\tau)|^p\rho(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}},
$$

where the constant C does not depend on t. Taking the limit as t approaches infinity in both sides of this equality we get the assertion. $\square$

Let $s \in (0,\infty)$ and $v \in W^{2,p}_\infty(\rho)$. Let us introduce the following functional

$$
\mathcal{F}(s;v) := \int_s^\infty \Big(\rho(t)|v''(t)|^p - \lambda\sigma(t)|v(t)|^p\Big)\,\mathrm{d}t.
$$

Lemma 2.4. *Let λ be eigenvalue of (1.1) and let there exist $T \in (0,\infty)$ such that for every $s \in (T,\infty)$ the following inequality*

$$
\mathcal{F}(s;v) > 0 \tag{2.11}
$$

holds for all $v \neq 0$, $v \in W^{2,p}_\infty(\rho)$. Then every eigenfunction associated with λ has finite number of zeros in $(0,\infty)$.

Proof. We argue by contradiction, i.e. let the assumptions of the lemma be satisfied, but suppose the problem has an oscillatory solution u. Using integration by parts and Proposition 2.3 we get

$$
\begin{aligned}
\mathcal{F}(s;u) &= \int_s^\infty \Big((\rho(t)\varphi(u''(t)))'' - \lambda\sigma(t)\varphi(u(t))\Big)u(t)\,\mathrm{d}t \\
&\quad + \Big[\rho(t)\varphi(u''(t))u'(t) - (\rho(t)\varphi(u''(t)))'u(t)\Big]_s^\infty \\
&= (\rho(s)\varphi(u''(s)))'u(s) - \rho(s)\varphi(u''(s))u'(s).
\end{aligned}
$$

Let $\{t_k\}_{k=1}^{\infty}$ be zero points of u. Then for every $T > 0$ there exists an interval $(t_k, t_{k+1}) \subset (T, \infty)$ such that $u(t) > 0$ for all $t \in (t_k, t_{k+1})$. From the positivity of the functional we get that at any zero point of u the first derivative of u could not be zero, which means that the function u will be negative in the next interval (t_{k+1}, t_{k+2}). It can be shown that there exist $a_k \in (t_k, t_{k+1})$ and $b_{k+1} \in (t_{k+1}, t_{k+2})$ such that $(\rho(t)\varphi(u''(t)))'|_{t=a_k} = 0$ and $u'(b_{k+1}) = 0$.

Integrating the equation in (1.1) twice, starting from a_k, we get

$$\rho(s)\varphi(u''(s)) = \lambda \int_{a_k}^{s} (s-t)\sigma(t)\varphi(u(t))\,\mathrm{d}t + A_k$$

and then

$$u''(s) = \rho^{1-p'}(s)\varphi^{-1}\left(\lambda \int_{a_k}^{s} (s-t)\sigma(t)\varphi(u(t))\,\mathrm{d}t + A_k\right),$$

where $A_k = \rho(a_k)\varphi(u''(a_k))$. Here $A_k < 0$, which follows from the positivity of the functional and the solution in (t_k, t_{k+1}).

Integrating the last equality twice, using the conditions at a_k and b_{k+1}, we get

$$u(x) = \int_{x}^{b_{k+1}} (s-x)\rho^{1-p'}(s)\varphi^{-1}\left(\lambda \int_{a_k}^{s} (s-t)\sigma(t)\varphi(u(t))\,\mathrm{d}t + A_k\right)\mathrm{d}s + B_{k+1},$$

where $B_{k+1} = u(b_{k+1})$. Since u changes its sign in (t_{k+1}, t_{k+2}) to the negative, i.e. $B_{k+1} < 0$, which implies that

$$u(x) < \lambda^{\frac{1}{p-1}} \int_{x}^{b_{k+1}} (s-x)\rho^{1-p'}(s)\varphi^{-1}\left(\int_{a_k}^{s} (s-t)\sigma(t)\varphi(u(t))\,\mathrm{d}t\right)\mathrm{d}s \tag{2.12}$$

for all $x \in (a_k, t_{k+1})$. From this we get that

$$u(x)\chi_{(a_k,t_{k+1})}(x) < \lambda^{\frac{1}{p-1}} \int_{x}^{b_{k+1}} (s-x)\rho^{1-p'}(s)\varphi^{-1}\left(\int_{a_k}^{s} (s-t)\sigma(t)\varphi(u(t)\chi_{(a_k,t_{k+1})}(t))\,\mathrm{d}t\right)\mathrm{d}s \tag{2.13}$$

holds for all $x \in (a_k, b_{k+1})$.

Multiplying both sides of the estimate by $\sigma(x)$ and integrating over the interval (a_k, b_{k+1}) we have

$$\begin{aligned}
& \int_{a_k}^{b_{k+1}} \sigma(x)\big(u(x)\chi_{(a_k,t_{k+1})}(x)\big)^p\,\mathrm{d}x \\
< \; & \lambda^{p'} \int_{a_k}^{b_{k+1}} \sigma(x)\left[\int_{x}^{b_{k+1}} (s-x)\rho^{1-p'}(s)\varphi^{-1}\left(\int_{a_k}^{s} (s-t)\sigma(t)\varphi(u(t)\chi_{(a_k,t_{k+1})}(t))\,\mathrm{d}t\right)\mathrm{d}s\right]^p \mathrm{d}x \\
< \; & \lambda^{p'} C_k \int_{a_k}^{b_{k+1}} \rho^{1-p'}(s)\left(\int_{a_k}^{s} (s-t)\sigma(t)\varphi(u(t)\chi_{(a_k,t_{k+1})}(t))\,\mathrm{d}t\right)^{p'} \mathrm{d}s \qquad (2.14) \\
< \; & \lambda^{p'} C_k^{p'} \int_{a_k}^{b_{k+1}} \sigma(x)\big(u(x)\chi_{(a_k,t_{k+1})}(x)\big)^p\,\mathrm{d}x. \qquad (2.15)
\end{aligned}$$

To get (2.14) and (2.15) we successively used (2.5) and (2.6) with respect to the functions $w(s) = \rho^{1-p'}(s)\varphi^{-1}\left(\int_{a_k}^{s}(s-t)\sigma(t)\varphi(u(t)\chi_{(a_k,t_{k+1})}(t))\,\mathrm{d}t\right)$ and $w(t) = \sigma(t)\varphi(u(t)\chi_{(a_k,t_{k+1})}(t))$, respectively.

From (2.15) we have that

$$C_k\lambda > 1. \tag{2.16}$$

From the other side (2.11) implies that

$$\lambda\int_s^\infty \sigma(t)|v(t)|^p\,\mathrm{d}t < \int_s^\infty \rho(t)|v''(t)|^p\,\mathrm{d}t$$

for all $v \neq 0$, $v \in W^{2,p}_\infty(\rho)$, where $s > T$ is arbitrary. Then using the boundary conditions at infinity we obtain

$$\int_s^\infty \sigma(t)\left|\int_t^\infty (t-s)v''(s)\,\mathrm{d}s\right|^p\,\mathrm{d}t < \frac{1}{\lambda}\int_s^\infty \rho(t)|v''(t)|^p\,\mathrm{d}t.$$

Since this inequality holds for all $s > T$ and $v \in W^{2,p}_\infty(\rho)$, $v \neq 0$. Which implies that is true for the functions $supp\, v \subset (a_k, b_k)$. From this and Lemma 2.2 we get that

$$C_k\lambda \leq 1,$$

which is a contradiction to (2.16). □

Theorem 2.5. *Let λ be an eigenvalue such that*

$$\max\{\limsup_{t\to\infty} A_1(0;t), \limsup_{t\to\infty} A_2(0;t)\} < \frac{2^{(1-p)p-1}p^{-p}(p')^{-p}}{\lambda}, \tag{2.17}$$

then every eigenfunction associated with λ has finite number of zeros.

Proof. We apply Lemma 2.4. From (2.17) we obtain that there exists $T \in (0,\infty)$ such that

$$A(s) \;<\; \frac{2^{(1-p)p-1}p^{-p}(p')^{-p}}{\lambda} \tag{2.18}$$

holds for all $s \in (T, \infty)$. Let $v \in W^{2,p}_\infty(\rho)$ be arbitrary but fixed. Now by using inequality (2.5) with $w = |v''|$, we estimate the following integral

$$\begin{aligned}\int_s^\infty \sigma(t)|v(t)|^p\,\mathrm{d}t &= \int_s^\infty \sigma(t)\left|\int_t^\infty (t-s)v''(s)\,\mathrm{d}s\right|^p\,\mathrm{d}t\\ &\leq C(s)\int_s^\infty \rho(t)|v''(t)|^p\,\mathrm{d}t\\ &< \frac{1}{\lambda}\int_s^\infty \rho(t)|v''(t)|^p\,\mathrm{d}t,\end{aligned} \tag{2.19}$$

i.e.

$$\mathcal{F}(s;v) = \int_s^\infty \rho(t)|v''(t)|^p \,\mathrm{d}t - \lambda \int_s^\infty \sigma(t)|v(t)|^p \,\mathrm{d}t > 0.$$

To get (2.19) from (2.18) we used (2.9) from Lemma 2.2 for the upper estimate of constant $C(s)$:

$$C(s) \;\le\; 2^{(p-1)p+1} p^p (p')^p A(s) < \frac{1}{\lambda}.$$

The assertion of Theorem 2.5 now follows from Lemma 2.4. □

Corollary 2.6. *Let (1.2) be satisfied. Let u and $\{u_n\}_{n=1}^\infty$ be eigenfunctions of (1.1) such that $u_n \to u$ in $W_\infty^{2,p}(\rho)$. Then there exists $n_0 > 0$ and $T > 0$ such that for all $n \ge n_0$ the functions u and u_n have definite signs in (T, ∞).*

Proof. Using

$$\int_0^\infty \rho(t)|u_n''(t)|^p \,\mathrm{d}t = \lambda_n \int_0^\infty \sigma(t)|u_n(t)|^p \,\mathrm{d}t$$

and $W_\infty^{2,p}(\rho) \hookrightarrow L^p(\sigma)$ we get

$$\lambda_n \to \lambda \ne 0 \qquad \text{as} \quad n \to \infty, \tag{2.20}$$

where λ_n and λ are eigenvalues corresponding to u_n and u, respectively. Then (1.2) and (2.18) follow the existences of $T > 0$ and $n_0 \in \mathbb{N}$ such that for all λ_n, $n \ge n_0$ the assumptions of Lemma 2.4 hold. Thus, we have that every eigenfunction has finite number of zeros.

Now we prove that zero points of $\{u_n\}$ are uniformly bounded from above with respect to n. We show this by contradiction, i.e., suppose that there exists a subsequence of the eigenfunctions $\{u_n\}$ such that the largest zero points diverges to infinity. Then repeating the same calculations as in the proof of the lemma, taking as t_k the largest zero of u_n and $t_{k+1} = b_{k+1} = \infty$ we get the same contradiction. □

3 Discreteness of the spectrum

The main result in this section is the following theorem.

Theorem 3.1. *Let $1 < p < \infty$ and suppose (1.2). Then the set of all eigenvalues of (1.1) forms an increasing sequence $\{\lambda_n\}_{n=1}^\infty$ such that $\lambda_1 > 0$ and $\lim_{n\to\infty} \lambda_n = \infty$. The eigenfunctions are isolated and to each λ_n there corresponds a finite number of normalized eigenfunctions.*

We postpone the proof to the end of this section. First we need a series of auxiliary assertions.

Remark 3.2. We prove the theorem for $p \ne 2$, since $p = 2$ is well known. Indeed, when $p = 2$ let $\mathcal{L}$ be the operator in $L^2(\sigma)$ given by the expression

$$\mathcal{L}u = \frac{1}{\sigma}(\rho u'')'', \quad u \in \mathrm{Dom}(\mathcal{L}),$$

with domain consisting of those functions $u \in L^2(\sigma)$ for which $(1/\sigma)(\rho u'')'' \in L^2(\sigma)$ and $u'(0) = (\rho u'')'(0) = u(\infty) = u'(\infty) = 0$. A simple calculation shows that $\mathcal{L}$ is a positive operator and that the domain of the quadratic form of $\mathcal{L}$ is contained in $W^{2,2}(\rho)$. Thus $\mathrm{Dom}(\mathcal{L}) \subseteq L^2(\sigma)$ a fortiori. However $W^{2,2}(\rho)$ is compactly embedded in $L^2(\sigma)$, which implies that $\mathcal{L}^{-1}$ is a compact positive self-adjoint operator. The result now follows from the Riesz-Schauder theorem.

Proposition 3.3. *Let u be an eigenfunction of (1.1). Then*

$$|u(t)| + |u'(t)| \neq 0 \tag{3.1}$$

and

$$|\rho(t)\varphi(u''(t))| + |(\rho(t)\varphi(u''(t)))'| \neq 0 \tag{3.2}$$

hold for all $t \in [0, \infty)$.

Proof. We prove the proposition by contradiction, i.e., we suppose that u is an eigenfunction and there exists $t_0 \in [0, \infty)$ such that at least in one of (3.1), (3.2) the equality holds. Without loss of generality we assume that equality holds in (3.1). The other case is treated similarly.

Let $t_0 \in (0, \infty)$ be such that $u(t_0) = u'(t_0) = 0$. Integrating twice both sides of the equation in (1.1) over (t_0, t) we get

$$\rho(t)\varphi(u''(t)) = \lambda \int_{t_0}^{t} (t - s)\sigma(s)\varphi(u(s))\,\mathrm{d}s + A(t - t_0) + B$$

and then

$$u(t) = \int_{t_0}^{t} (t - \theta)\rho^{1-p'}(\theta)\varphi^{-1}\left(\lambda \int_{t_0}^{\theta} (\theta - s)\sigma(s)\varphi(u(s))\,\mathrm{d}s + A(\theta - t_0) + B\right)\mathrm{d}\theta, \tag{3.3}$$

where $A = (\rho(t)\varphi(u''(t)))'\,|_{t=t_0}$ and $B = \rho(t_0)\varphi(u''(t_0))$.

Now we distinguish among the following cases:
(i) $A \geq 0$, $B > 0$;
(ii) $A > 0$, $B = 0$;
(iii) $A \geq 0$, $B < 0$;
(iv) $A = B = 0$.
The other cases can be treated similarly.

(i) From (3.3) we get that u is positive monotone increasing function in (t_0, ∞), which implies that $u(\infty) > 0$. This is a contradiction with the boundary condition $u(\infty) = 0$.

(ii) This case can be treated analogously to(i).

(iii) Let $t < t_0$ and rewrite (3.3) in the form

$$u(t) = \int_{t}^{t_0} (\theta - t)\rho^{1-p'}(\theta)\varphi^{-1}\left(\lambda \int_{\theta}^{t_0} (s - \theta)\sigma(s)\varphi(u(s))\,\mathrm{d}s - A(t_0 - \theta) + B\right)\mathrm{d}\theta.$$

Then we get that the function u is negative and monotone increasing function in $(0, t_0)$. Using these properties it can be shown that u' is positive and monotone decreasing function in the interval, which implies that $u'(0) > 0$. This is also a contradiction with the boundary condition $u'(0) = 0$.

(iv) In this case (3.3) takes the form

$$u(t) = \int_{t_0}^{t} (t-\theta)\rho^{1-p'}(\theta)\varphi^{-1}\left(\lambda \int_{t_0}^{\theta} (\theta - s)\sigma(s)\varphi(u(s))\,\mathrm{d}s\right) \mathrm{d}\theta,$$

and we get

$$|u(t)| \le \int_{t_0}^{t} (t-\theta)\rho^{1-p'}(\theta)\,\mathrm{d}\theta \left(\lambda \int_{t_0}^{t} (t-s)\sigma(s)|u(s)|^{p-1}\,\mathrm{d}s\right)^{\frac{1}{p-1}},$$

that is

$$|u(t)|^{p-1} \le \lambda \left(\int_{t_0}^{t} (t-\theta)\rho^{1-p'}(\theta)\,\mathrm{d}\theta\right)^{p-1} \int_{t_0}^{t} (t-s)\sigma(s)|u(s)|^{p-1}\,\mathrm{d}s$$

for all $t \in (0,\infty)$. Now using the Gronwall inequality (see Theorem 16 in [5]) we obtain $u \equiv 0$, which is not possible, since u is an eigenfunction.

Let $t_0 = 0$, i.e. $u(0) = u'(0) = 0$. Then (3.3) takes the form

$$u(t) = \int_{0}^{t} (t-\theta)\rho^{1-p'}(\theta)\varphi^{-1}\left(\lambda \int_{0}^{\theta} (\theta - s)\sigma(s)\varphi(u(s))\,\mathrm{d}s + B\right) \mathrm{d}\theta.$$

(i) Let $B > 0$, then the function u is positive and monotone increasing in $(0,\infty)$, which implies that $u(\infty) > 0$. This is a contradiction with $u(\infty) = 0$.

(ii) The case $B < 0$ is treated analogously to (i).

(iii) If $B = 0$ then we get

$$u(t) = \int_{0}^{t} (t-\theta)\rho^{1-p'}(\theta)\varphi^{-1}\left(\lambda \int_{0}^{\theta} (\theta - s)\sigma(s)\varphi(u(s))\,\mathrm{d}s\right) \mathrm{d}\theta.$$

Similarly to (iv) above we get a contradiction using the Gronwall inequality.

The proof is complete. □

Corollary 3.4. *Let u be an eigenfunction and let $t_0 \in (0,\infty)$ be such that $u(t_0) = 0$ (or $u''(t_0) = 0$). Then there exist $c_1, c_2 > 0$ and $\delta > 0$ such that*

$$c_1|t - t_0| \le |u(t)| \le c_2|t - t_0|$$

$$\left(\text{or} \quad c_1|t - t_0|^{\frac{1}{p-1}} \le |u''(t)| \le c_2|t - t_0|^{\frac{1}{p-1}}\right)$$

holds for all $t \in (t_0 - \delta, t_0 + \delta)$.

Proof. The proof follows from the Lagrange mean value theorem and the previous proposition. □

Lemma 3.5. *Let u and $\{u_n\}_{n=1}^{\infty}$ be eigenfunctions of (1.1) such that $u_n \to u$ in $W^{2,p}_{\infty}(\rho)$. Then there exists $n_0 > 0$ such that for all $n \geq n_0$ the functions u and u_n (u'' and u''_n) have the same number of zero points. Moreover, zero points of u_n (u''_n) converge to zero points of u (u'').*

Proof. Using the boundary conditions and Hölder's inequality we get for $t \in [0, \infty)$

$$\begin{aligned}
|u_n(t) - u(t)| &= \left| \int_t^{\infty} (s-t)(u''_n(s) - u''(s))\,\mathrm{d}s \right| \\
&\leq \left(\int_t^{\infty} (s-t)^{p'} \rho^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{1}{p'}} \left(\int_t^{\infty} |u''_n(s) - u''(s)|^p \rho(s)\,\mathrm{d}s \right)^{\frac{1}{p}} \\
&\leq \left(\int_0^{\infty} s^{p'} \rho^{1-p'}(s)\,\mathrm{d}s \right)^{\frac{1}{p'}} \|u_n - u\|_{2,p,\rho}, \qquad (3.4)
\end{aligned}$$

which implies the uniform convergence of $\{u_n\}$ to u in $(0, \infty)$.
From this we obtain that

$$\int_0^t \sigma(s)\varphi(u_n(s))\,\mathrm{d}s \quad \to \quad \int_0^t \sigma(s)\varphi(u(s))\,\mathrm{d}s$$

and

$$\lambda_n \int_0^t (t-s)\sigma(s)\varphi(u_n(s))\,\mathrm{d}s \quad \to \quad \lambda \int_0^t (t-s)\sigma(s)\varphi(u(s))\,\mathrm{d}s$$

for all $t \in (0, \infty)$, moreover, these convergence are uniform in each bounded subinterval of $[0, \infty)$. This and the following estimate

$$\begin{aligned}
\|u_n - u\|^p_{2,p,\rho} &\geq \int_0^T \rho(s)|u''_n(t) - u''(t)|^p\,\mathrm{d}t \\
&= \int_0^T \rho^{1-p'}(s) \left| \rho^{\frac{1}{p-1}}(t) u''_n(t) - \rho^{\frac{1}{p-1}}(t) u''(t) \right|^p \,\mathrm{d}t \\
&= \int_0^T \rho^{1-p'}(s) \left| \varphi^{-1}(\rho(t)\varphi(u''_n(t)) - \varphi^{-1}(\rho(t)\varphi(u''(t)) \right|^p \,\mathrm{d}t \\
&= \int_0^T \rho^{1-p'}(s) \left| \varphi^{-1} \left(\int_0^t (t-s)\sigma(s)\varphi(u_n(s))\,\mathrm{d}s + \rho(0)\varphi(u''_n(0)) \right) \right. \\
&\qquad \left. -\varphi^{-1} \left(\int_0^t (t-s)\sigma(s)\varphi(u(s))\,\mathrm{d}s + \rho(0)\varphi(u''(0)) \right) \right|^p \,\mathrm{d}t
\end{aligned}$$

imply that

$$\rho(0)\varphi(u''_n(0)) \to \rho(0)\varphi(u''(0))$$

as $n \to \infty$. Integrating both sides of the equation in (1.1) over the interval $(0, t)$ once and twice, we obtain from the above convergence that also

$$(\rho(t)\varphi(u''_n(t)))' \quad \to \quad (\rho(t)\varphi(u''(t)))' \qquad (3.5)$$

and

$$\rho(t)\varphi(u_n''(t)) \quad \to \quad \rho(t)\varphi(u''(t)), \tag{3.6}$$

respectively, for $t \in [0,\infty)$. The convergences are also uniform in an arbitrary bounded subinterval of $[0,\infty)$.

Recall that there exists $T_1 > 0$ such that

$$(\rho(t)\varphi(u''(t)))'|_{t=T_1} > 0$$

(see the proof of Theorem 2.1). For $n \geq n_0$ we have

$$\begin{aligned}
& \rho(t)\varphi(u_n''(t)) = \\
= \; & \lambda_n \int_{T_1}^{t} (t-\tau)\sigma(\tau)\varphi(u_n(\tau))\mathrm{d}\tau + (\rho(t)\varphi(u_n''(t)))'|_{t=T_1}(t-T_1) + \rho(T_1)\varphi(u_n''(T_1)) \\
\geq \; & \lambda_n \int_{T_1}^{t} (t-\tau)\sigma(\tau)\varphi(u_n(\tau))\mathrm{d}\tau + \frac{1}{2}(\rho(t)\varphi(u''(t)))'|_{t=T_1}(t-T_1) - 2\rho(T_1)\varphi(|u''(T_1)|) \\
\geq \; & \lambda_n \int_{T_1}^{t} (t-\tau)\sigma(\tau)\varphi(u_n(\tau))\mathrm{d}\tau
\end{aligned} \tag{3.7}$$

for all $t \geq T$, where $T > T_1$ is taken from

$$\frac{1}{2}(\rho(t)\varphi(u''(t)))'|_{t=T_1}(T-T_1) - 2\rho(T_1)\varphi(|u''(T_1)|) = 0,$$

which does not depend on n. Using Corollary 2.6 we get the existences of another $T > 0$ and $n_0 > 0$ such that for all $n \geq n_0$, u_n are positive in (T,∞). If we choose as T and n_0 the larger once all results here will be saved and imply that the second derivatives u_n'', $n \geq n_0$, do not change the sign in (T,∞).

Consequently we obtain the convergence of $\{u_n\}$ to u (and $\{\rho\varphi(u_n'')\}$ to $\rho\varphi(u'')$) in $C^1[0,T]$ which imply together with Proposition 3.3 that the functions u_n and u (u_n'' and u''), $n \geq n_0$ have the same number of zero points and zero points of u_n (u_n'') converge to zero points of u (u''). The proof is complete. □

Let us denote

$$\tilde{u}_n(\tau,t) = u(t) + \tau(u_n(t) - u(t)) \quad \text{and} \quad \tilde{u}_n''(\tau,t) = u''(t) + \tau(u_n''(t) - u''(t)).$$

Corollary 3.6. *For arbitrary $T > 0$ the followings hold:*

(i) Let $n_0 > 0$ be from Lemma 3.5. Then

$$\max_{\tau\in[0,1]} \int_0^T |\tilde{u}_n(\tau,t)|^{p-2}\,\mathrm{d}t \leq C$$

and

$$\max_{\tau\in[0,1]} \int_0^T |\tilde{u}_n''(\tau,t)|^{2-p}\,\mathrm{d}t \leq C$$

hold for all $n \geq n_0$, where C is a constant independent of n.

(ii) For arbitrary $\varepsilon > 0$ there exist $n_0 > 0$ such that

$$\max_{\tau\in[0,1]} \int_0^T \left| |\tilde{u}_n(\tau,t)|^{p-2} - |u(t)|^{p-2} \right| \mathrm{d}t < \varepsilon$$

and

$$\max_{\tau\in[0,1]} \int_0^T \left| |\tilde{u}_n''(\tau,t)|^{2-p} - |u''(t)|^{2-p} \right| \mathrm{d}t < \varepsilon$$

hold for all $n \geq n_0$.

(iii) Let $n_0 > 0$ be from Lemma 3.5. Then for every $\varepsilon > 0$ there exists $\delta > 0$ that

$$\max_{\tau\in[0,1]} \int_0^T \left| |\tilde{u}_n(\tau,t+h)|^{p-2} - \tilde{u}_n(\tau,t)|^{p-2} \right| \mathrm{d}t < \varepsilon$$

holds for all $|h| < \delta$ and $n \geq n_0$.

Proof. Let $\varepsilon > 0$ be arbitrary. From Lemma 3.5 follows that u_n $(n \geq n_0)$ and u have the same number of zeros in $(0,\infty)$. Now we show only

$$\max_{\tau\in[0,1]} \int_0^T \left| |\tilde{u}_n(\tau,t)|^{p-2} - |u(t)|^{p-2} \right| \mathrm{d}t < \varepsilon,$$

since the other estimates can be proved analogously. Let $T_1 > T$ be such that the interval $(0, T_1)$ contains all zero points $\{t_i^n\}_{i=1}^m$ and $\{t_i\}_{i=1}^m$ of functions u_n and u, respectively. Then

$$\begin{aligned}
& \int_0^T \left| |\tilde{u}_n(\tau,t)|^{p-2} - |u(t)|^{p-2} \right| \mathrm{d}t \\
\leq\ & \int_0^{T_1} \left| |\tilde{u}_n(\tau,t)|^{p-2} - |u(t)|^{p-2} \right| \mathrm{d}t \\
=\ & \left(\int_{\cup_{i=1}^m (t_i-\delta,t_i+\delta)} + \int_{(0,T_1)\setminus\cup_{i=1}^m (t_i-\delta,t_i+\delta)} \right) \left| |\tilde{u}_n(\tau,t)|^{p-2} - |u(t)|^{p-2} \right| \mathrm{d}t,
\end{aligned}$$

where $\delta > 0$ is such that $t_i^n \in (t_i - \delta, t_i + \delta)$ for $i = 1, ..., m$. To estimate the first integral we use Corollary 3.4, i.e.,

$$\begin{aligned}
& \int_{\cup_{i=1}^m (t_i-\delta,t_i+\delta)} \left| |\tilde{u}_n(\tau,t)|^{p-2} - |u(t)|^{p-2} \right| \mathrm{d}t \\
=\ & \sum_{i=1}^m \int_{t_i-\delta}^{t_i+\delta} \left| |\tilde{u}_n(\tau,t)|^{p-2} - |u(t)|^{p-2} \right| \mathrm{d}t \\
\leq\ & C_1 \sum_{i=1}^m \int_{t_i-\delta}^{t_i+\delta} \left[|t - t_i|^{p-2} + |t - t_i^n|^{p-2} \right] \mathrm{d}t \\
\leq\ & C_2 \delta^{p-1},
\end{aligned}$$

where C_1 and C_2 are constants independent of n and δ. If we choose n_0 sufficiently large then by Lemma 3.5 for all $n \geq n_0$ the zero points t_i^n and t_i are sufficiently close to each other, which give us a chance to choose δ small such that $C_2\delta^{p-1} < \varepsilon/2$.

Let δ be fixed. Using the uniform convergence of $\{\tilde{u}_n\}$ to u (see Lemma 3.5) we can choose n_0 greater than in the previous case such that the second integral is also less than $\varepsilon/2$ for all $n \geq n_0$.

The proof is complete. □

Lemma 3.7. *Let assumptions of Lemma 3.5 be satisfied. Then there exists $T > 0$ such that for every $\varepsilon > 0$ there exists $n_0 > 0$ such that*

$$\int_T^\infty s^2\rho^{-1}(s)\big||u''(s)|^{2-p} - |\tilde{u}_n''(\tau,s)|^{2-p}\big|\,\mathrm{d}s < \varepsilon$$

holds for all $n > n_0$ and $\tau \in [0,1]$.

Proof. By Lemma 3.5 we get that there exist $T > 0$ and $n_0 > 0$ such that u and u_n have definite sign in (T,∞) for all $n > n_0$. Without loss of generality we can assume that are positive. Then from (3.7) we have

$$\rho(s)\varphi(u_n''(s)) \geq \lambda \int_{T_1}^T (s-\tau)\sigma(\tau)\varphi(u_n(\tau))\mathrm{d}\tau \geq C_1 s$$

i.e.,

$$u_n''(s) \geq \left(\frac{C_1 s}{\rho(s)}\right)^{\frac{1}{p-1}}$$

for all $s > T$. The same estimate can be also obtained for u. Using this estimate we get

$$\begin{aligned}
&\big||u''(s)|^{2-p} - |\tilde{u}_n''(\tau,s)|^{2-p}\big| = \Big|\int_{|\tilde{u}_n''(\tau,s)|}^{|u''(s)|} (2-p)t^{1-p}\,\mathrm{d}t\Big| \\
\leq\ & |2-p|\big||u''(s)| - |\tilde{u}_n''(\tau,s)|\big|\big(|u''(s)|^{1-p} + |\tilde{u}_n''(\tau,s)|^{1-p}\big) \\
\leq\ & C_2|u_n''(s) - u''(s)|\big(|u''(s)|^{1-p} + |u_n''(s)|^{1-p}\big) \\
\leq\ & C_3\frac{\rho(s)}{s}|u_n''(s) - u''(s)|,
\end{aligned}$$

i.e.,

$$\big||u''(s)|^{2-p} - |\tilde{u}_n''(\tau,s)|^{2-p}\big| \leq C_3\frac{\rho(s)}{s}|u_n''(s) - u''(s)|.$$

From this we have that

$$\int_T^\infty s^2\rho^{-1}(s)\big||u''(s)|^{2-p} - |\tilde{u}_n''(\tau,s)|^{2-p}\big|\,\mathrm{d}s \leq C_3\int_T^\infty s|u_n''(s) - u''(s)|\,\mathrm{d}s$$

and using Hölder's inequality in the right hand side of the estimate as

$$\int_T^\infty s|u_n''(s) - u''(s)|\,\mathrm{d}s \leq \left(\int_T^\infty s^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{1}{p'}}\left(\int_T^\infty \rho(s)|u_n''(s) - u''(s)|^{p'}\,\mathrm{d}s\right)^{\frac{1}{p}}$$

we have the proof. □

Lemma 3.8. *Let u and $\{u_n\}_{n=1}^\infty$ be eigenfunctions of (1.1) such that $u_n \to u$ in $W^{2,p}_\infty(\rho)$. Then there exists $n_0 > 0$ such that*

$$\int_0^\infty \sigma\,[\varphi(u_n) - \varphi(u)]\,(u_n - u)\,\mathrm{d}t \le C \int_0^\infty \rho\,[\varphi(u_n'') - \varphi(u'')]\,(u_n'' - u'')\,\mathrm{d}t \tag{3.8}$$

holds for all $n \ge n_0$, where C is a constant independent of n.

Proof. Let us rewrite the integrals in (3.8) as

$$\int_0^\infty \sigma\,[\varphi(u_n) - \varphi(u)]\,(u_n - u)\,\mathrm{d}t = (p-1)\int_0^1 \int_0^\infty \sigma|\tilde{u}_n|^{p-2}(u_n - u)^2\,\mathrm{d}t\,\mathrm{d}\tau \tag{3.9}$$

and

$$\int_0^\infty \rho\,[\varphi(u_n'') - \varphi(u'')]\,(u_n'' - u'')\,\mathrm{d}t = (p-1)\int_0^1 \int_0^\infty \rho|\tilde{u}_n''|^{p-2}(u_n'' - u'')^2\,\mathrm{d}t\,\mathrm{d}\tau. \tag{3.10}$$

Then instead of (3.8) it is sufficient to show the estimate

$$\int_0^\infty \sigma(t)|\tilde{u}_n(\tau,t)|^{p-2}(u_n(t) - u(t))^2\,\mathrm{d}t \le C\int_0^\infty \rho(t)|\tilde{u}_n''(\tau,t)|^{p-2}(u_n''(t) - u''(t))^2\,\mathrm{d}t \tag{3.11}$$

for all $\tau \in [0,1]$, which is written as

$$\int_0^\infty \left(\int_t^\infty (s-t)w(s)\,\mathrm{d}s\right)^2 \sigma(t)|\tilde{u}_n(\tau,t)|^{p-2}\,\mathrm{d}t \le C\int_0^\infty w(t)^2\rho(t)|\tilde{u}_n''(\tau,t)|^{p-2}\,\mathrm{d}t \tag{3.12}$$

where $w = u_n'' - u''$. Further, we show that (3.12) holds and the constant C is independent of n and τ.

Let $a \in [0,\infty)$ and $\tau \in [0,1]$. If we denote

$$\tilde{A}_1(a;\tau,t) = \left(\int_t^\infty (s-t)^2\rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}t\right)\left(\int_a^t \sigma(s)|\tilde{u}_n(\tau,s)|^{p-2}\,\mathrm{d}s\right)$$

and

$$\tilde{A}_2(a;\tau,t) = \left(\int_t^\infty \rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}t\right)\left(\int_a^t (t-s)^2\sigma(s)|\tilde{u}_n(\tau,s)|^{p-2}\,\mathrm{d}s\right)$$

then using Lemma 2.2 with $p := 2$, weight functions $\sigma := \sigma|\tilde{u}_n|^{p-2}$ and $\rho := \rho|\tilde{u}_n''|^{p-2}$, we get that (3.12) holds if there exists $\tilde{C}$ independent of n and τ such that

$$\sup_{t>0} \tilde{A}_i(0;\tau,t) \le \tilde{C}, \qquad i = 1,2. \tag{3.13}$$

Hence it remains to prove (3.13). Let $T > 0$ and $n_0 > 0$ be such that u'' and u_n'' do not change their sings in (T,∞) for all $n \ge n_0$. Without loss of generality we will assume that both are positive. With $u(\infty) = u_n(\infty) = 0$ this implies that u and u_n

are also positive and, moreover, u and u_n are monotone decreasing functions in the interval (T,∞).

If $0<t<T$ then using Lemma 3.7 and Corollary 3.6 we get that

$$\begin{aligned}\tilde{A}_1(0;\tau,t) &\le \left(\int_0^\infty s^2\rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}s\right)\left(\int_0^T |\tilde{u}_n(\tau,s)|^{p-2}\,\mathrm{d}s\right)\\ &\le \tilde{C} \end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\tilde{A}_2(0;\tau,t) &\le \left(\int_t^\infty \rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}s\right)\left(t^2\int_0^t \sigma(s)|\tilde{u}_n(\tau,s)|^{p-2}\,\mathrm{d}s\right)\\ &\le \left(\int_0^\infty s^2\rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}s\right)\left(\int_0^T |\tilde{u}_n(\tau,s)|^{p-2}\,\mathrm{d}s\right)\\ &\le \tilde{C}. \end{aligned} \tag{3.15}$$

If $T\le t<\infty$ then by Corollary 3.6 we have

$$\begin{aligned}&\tilde{A}_1(0;\tau,t)\\ =\ &\tilde{A}_1(T;\tau,t)+\left(\int_t^\infty (s-t)^2\rho^{-1}(s)(\tilde{u}_n''(\tau,s))^{2-p}\,\mathrm{d}s\right)\left(\int_0^T \sigma(s)|\tilde{u}_n(\tau,s)|^{p-2}\,\mathrm{d}s\right)\\ \le\ &\tilde{A}_1(T;\tau,t)+C\int_t^\infty (s-t)^2\rho^{-1}(s)(\tilde{u}_n''(\tau,s))^{2-p}\,\mathrm{d}s \end{aligned} \tag{3.16}$$

and

$$\begin{aligned}&\tilde{A}_2(0;\tau,t)\\ \le\ &\tilde{A}_2(T;\tau,t)+\left(\int_t^\infty \rho^{-1}(s)(\tilde{u}_n''(\tau,s))^{2-p}\,\mathrm{d}s\right)\left(t^2\int_0^T \sigma(s)|\tilde{u}_n(\tau,s)|^{p-2}\,\mathrm{d}s\right)\\ \le\ &\tilde{A}_2(T;\tau,t)+C\int_t^\infty s^2\rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}s. \end{aligned} \tag{3.17}$$

Further, we consider the cases $1<p<2$ and $2<p<\infty$, separately.

Let $1<p<2$. Then using Hölder's inequality with exponents $\frac{1}{p-1}$, $\left(\frac{1}{p-1}\right)'=\frac{1}{2-p}$ in the first integrals and the monotonicity of $\tilde{u}_n$ with respect to s in the second integrals of $\tilde{A}_1(T;\tau,t)$, $\tilde{A}_2(T;\tau,t)$ we get

$$\begin{aligned}\tilde{A}_1(T;\tau,t) &= \left(\int_t^\infty (s-t)^2\rho^{-1}(s)(\tilde{u}_n''(\tau,s))^{2-p}\,\mathrm{d}s\right)\left(\int_T^t \sigma(s)(\tilde{u}_n(\tau,s))^{p-2}\,\mathrm{d}s\right)\\ &\le \left(\int_t^\infty (s-t)^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\right)^{p-1}\left(\int_t^\infty (s-t)\tilde{u}_n''(\tau,s)\,\mathrm{d}s\right)^{2-p}\\ &\qquad\times\left(\int_T^t \sigma(s)\,\mathrm{d}s\right)(\tilde{u}_n(\tau,t))^{p-2}\\ &= A_1(T;t)(\tilde{u}_n(\tau,t))^{2-p}(\tilde{u}_n(\tau,t))^{p-2}\\ &\le A_1(0;t) \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}
\tilde{A}_2(T;\tau,t) &= \left(\int_t^\infty \rho^{-1}(s)(\tilde{u}_n''(\tau,s))^{2-p}\,\mathrm{d}t\right)\left(\int_T^t (t-s)^2\sigma(s)(\tilde{u}_n(\tau,s))^{p-2}\,\mathrm{d}s\right) \\
&\le \left(\int_t^\infty \rho^{1-p'}(s)\,\mathrm{d}s\right)^{p-1}\left(\int_t^\infty (\tilde{u}_n''(\tau,s))\,\mathrm{d}s\right)^{2-p} \\
&\quad \times \left(\int_T^t (t-s)^p\sigma(s)\,\mathrm{d}s\right)(t-T)^{2-p}|\tilde{u}_n(\tau,t)|^{p-2} \\
&= A_2(T;t)\left|\frac{\tilde{u}_n'(\tau,t)(t-T)}{\tilde{u}_n(\tau,t)}\right|^{2-p} \\
&\le A_2(T;t)\left|\frac{\tilde{u}_n'(\tau,t)(t-T)}{\tilde{u}_n(\tau,t)-\tilde{u}_n(\tau,T)}\right|^{2-p} \\
&\le A_2(T;t) \le A_2(0;t). \qquad (3.19)
\end{aligned}$$

Here, we used Lagrange's mean value theorem and monotonicity of $\tilde{u}_n'(\tau,t) = u'(t) + \tau(u_n'(t) - u'(t))$ with respect to t.
To estimate the integral

$$\int_t^\infty (s-t)^2\rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}s$$

we use Hölder's inequality with exponents $\frac{p'}{2}$ and $(\frac{p'}{2})' = \frac{p}{2-p}$:

$$\begin{aligned}
&\int_t^\infty (s-t)^2\rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}s \\
\le\ & \left(\int_t^\infty (s-t)^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{2}{p'}}\left(\int_t^\infty \rho(s)|\tilde{u}_n''(\tau,s)|^p\,\mathrm{d}s\right)^{\frac{p}{2-p}} \\
\le\ & C\left(\int_t^\infty (s-t)^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{2}{p'}}.
\end{aligned}$$

Consequently, from (3.16), (3.18) and (3.17), (3.19) we obatin

$$\tilde{A}_i(0;\tau,t) \le A_i(0;t) + C\left(\int_t^\infty (s-t)^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{2}{p'}} \qquad (3.20)$$

for $i = 1, 2$.

Let $2 < p < \infty$. Then using (3.7) we get

$$\begin{aligned}
(\tilde{u}_n''(\tau,s))^{p-1} &= ((1-\tau)u''(s) + \tau u_n''(s))^{p-1} \\
&\geq \frac{1}{2}\left[((1-\tau)u''(s))^{p-1} + (\tau u_n''(s))^{p-1}\right] \\
&= \frac{1}{2\rho(s)}\left[(1-\tau)^{p-1}\rho(s)\varphi(u''(s)) + \tau^{p-1}\rho(s)\varphi(u_n''(s))\right] \\
&\geq \frac{1}{2\rho(s)}\left[(1-\tau)^{p-1}\lambda\int_{T_1}^{s}(s-\theta)\sigma(\theta)\varphi\left(u(\theta)\right)\,\mathrm{d}\theta\right. \\
&\quad \left. + \tau^{p-1}\lambda_n\int_{T_1}^{s}(s-\theta)\sigma(\theta)\varphi\left(u_n(\theta)\right)\,\mathrm{d}\theta\right] \\
&\geq \frac{\lambda-\delta}{2\rho(s)}\int_{T_1}^{s}(s-\theta)\sigma(\theta)\left[(1-\tau)^{p-1}\varphi\left(u(\theta)\right) + \tau^{p-1}\varphi\left(u_n(\theta)\right)\right]\,\mathrm{d}\theta \\
&\geq \frac{\lambda-\delta}{2\rho(s)}\int_{T_1}^{s}(s-\theta)\sigma(\theta)\left[\varphi\left((1-\tau)u(\theta)\right) + \varphi\left(\tau u_n(\theta)\right)\right]\,\mathrm{d}\theta \\
&\geq \frac{\lambda-\delta}{2^p\rho(s)}\int_{T_1}^{s}(s-\theta)\sigma(\theta)\varphi\left((1-\tau)u(\theta) + \tau u_n(\theta)\right)\,\mathrm{d}\theta \\
&= \frac{\lambda-\delta}{2^p\rho(s)}\int_{T_1}^{s}(s-\theta)\sigma(\theta)\varphi\left(\tilde{u}_n(\tau,\theta)\right)\,\mathrm{d}\theta \qquad (3.21)
\end{aligned}$$

which holds for all $s > T$, where $\delta \in (0,\lambda)$ is such that $\lambda - \delta \leq \lambda_n$ for all $n \geq n_0$ and $T_1 < T$.

This and positivity of $\tilde{u}_n$ in (T,∞) imply that

$$\begin{aligned}
&\int_t^\infty (s-t)^2\rho(s)^{-1}(\tilde{u}_n''(\tau,s))^{2-p}\,\mathrm{d}s \\
\leq\ & \frac{2^{p'(p-2)}}{(\lambda-\delta)^{\frac{p-2}{p-1}}}\int_t^\infty (s-t)^2\rho(s)^{1-p'}\left[\int_{T_1}^{s}(s-\theta)\sigma(\theta)\varphi\left(\tilde{u}_n(\tau,\theta)\right)\,\mathrm{d}\theta\right]^{\frac{2-p}{p-1}}\,\mathrm{d}s \\
\leq\ & \frac{2^{p'(p-2)}}{(\lambda-\delta)^{\frac{p-2}{p-1}}}\int_t^\infty (s-t)^2\rho(s)^{1-p'}\left[\int_{T_1}^{t}(s-\theta)\sigma(\theta)\varphi\left(\tilde{u}_n(\tau,\theta)\right)\,\mathrm{d}\theta\right]^{\frac{2-p}{p-1}}\,\mathrm{d}s \\
\leq\ & \frac{2^{p'(p-2)}}{(\lambda-\delta)^{\frac{p-2}{p-1}}}\int_t^\infty (s-t)^{p'}\rho(s)^{1-p'}\left[\int_{T_1}^{t}\sigma(\theta)\varphi\left(\tilde{u}_n(\tau,\theta)\right)\,\mathrm{d}\theta\right]^{\frac{2-p}{p-1}}\,\mathrm{d}s \\
=\ & \frac{2^{p'(p-2)}}{(\lambda-\delta)^{\frac{p-2}{p-1}}}\int_t^\infty (s-t)^{p'}\rho(s)^{1-p'}\,\mathrm{d}s\left[\int_{T_1}^{t}\sigma(\theta)\varphi\left(\tilde{u}_n(\tau,\theta)\right)\,\mathrm{d}\theta\right]^{\frac{2-p}{p-1}}. \qquad (3.22)
\end{aligned}$$

From this we have

$$
\begin{aligned}
& \int_t^\infty (s-t)^2 \rho^{-1}(s) |\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}s \\
\leq\ & C \int_t^\infty (s-t)^{p'} \rho(s)^{1-p'}\,\mathrm{d}s \left[\int_{T_1}^T \sigma(\theta)\varphi\left(\tilde{u}_n(\tau,\theta)\right)\,\mathrm{d}\theta\right]^{\frac{2-p}{p-1}} \\
\leq\ & C' \int_t^\infty (s-t)^{p'} \rho(s)^{1-p'}\,\mathrm{d}s, \qquad (3.23)
\end{aligned}
$$

here we used the convergence of the sequence $\left\{\int_{T_1}^T \sigma(\theta)\varphi\left(\tilde{u}_n(\tau,\theta)\right)\,\mathrm{d}\theta\right\}_{n=1}^\infty$ to $\int_{T_1}^T \sigma(\theta)\varphi\left(u(\theta)\right)\,\mathrm{d}\theta$ as $n\to\infty$ and the positivity of $\int_{T_1}^T \sigma(\theta)\varphi\left(u(\theta)\right)\,\mathrm{d}\theta$. The convergence of the sequence follows from uniform convergence of $\{\tilde{u}_n(\tau,\theta)\}_{n=1}^\infty$ to $u(\theta)$ in $[T_1,T]$ at every $\tau\in[0,1]$.

Using (3.22) in the first integral of $\tilde{A}_1(T;\tau,t)$ and Hölder's inequality with exponents $p-1$, $\frac{p-1}{p-2}$ in the second integral of $\tilde{A}_1(T;\tau,t)$ we get

$$
\begin{aligned}
\tilde{A}_1(T;\tau,t)\ \leq\ & C\left(\int_t^\infty (s-t)^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_T^t \sigma(s)(\tilde{u}_n(\tau,s))^{p-1}\,\mathrm{d}s\right)^{\frac{2-p}{p-1}} \\
& \times\left(\int_T^t \sigma(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}}\left(\int_T^t \sigma(s)(\tilde{u}_n(\tau,s))^{p-1}\,\mathrm{d}s\right)^{\frac{p-2}{p-1}} \\
\leq\ & C A_1^{\frac{1}{p-1}}(0;\ t). \qquad (3.24)
\end{aligned}
$$

Using (3.21) in the first integral and Hölder's inequality with exponents $p-1$, $\frac{p-1}{p-2}$ in the second integral of $\tilde{A}_2(T;\tau,t)$ we have

$$
\begin{aligned}
\tilde{A}_2(T;\tau,t)\ \leq\ & C\left(\int_t^\infty \rho^{1-p'}(s)\left(\int_T^s (s-\theta)\sigma(\theta)(\tilde{u}_n(\tau,\theta))^{p-1}\,\mathrm{d}\theta\right)^{\frac{2-p}{p-1}}\,\mathrm{d}s\right) \\
& \times\left(\int_T^t (t-s)^p\sigma(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}}\left(\int_T^t (t-s)\sigma(t)(\tilde{u}_n(\tau,t))^{p-1}\,\mathrm{d}s\right)^{\frac{p-2}{p-1}} \\
\leq\ & C\left(\int_t^\infty \rho^{1-p'}(s)\,\mathrm{d}s\right)\left(\int_T^t (t-s)\sigma(s)(\tilde{u}_n(\tau,s))^{p-1}\,\mathrm{d}s\right)^{\frac{2-p}{p-1}} \\
& \times\left(\int_T^t (t-s)^p\sigma(s)\,\mathrm{d}s\right)^{\frac{1}{p-1}}\left(\int_T^t (t-s)\sigma(t)(\tilde{u}_n(\tau,t))^{p-1}\,\mathrm{d}s\right)^{\frac{p-2}{p-1}} \\
\leq\ & C A_2^{\frac{1}{p-1}}(0;t). \qquad (3.25)
\end{aligned}
$$

Consequently we have from (3.16), (3.23), (3.24) and (3.17), (3.23), (3.25) we have

$$
\tilde{A}_i(0;\tau,t)\leq C\left[A_i^{\frac{1}{p-1}}(0;t)+\int_t^\infty (s-t)^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\right] \qquad (3.26)
$$

for $i = 1, 2$.

Finally, from (3.14), (3.15), (3.20) and (3.26) imply (3.13). Moreover, if we use the upper estimate for the best constant in Lemma 2.2, which in our case takes the form (note that $p = 2$ in this case!)

$$C \leq 2^7 \max\{\sup_{t>0} \tilde{A}_1(0;\tau,t), \sup_{t>0} \tilde{A}_2(0;\tau,t)\}$$

we get the independence of C in (3.8) both on n and τ.

The proof is complete. □

In the proof of Theorem 3.1 we shall use relative compactness of the following sequence

$$\left\{ \left[\frac{\sigma(\varphi(u_n) - \varphi(u))(u_n - u)}{\int_0^\infty \rho\left[\varphi(u_n'') - \varphi(u'')\right](u_n'' - u'')\,\mathrm{d}t} \right]^{\frac{1}{2}} \right\}_{n=n_0}^{\infty} \tag{3.27}$$

in $L^2(0,\infty)$, where n_0 is from Lemma 3.5. To prove this fact we use the criterion for relatively compact sets in Lebesgue's spaces. Namely, our set is relatively compact if and only if it is bounded and $2-$mean equicontinuous. Boundedness of the sequence follows from Lemma 3.8. The following lemma establishes its $2-$mean equicontinuity.

Lemma 3.9. *Let u and $\{u_n\}_{n=1}^\infty$ be eigenfunctions of (1.1) such that $u_n \to u$ in $W_\infty^{2,p}(\rho)$. Then (3.27) is $2-$mean equicontinuous sequence.*

Proof. Let us denote

$$U_n(\tau,t) = \sigma(t)^{\frac{1}{2}} |\tilde{u}_n(\tau,t)|^{\frac{p-2}{2}} (u_n(t) - u(t)).$$

Taking into account (3.9) and (3.10) it is sufficient to prove for every $\varepsilon > 0$ the existence of $\delta > 0$ such that for all $h, |h| < \delta$ the estimate

$$\int_0^\infty \left[U_n(\tau,t+h) - U_n(\tau,t)\right]^2 \mathrm{d}t < C\varepsilon \int_0^\infty \rho(t)|\tilde{u}_n''|^{p-2}(u_n'' - u'')^2\,\mathrm{d}t \tag{3.28}$$

holds for all $n \geq n_0$, where C is a constant independent of n, τ, δ and ε. Indeed, if we consider the following

$$\begin{aligned}
&\int_0^\infty \Big(\Big[\sigma(t+h)\big(\varphi(u_n(t+h)) - \varphi(u(t+h))\big)(u_n(t+h) - u(t+h))\Big]^{\frac{1}{2}} \\
&\qquad - \Big[\sigma(t)(\varphi(u_n(t)) - \varphi(u(t)))(u_n(t) - u(t))\Big]^{\frac{1}{2}} \Big)^2 \mathrm{d}\tau \\
= &\ (p-1)\int_0^\infty \Big(\Big[\int_0^1 U_n(\tau,t+h)^2\,\mathrm{d}\tau\Big]^{\frac{1}{2}} - \Big[\int_0^1 U_n(\tau,t)^2\,\mathrm{d}\tau\Big]^{\frac{1}{2}} \Big)^2 \mathrm{d}t \\
\leq &\ (p-1)\int_0^\infty \Big(\int_0^1 \Big[U_n(\tau,t+h) - U_n(\tau,t)\Big]^2 \mathrm{d}\tau \Big)\,\mathrm{d}t \\
= &\ (p-1)\int_0^1 \Big(\int_0^\infty \Big[U_n(\tau,t+h) - U_n(\tau,t)\Big]^2 \mathrm{d}t \Big)\,\mathrm{d}\tau.
\end{aligned}$$

If (3.28) holds then

$$
\begin{aligned}
&\int_0^\infty \Big(\Big[\sigma(t+h)[\varphi(u_n(t+h))-\varphi(u(t+h))](u_n(t+h)-u(t+h))\Big]^{\frac{1}{2}} \\
&\qquad -\Big[\sigma(t)[\varphi(u_n(t))-\varphi(u(t))](u_n(t)-u(t))\Big]^{\frac{1}{2}}\Big)^2 \,\mathrm{d}\tau \\
\leq\ & C\varepsilon(p-1)\int_0^1 \Big[\int_0^\infty \rho(t)|\tilde{u}_n''|^{p-2}(u_n''-u'')^2 \,\mathrm{d}t\Big]\,\mathrm{d}\tau \\
=\ & C\varepsilon(p-1)\int_0^\infty \Big[\int_0^1 \rho(t)|\tilde{u}_n''|^{p-2}(u_n''-u'')^2 \,\mathrm{d}\tau\Big]\,\mathrm{d}t \\
=\ & C\varepsilon\int_0^\infty \rho\,[\varphi(u_n'')-\varphi(u'')]\,(u_n''-u'')\,\mathrm{d}t,
\end{aligned}
$$

and the equicontinuity of (3.27) follows.

Let $\varepsilon > 0$ be arbitrary, but fixed. To get (3.28) we split the integral on the left hand side of the estimate into two integrals

$$
\int_0^\infty \big(U_n(\tau,t+h)-U_n(\tau,t)\big)^2\,\mathrm{d}t \ = \ \int_0^T + \int_T^\infty = I_1 + I_2
$$

and estimate them separately, where T is arbitrary for now.

We use also the following estimate which follows from Hölder's inequality and Lemma 3.7:

$$
\begin{aligned}
&|u_n(t)-u(t)| = \left|\int_t^\infty (s-t)(u_n''(s)-u''(s))\,\mathrm{d}s\right| \\
&\leq \left(\int_t^\infty (s-t)^2\rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_t^\infty \rho|\tilde{u}_n''|^{p-2}(u_n''-u'')^2\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq \left(\int_0^\infty s^2\rho^{-1}(s)|\tilde{u}_n''(\tau,s)|^{2-p}\,\mathrm{d}s\right)^{\frac{1}{2}} \left(\int_0^\infty \rho|\tilde{u}_n''|^{p-2}(u_n''-u'')^2\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\leq C\left(\int_0^\infty \rho|\tilde{u}_n''|^{p-2}(u_n''-u'')^2\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}
\tag{3.29}
$$

First we find the correct T by estimating I_2 as follows:

$$
\begin{aligned}
I_2 \ &= \ \int_T^\infty \big(U_n(\tau,t+h)-U_n(\tau,t)\big)^2\,\mathrm{d}t \\
&\leq \ 2\int_T^\infty U_n(\tau,t+h)^2\,\mathrm{d}t + 2\int_T^\infty U_n(\tau,t)^2\,\mathrm{d}t \\
&= \ 2\int_{T+h}^\infty U_n(\tau,t)^2\,\mathrm{d}t + 2\int_T^\infty U_n(\tau,t)^2\,\mathrm{d}t \\
&\leq \ 4\int_{T/2}^\infty U_n(\tau,t)^2\,\mathrm{d}t.
\end{aligned}
$$

To estimate the last integral we use the Hardy inequality (see Lemma 2.2) for $p := 2$, with weight functions $\sigma := \sigma|\tilde{u}_n|^{p-2}$ and $\rho := \rho|\tilde{u}_n''|^{p-2}$, i.e.,

$$\begin{aligned}\int_{T/2}^{\infty} U_n(\tau,t)^2 \,\mathrm{d}t &= \int_{T/2}^{\infty} \sigma(t)|\tilde{u}_n(\tau,t)|^{p-2}(u_n(t)-u(t))^2 \,\mathrm{d}t \\ &\le C\int_{T/2}^{\infty} \rho(t)|\tilde{u}_n''(\tau,t)|^{p-2}(u_n''(t)-u''(t))^2 \,\mathrm{d}t,\end{aligned}$$

where $C = C(T,n,\tau)$ is a constant which satisfies

$$C(T,n,\tau) \le 2^7 \sup_{t\ge \frac{T}{2}} \tilde{A}_i(0;\tau;t)$$

for $i = 1,2$. Taking into account (3.20) and (3.26) we get that

$$\begin{aligned}C(T,n,\tau) \le C_1 \Bigg(\sup_{t>T/2}\Bigg[&A_i(0;t) + A_i^{\frac{1}{p-1}}(0;t) \\ &+ \int_t^{\infty} s^{p'}\rho^{1-p'}(s)\,\mathrm{d}s + \left(\int_t^{\infty} s^{p'}\rho^{1-p'}(s)\,\mathrm{d}s\right)^{\frac{2}{p'}}\Bigg]\Bigg) \qquad (3.30)\end{aligned}$$

for $i = 1,2$, where C_1 is a constant independent of T, n and τ. Using (1.2) we choose T in (3.30) sufficiently large such that

$$C(T,n,\tau) < \varepsilon$$

which implies that

$$I_2 < \varepsilon \int_0^{\infty} \rho|\tilde{u}_n''|^{p-2}(u_n''-u'')^2 \,\mathrm{d}s. \qquad (3.31)$$

Now we fix $T > 0$ and estimate I_1 in the form

$$\begin{aligned}I_1 &= \int_0^T \big(U_n(\tau,t+h) - U_n(\tau,t)\big)^2 \,\mathrm{d}t \\ &\le 2\Bigg(\int_0^T (u_n(t+h)-u(t+h))^2\big(\sigma(t+h)^{\frac{1}{2}}|\tilde{u}_n(t+h)|^{\frac{p-2}{2}} - \sigma(t)^{\frac{1}{2}}|\tilde{u}_n(\tau,t)|^{\frac{p-2}{2}}\big)^2 \,\mathrm{d}t \\ &\quad + \int_0^T \sigma(t)|\tilde{u}_n(\tau,t)|^{p-2}\big([u_n(t+h)-u(t+h)] - [u_n(t)-u(t)]\big)^2 \,\mathrm{d}t\Bigg) \\ &= 2\left(I_{1,1} + I_{1,2}\right).\end{aligned}$$

To estimate $I_{1,1}$ we proceed as follows:

$$\begin{aligned}
I_{1,1} &= \int_0^T \big(u_n(t+h)-u(t+h)\big)^2\big(\sigma(t+h)^{\frac12}|\tilde u_n(\tau,t+h)|^{\frac{p-2}{2}} - \sigma(t)^{\frac12}|\tilde u_n(\tau,t)|^{\frac{p-2}{2}}\big)^2\,\mathrm{d}t\\
&\le 2\int_0^T \big(u_n(t+h)-u(t+h)\big)^2(\sigma(t+h)^{\frac12}-\sigma(t)^{\frac12})^2|\tilde u_n(\tau,t+h)|^{p-2}\,\mathrm{d}t\\
&\quad +2\int_0^T \big(u_n(t+h)-u(t+h)\big)^2\sigma(t+h)\big(|\tilde u_n(\tau,t+h)|^{\frac{p-2}{2}}-|\tilde u_n(\tau,t)|^{\frac{p-2}{2}}\big)^2\,\mathrm{d}t\\
&\le 2\max_{[0,T]}(\sigma(t+h)^{\frac12}-\sigma(t)^{\frac12})^2\max_{[h,T+h]}(u_n(t)-u(t))^2\int_h^{T+h}|\tilde u_n(\tau,t)|^{p-2}\,\mathrm{d}t\\
&\quad +2\max_{[h,T+h]}\sigma(t)\max_{[h,T+h]}(u_n(t)-u(t))^2\int_0^T\big(|\tilde u_n(\tau,t+h)|^{\frac{p-2}{2}}-|\tilde u_n(\tau,t)|^{\frac{p-2}{2}}\big)^2\,\mathrm{d}t\\
&\le \Big[2\max_{[0,T]}|\sigma(t+h)-\sigma(t)|\int_0^{2T}|\tilde u_n(\tau,t)|^{p-2}\,\mathrm{d}t\\
&\quad +2\max_{[0,2T]}\sigma(t)\int_0^T\big||\tilde u_n(\tau,t+h)|^{p-2}-|\tilde u_n(\tau,t)|^{p-2}\big|\,\mathrm{d}t\Big]\max_{[h,T+h]}(u_n(t)-u(t))^2.
\end{aligned}$$

Using uniform continuity of σ in $[0,T]$, Corollary 3.6 and (3.29) we get that there exists $\delta_1>0$ such that for all $0<h<\delta_1$ the estimate

$$I_{1,1} \;<\; C_{1,1}\varepsilon\int_0^\infty \rho(t)|\tilde u_n''|^{p-2}(u_n''-u'')^2\,\mathrm{d}t$$

holds for all $n\ge n_0$, where $C_{1,1}$ is a constant independent of n,τ,δ and ε. Using Lagrange's mean value theorem and Hölder's inequality we estimate $I_{1,2}$ in the form:

$$\begin{aligned}
I_{1,2} &= \int_0^T \sigma(t)|\tilde u_n(\tau,t)|^{p-2}\big([u_n(t+h)-u(t+h)]-[u_n(t)-u(t)]\big)^2\,\mathrm{d}t\\
&\le h^2\int_0^T\sigma(t)|\tilde u_n(\tau,t)|^{p-2}\max_{[t,t+h]}\big(u_n'(s)-u'(s)\big)^2\,\mathrm{d}t\\
&\le h^2\int_0^T\sigma(t)|\tilde u_n(\tau,t)|^{p-2}\,\mathrm{d}t\max_{[0,2T]}\big(u_n'(s)-u'(s)\big)^2\\
&= h^2\int_0^T\sigma(t)|\tilde u_n(\tau,t)|^{p-2}\,\mathrm{d}t\max_{[0,2T]}\left(\int_s^\infty(u_n''(\theta)-u''(\theta))\,\mathrm{d}\theta\right)^2\\
&\le h^2\int_0^T\sigma(t)|\tilde u_n(\tau,t)|^{p-2}\,\mathrm{d}t\max_{[0,2T]}\left(\int_s^\infty\rho(\theta)^{-1}|\tilde u_n''|^{2-p}\,\mathrm{d}\theta\int_s^\infty\rho(\theta)|\tilde u_n''|^{p-2}(u_n''-u'')^2\,\mathrm{d}\theta\right)\\
&= h^2\int_0^T\sigma(t)|\tilde u_n(\tau,t)|^{p-2}\,\mathrm{d}t\left(\int_0^\infty\rho(\theta)^{-1}|\tilde u_n''|^{2-p}\,\mathrm{d}\theta\right)\left(\int_0^\infty\rho(\theta)|\tilde u_n''|^{p-2}(u_n''-u'')^2\,\mathrm{d}\theta\right).
\end{aligned}$$

For the boundedness of the first integral we use Corollary 3.6. The boundedness of the second integral follows from the estimate

$$\begin{aligned}\int_0^\infty \rho(\theta)^{-1}|\tilde{u}_n''|^{2-p}\,\mathrm{d}\theta &= \int_0^1 \rho(\theta)^{-1}|\tilde{u}_n''|^{2-p}\,\mathrm{d}\theta + \int_1^\infty \rho(\theta)^{-1}|\tilde{u}_n''|^{2-p}\,\mathrm{d}\theta \\ &\le \max_{s\in[0,1]} \rho(s)^{-1} \int_0^1 |\tilde{u}_n''|^{2-p}\,\mathrm{d}\theta + \int_1^\infty \theta^2\rho(\theta)^{-1}|\tilde{u}_n''|^{2-p}\,\mathrm{d}\theta \quad (3.32)\end{aligned}$$

Corollary 3.6 and Lemma 3.7. Then we choose $\delta_2 > 0$ such that for all $0 < h < \delta_2$ the following

$$I_{1,2} \;<\; C_{1,2}\varepsilon \int_0^\infty \rho(t)|\tilde{u}_n''|^{p-2}(u_n'' - u'')^2\,\mathrm{d}t$$

holds for all $n \ge n_0$, where $C_{1,2}$ is a constant independent of n, τ, δ and ε.

Summing up all the foregoing estimates for $I_{1,1}, I_{1,2}$ and I_2 we get that there exists $\delta = \min\{\delta_1, \delta_2\}$ such that for all $0 < h < \delta$ the following estimate

$$I \le I_{1,1} + I_{1,2} + I_2 < C\varepsilon \int_0^\infty \rho(t)|\tilde{u}_n''|^{p-2}(u_n'' - u'')^2\,\mathrm{d}t$$

holds for all $n \ge n_0$, where C is a constant independent of n, τ and ε.

The proof is complete. □

Lemma 3.10. *Let u and $\{u_n\}_{n=1}^\infty$ be eigenfunctions of (1.1) such that $u_n \to u$ in $W_\infty^{2,p}(\rho)$. Suppose $\|u\|_{p,\sigma} = \|u_n\|_{p,\sigma} = 1$. Then there exists $n_0 > 0$ such that*

$$|\lambda - \lambda_n| \le C \int_0^\infty \rho(t)\,[\varphi(u_n'') - \varphi(u'')]\,(u_n'' - u'')\,\mathrm{d}t \qquad (3.33)$$

holds for all $n \ge n_0$, where C is a constant independent of n.

Proof. λ and u are eigenvalue and eigenfunction of (1.1) if and only if for the functional

$$\Phi(u) = \int_0^\infty \rho(t)|u''(t)|^p\,\mathrm{d}t$$

the following equalities hold

$$\Phi(u) = \lambda, \qquad D\Phi(u, v) = 0$$

for all $v \in W^{2,p}_\infty(\rho)$. Since the functional Φ is continuously differentiable, we have

$$\begin{aligned}
|\lambda_n - \lambda| &= |\Phi(u_n) - \Phi(u)| \\
&= \left| \int_0^1 D\Phi(u + \eta(u_n - u), u_n - u)\,\mathrm{d}\eta \right| \\
&= \left| \int_0^1 \left[D\Phi(u + \eta(u_n - u), u_n - u) - D\Phi(u, u_n - u) \right] \mathrm{d}\eta \right| \\
&= \left| \int_0^1 \left[\int_0^\eta D^2\Phi(u + \tau(u_n - u), u_n - u, u_n - u)\,\mathrm{d}\tau \right] \mathrm{d}\eta \right| \\
&= \left| \int_0^1 (1-\tau) D^2\Phi(u + \tau(u_n - u), u_n - u, u_n - u)\,\mathrm{d}\tau \right| \\
&\le \int_0^1 \left| D^2\Phi(u + \tau(u_n - u), u_n - u, u_n - u) \right| \mathrm{d}\tau \\
&= p(p-1) \int_0^1 \int_0^\infty \rho(t) |\tilde{u}''_n(\tau, t)|^{p-2} (u''_n(t) - u''(t))^2\,\mathrm{d}t\,\mathrm{d}\tau \\
&= p \int_0^\infty \rho(t) \left[\varphi(u''_n) - \varphi(u'') \right] (u''_n - u'')\,\mathrm{d}t.
\end{aligned}$$

The lemma is proved. □

Let u be an eigenfunction and denote

$$\rho_1(t) = \rho(t)|u''(t)|^{p-2} \qquad \text{and} \qquad \sigma_1(t) = \sigma(t)|u(t)|^{p-2}.$$

Consider the following linear problem

$$\left\{ \begin{array}{ll} (\rho_1(t) w''(t))'' - \lambda \sigma_1(t) w(t) = 0, & t > 0, \\ w'(0) = (\rho_1(t) w''(t))'\,|_{t=0} = 0, & w(\infty) = w'(\infty) = 0. \end{array} \right. \tag{3.34}$$

By Corollary 3.6, $\sigma_1, \rho_1 \in L^1[0, x]$ for all $x > 0$.

By a *solution* of (3.34) we understand a function $w \in C^2(0, \infty)$ such that $\rho_1 w'' \in C^2(0, \infty)$, the equation in (3.34) holds at every point, the boundary conditions are satisfied and the Dirichlet integral $\int_0^\infty \rho_1(t)|w''(t)|^2 \mathrm{d}t$ is finite.

The parameter λ is called an *eigenvalue* of (3.34) if this problem has a nontrivial (i.e. nonzero) solution. This solution is then called an *eigenfunction* of (3.34) associated with λ.

Define a Hilbert space $W^{2,2}_\infty(\rho_1)$ of all functions $v \in C^1(0, \infty)$ and v' is absolutely continuous functions such that $v'(0) = v(\infty) = v'(\infty) = 0$ and

$$\|v\|_{2,2;\rho_1} := \left(\int_0^\infty \rho_1(t) |v''(t)|^2\,\mathrm{d}t \right)^{\frac{1}{2}} < \infty.$$

A function $w \in W^{2,2}_{\infty}(\rho_1)$ is called a *weak solution* of (3.34) if the integral identity

$$\int_0^\infty \rho_1(t) w''(t) v''(t)\,\mathrm{d}t = \lambda \int_0^\infty \sigma_1(t) w(t) v(t)\,\mathrm{d}t \tag{3.35}$$

holds for all $v \in W^{2,2}_{\infty}(\rho_1)$.

Remark 3.11. Here also it can be shown that every weak solution is a solution.

Lemma 3.12. *Eigenfunctions of (3.34) associated with eigenvalue λ are mutually proportional.*

Proof. We prove the lemma by contradiction, i.e., let there exist w_1 and w_2 two eigenfunctions associated with λ, which are not mutually proportional. Then

$$w = c_1 w_1 + c_2 w_2$$

is also eigenfunction associated with λ. If we choose c_1 and c_2 such that $w''(0) = 0$, then from (3.34) we get

$$w(t) = w(0) + \lambda \int_0^t \left[\int_\tau^t \frac{(s-\tau)(t-s)}{\rho_1(s)}\,\mathrm{d}s \right] \sigma_1(\tau) w(\tau)\,\mathrm{d}\tau. \tag{3.36}$$

- If $w(0) = 0$ then (3.36) implies

  $$|w(t)| \le \lambda \int_0^t \left[\int_\tau^t \frac{(s-\tau)(t-s)}{\rho_1(s)}\,\mathrm{d}s \right] \sigma_1(\tau) |w(\tau)|\,\mathrm{d}\tau.$$

  Using the Gronwall inequality (Theorem 16 in [5]) we obtain that $w = 0$, which contradicts the linear independence of w_1 and w_2.

- Suppose w does not remain positive. Then there exists $t^* > 0$ such that $w(t) > 0$ for all $t \in [0, t^*)$, $w(t^*) = 0$. Using (3.36) with $t = t^*$ gives a contradiction. Then $w > 0$ everywhere and (3.36) shows that w is increasing contradicting $w(\infty) = 0$. Similarly, we can treat the case $w(0) < 0$.

The lemma is proved. □

Proof of Theorem 3.1. First we prove the easier part: if the normalized eigenfunctions are isolated, then the set of all eigenvalues is isolated and to every λ_i there corresponds a finite number of normed eigenfunctions. In fact, if $\lambda_{n_k} \to \lambda$, then we can suppose that $u_{n_k} \rightharpoonup u$ in $W^{2,p}_{\infty}(\rho)$ and the compact imbedding

$$W^{2,p}_{\infty}(\rho) \hookrightarrow\hookrightarrow L^p(\sigma)$$

implies that $u_{n_k} \to u$ in $L^p(\sigma)$. Using

$$\int_0^\infty \rho(t) \varphi(u''_{n_k}(t)) v''(t)\,\mathrm{d}t = \lambda_{n_k} \int_0^\infty \sigma(t) \varphi(u_{n_k}(t)) v(t)\,\mathrm{d}t$$

and

$$\begin{aligned}&\int_0^\infty \rho(t)\left[\varphi(u''_{n_k}(t))-\varphi(u''_{n_m}(t))\right](u''_{n_k}(t)-u''_{n_m}(t))\,\mathrm{d}t\\ \geq\ & (\|u_{n_k}\|^{p-1}_{2,p;\rho}-\|u_{n_m}\|^{p-1}_{2,p;\rho})(\|u_{n_k}\|_{2,p;\rho}-\|u_{n_m}\|_{2,p;\rho})\end{aligned}$$

we get that $\|u_{n_k}\|_{2,p;\rho}\longrightarrow\|u\|_{2,p;\rho}$. Since $W^{2,p}_\infty(\rho)$ is uniformly convex, we get $u_{n_k}\to u$ in $W^{2,p}_\infty(\rho)$, which implies that $u\neq 0$ is an eigenfunction and λ is the corresponding eigenvalue. This is a contradiction, since eigenfunctions are isolated.

Due to the compact embedding $W^{2,p}_\infty(\rho)\hookrightarrow\hookrightarrow L^p(\sigma)$ (Remark 1.2) we can use Ljusternik-Schnirelmann variational characterization and construct an infinite sequence of (variational) eigenvalues of (1.1) which approach infinity.

In particular, it follows that $\{\lambda_n\}_{n=1}^\infty$ is an isolated set, $\lim\lambda_n=\infty$ and, moreover, to each eigenvalue λ_n there corresponds only a finite number of normalized eigenfunctions.

It remains to show that every normalized eigenfunction is isolated. We prove this fact by contradiction. Let there exist normalized eigenfunctions $u_n, u\in W^{2,p}_\infty(\rho)$ such that $u_n\neq u$, $\|u_n-u\|_{2,p,\rho}\to 0$. Let λ_n and λ be the eigenvalues associated with u_n and u, respectively. From (3.33) it follows $\lambda_n\to\lambda$.

Without loss of generality it can be assumed that $\|u\|_{p,\sigma}=\|u_n\|_{p,\sigma}=1$ for all $n=1,2,\ldots$. From the definition of weak solutions u_n and u we get:

$$\begin{aligned}\int_0^\infty \rho(t)\left[\varphi(u''_n(t))-\varphi(u''(t))\right]v''(t)\,\mathrm{d}t &= \lambda\int_0^\infty\sigma(t)\left[\varphi(u_n(t))-\varphi(u(t))\right]v(t)\,\mathrm{d}t\\ &\quad+(\lambda_n-\lambda)\int_0^\infty\sigma(t)\varphi(u_n(t))v(t)\,\mathrm{d}t\end{aligned}\tag{3.37}$$

for all $v\in W^{2,p}_\infty(\rho)$. If we denote

$$w''_n(t)=\Big(\int_0^1|\tilde u''_n(\tau,t)|^{p-2}\,\mathrm{d}\tau\Big)^{\frac12}|u''(t)|^{\frac{2-p}{2}}(u''_n(t)-u''(t))$$

then

$$\begin{aligned}\|w_n\|^2_{2,2,\rho_1} &= \int_0^\infty\rho_1(t)\big(w''_n(t)\big)^2\,\mathrm{d}t=\int_0^\infty\rho(t)|u''(t)|^{p-2}\big(w''_n(t)\big)^2\,\mathrm{d}t\\ &= \int_0^\infty\rho(t)\left(\int_0^1|\tilde u''_n(\tau,t)|^{p-2}\,\mathrm{d}\tau\right)\big(u''_n(t)-u''(t)\big)^2\,\mathrm{d}t\\ &= \frac{1}{p-1}\int_0^\infty\rho(t)\left[\varphi(u''_n(t))-\varphi(u''(t))\right](u''_n(t)-u''(t))\,\mathrm{d}t.\end{aligned}$$

Dividing both sides of (3.37) by $\|w_n\|_{2,2,\rho_1}$ we have

$$\begin{aligned}\int_0^\infty\rho(t)\frac{\varphi(u''_n)-\varphi(u'')}{\|w_n\|_{2,2,\rho_1}}v''\,\mathrm{d}t &= \lambda\int_0^\infty\sigma(t)\frac{\varphi(u_n)-\varphi(u)}{\|w_n\|_{2,2,\rho_1}}v\,\mathrm{d}t\\ &\quad+\frac{\lambda_n-\lambda}{\|w_n\|_{2,2,\rho_1}}\int_0^\infty\sigma(t)\varphi(u_n)v\,\mathrm{d}t.\end{aligned}\tag{3.38}$$

Let us choose $v = (u_n - u)/\|w_n\|_{2,2,\rho_1}$ in (3.38). Then we have

$$\begin{aligned} p-1 &= \lambda \int_0^\infty \frac{\sigma(t)(\varphi(u_n)-\varphi(u))(u_n-u)}{\|w_n\|^2_{2,2,\rho_1}}\,\mathrm{d}\tau \\ &\quad + \frac{\lambda_n - \lambda}{\|w_n\|^2_{2,2,\rho_1}} \int_0^\infty \sigma(t)\varphi(u_n)(u_n-u)\,\mathrm{d}t. \qquad (3.39) \end{aligned}$$

Note that Lemmas 3.8 and 3.9 guarantee that the sequence

$$\left\{ \frac{[\sigma(\varphi(u_n)-\varphi(u))(u_n-u)]^{\frac{1}{2}}}{\|w_n\|_{2,2,\rho_1}} \right\}_{n=n_0}^\infty$$

is relatively compact in $L^2(0,\infty)$, which implies the existence of a subsequence converging to some $\overline{h} \in L^2(0,\infty)$. From this we get that

$$\frac{[(\varphi(u_n)-\varphi(u))(u_n-u)]^{\frac{1}{2}}}{|u|^{\frac{p-2}{2}}\|w_n\|_{2,2,\rho_1}} \to h \quad \text{in} \quad L^2(\sigma_1),$$

where $h = \overline{h}/\sqrt{\sigma_1} \in L^2(\sigma_1)$.
Passing to the limit in (3.39) and using boundedness of $\frac{\lambda_n - \lambda}{\|w_n\|^2_{2,2,\rho_1}}$ (Lemma 3.10) and

$$\int_0^\infty \sigma(t)\varphi(u_n)(u_n-u)\,\mathrm{d}t \longrightarrow 0$$

we obtain

$$\int_0^\infty \sigma(t)|u(t)|^{p-2}(h(t))^2\,\mathrm{d}t = \frac{p-1}{\lambda}.$$

Moreover, there exists $w \in W^{2,2}_\infty(\rho_1)$ such that

$$\frac{w_n}{\|w_n\|_{2,2,\rho_1}} \rightharpoonup w \quad \text{in} \quad W^{2,2}_\infty(\rho_1). \qquad (3.40)$$

Further, we show that (3.38) converges to some linear problem, the function w is its weak solution. Moreover, it will be shown that $w = h$, which implies that $w \neq 0$.

Without loss of generality it can be assumed for the function v in (3.38) that $v \in W^{2,2}_\infty(\rho_1)$ and there exists $T > 0$ such *supp* $v'' \subset [0,T]$, and $v'' \in C[0,T]$. Since, the set of such functions is dense in $W^{2,2}_\infty(\rho_1)$. Indeed, if $v \in W^{2,2}_\infty(\rho_1)$ then $v'' \in L^1(0,\infty)$, which follows from

$$\int_0^\infty |v''(t)|\,\mathrm{d}t \le \left(\int_0^\infty \rho_1(t)(v''(t))^2\,\mathrm{d}t\right)^{\frac{1}{2}} \left(\int_0^\infty \rho^{-1}(t)|u''(t)|^{2-p}\,\mathrm{d}t\right)^{\frac{1}{2}}$$

and (3.32). Then we use the density in $L^1(0,\infty)$ of the set of all continuous functions with compact support in $(0,\infty)$.

Using Hölder's inequality we get

$$\begin{aligned}
& \left|\int_0^\infty \rho(t)(\varphi(u_n'') - \varphi(u''))v''\,\mathrm{d}t - (p-1)\int_0^\infty \rho(t)|u''|^{p-2}w_n''v''\,\mathrm{d}t\right| \\
= \;& (p-1)\left|\int_0^\infty \rho(t)\left[\int_0^1 |\tilde{u}_n''|^{p-2}\,\mathrm{d}\tau - \Big(\int_0^1 |\tilde{u}_n''|^{p-2}\,\mathrm{d}\tau\Big)^{\frac{1}{2}}|u''|^{\frac{p-2}{2}}\right](u_n''-u'')v''\,\mathrm{d}t\right| \\
= \;& (p-1)\left|\int_0^\infty \rho(t)\left[\Big(\int_0^1 |\tilde{u}_n''|^{p-2}\,\mathrm{d}\tau\Big)^{\frac{1}{2}} - |u''|^{\frac{p-2}{2}}\right]v''\Big(\int_0^1 |\tilde{u}_n''|^{p-2}\,\mathrm{d}\tau\Big)^{\frac{1}{2}}(u_n''-u'')\,\mathrm{d}t\right| \\
\le \;& (p-1)\|w_n\|_{2,2,\rho_1}\left(\int_0^\infty \rho(t)\left[\Big(\int_0^1 |\tilde{u}_n''|^{p-2}\,\mathrm{d}\tau\Big)^{\frac{1}{2}} - |u''|^{\frac{p-2}{2}}\right]^2 (v'')^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
= \;& (p-1)\|w_n\|_{2,2,\rho_1}\left(\int_0^T \rho(t)\left[\Big(\int_0^1 |\tilde{u}_n''|^{p-2}\,\mathrm{d}\tau\Big)^{\frac{1}{2}} - |u''|^{\frac{p-2}{2}}\right]^2 (v'')^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
\le \;& (p-1)\|w_n\|_{2,2,\rho_1}\left(\int_0^T \rho(t)\left|\int_0^1 |\tilde{u}_n''|^{p-2}\,\mathrm{d}\tau - |u''|^{p-2}\right|(v'')^2\,\mathrm{d}t\right)^{\frac{1}{2}} \\
\le \;& (p-1)\|w_n\|_{2,2,\rho_1}\max_{t\in[0,T]}|v''(t)|\max_{\tau\in[0,1]}\left(\int_0^T \rho(t)\left||\tilde{u}_n''|^{p-2} - |u''|^{p-2}\right|\,\mathrm{d}t\right)^{\frac{1}{2}}
\end{aligned}$$

which with Corollary 3.6 imply that

$$\begin{aligned}
\lim_{n\to\infty}\int_0^\infty \rho\frac{\varphi(u_n'')-\varphi(u'')}{\|w_n\|_{2,2,\rho_1}}v''\,\mathrm{d}t &= \lim_{n\to\infty}(p-1)\int_0^\infty \rho|u''|^{p-2}\frac{w_n''}{\|w_n\|_{2,2,\rho_1}}v''\,\mathrm{d}t \\
&= (p-1)\int_0^\infty \rho|u''|^{p-2}w''v''\,\mathrm{d}t.
\end{aligned}$$

If we denote

$$h_n = \Big(\int_0^1 |\tilde{u}_n|^{p-2}\,\mathrm{d}\tau\Big)^{\frac{1}{2}}|u|^{\frac{2-p}{2}}(u_n-u)$$

then similarly as above and using (3.8) we get

$$\begin{aligned}
& \left|\int_0^\infty \sigma(t)(\varphi(u_n)-\varphi(u))v\,\mathrm{d}t - (p-1)\int_0^\infty \sigma(t)|u|^{p-2}h_nv\,\mathrm{d}t\right| \\
\le \;& (p-1)\|h_n\|_{2,\sigma_1}\max_{t\in[0,T]}|v(t)|\max_{\tau\in[0,1]}\left(\int_0^T \sigma(t)\left||\tilde{u}_n|^{p-2} - |u|^{p-2}\right|\,\mathrm{d}t\right)^{\frac{1}{2}} \\
\le \;& C\|w_n\|_{2,2,\rho_1}\max_{\tau\in[0,1]}\left(\int_0^T \sigma(t)\left||\tilde{u}_n|^{p-2} - |u|^{p-2}\right|\,\mathrm{d}t\right)^{\frac{1}{2}},
\end{aligned}$$

which with Corollary 3.6 and (3.40) imply

$$\begin{aligned}
\lim_{n\to\infty}\int_0^\infty \sigma\frac{\varphi(u_n)-\varphi(u)}{\|w_n\|_{2,2,\rho_1}}v\,\mathrm{d}t &= \lim_{n\to\infty}(p-1)\int_0^\infty \sigma|u|^{p-2}\frac{h_n}{\|w_n\|_{2,2,\rho_1}}v\,\mathrm{d}t \\
&= (p-1)\int_0^\infty \sigma|u|^{p-2}hv\,\mathrm{d}t.
\end{aligned}$$

Consequently, we obtain from (3.38) that

$$\int_0^\infty \rho|u''|^{p-2}w''v''\,\mathrm{d}t = \lambda\int_0^\infty \sigma|u|^{p-2}hv\,\mathrm{d}t.$$

Next show that $\int_0^\infty \sigma|u|^{p-2}hv\,\mathrm{d}t = \int_0^\infty \sigma|u|^{p-2}wv\,\mathrm{d}t.$

For this aim we use (3.29) to estimate

$$\begin{aligned}
&\left|\int_0^\infty \sigma(t)\Big(\int_0^1 |\tilde{u}_n(\tau,t)|^{p-2}\,\mathrm{d}\tau\Big)(u_n(t)-u(t))v(t)\,\mathrm{d}t - \int_0^\infty \sigma(t)|u(t)|^{p-2}w_n(t)v(t)\,\mathrm{d}t\right| \\
= &\left|\int_0^T \sigma(t)\Big(\int_0^1 |\tilde{u}_n(\tau,t)|^{p-2}\,\mathrm{d}\tau\Big)(u_n(t)-u(t))v(t)\,\mathrm{d}t - \int_0^T \sigma(t)|u(t)|^{p-2}w_n(t)v(t)\,\mathrm{d}t\right| \\
\le &\int_0^T \sigma(t)\left|\int_0^1 |\tilde{u}_n(\tau,t)|^{p-2}\,\mathrm{d}\tau - |u(t)|^{p-2}\right||u_n(t)-u(t)||v(t)|\,\mathrm{d}t \\
&+\int_0^T \sigma(t)|u(t)|^{p-2}\,|(u_n(t)-u(t))-w_n(t)|\,|v(t)|\,\mathrm{d}t \\
\le &\; C\int_0^T \sigma(t)\left|\int_0^1 |\tilde{u}_n(\tau,t)|^{p-2}\,\mathrm{d}\tau - |u(t)|^{p-2}\right||v(t)|\,\mathrm{d}t\|w_n\|_{2,2,\rho_1} \\
&+\int_0^T \sigma(t)|u(t)|^{p-2}\left|\int_t^\infty (s-t)\left[(u_n''(s)-u''(s))-w_n''(s)\right]\,\mathrm{d}s\right||v(t)|\,\mathrm{d}t = I_1+I_2.
\end{aligned}$$

For I_1 we can write the following estimate, which follows from boundedness of σ and v in $[0,T]$:

$$\begin{aligned}
I_1 &\le C\max_{[0,T]}|v(t)|\int_0^1\int_0^T \sigma(t)\left||\tilde{u}_n(\tau,t)|^{p-2}-|u(t)|^{p-2}\right|\,\mathrm{d}t\,\mathrm{d}\tau\|w_n\|_{2,2,\rho_1} \\
&\le C\max_{\tau\in[0,1]}\int_0^T \left||\tilde{u}_n(\tau,t)|^{p-2}-|u(t)|^{p-2}\right|\,\mathrm{d}t\|w_n\|_{2,2,\rho_1}.
\end{aligned}$$

Now using Corollary 3.6 we get that $\frac{I_1}{\|w_n\|_{2,2,\rho_1}}$ converges to zero when $n\to\infty$.

In order to estimate I_2, we use boundedness of v, Corollary 3.6 and Hölder's inequality

$$\begin{aligned}
I_2 &\le \int_0^T \sigma(t)|u(t)|^{p-2}\left(\int_t^\infty (s-t)\,|(u_n''(s)-u''(s))-w_n''(s)|\,\mathrm{d}s\right)|v(t)|\,\mathrm{d}t \\
&\le \int_0^T \sigma(t)|u(t)|^{p-2}|v(t)|\,\mathrm{d}t\left(\int_0^\infty s\,|(u_n''(s)-u''(s))-w_n''(s)|\,\mathrm{d}s\right) \\
&\le C\int_0^\infty s\left||u''(s)|^{\frac{2-p}{2}}-|\tilde{u}_n''(\tau,s)|^{\frac{2-p}{2}}\right||\tilde{u}_n''(\tau,s)|^{\frac{p-2}{2}}|u_n''(s)-u''(s)|\,\mathrm{d}s \\
&\le C\|w_n\|_{2,2,\rho_1}\left(\int_0^\infty \rho^{-1}(s)s^2\left[|u''(s)|^{\frac{2-p}{2}}-|\tilde{u}_n''(\tau,s)|^{\frac{2-p}{2}}\right]^2\,\mathrm{d}s\right)^{\frac{1}{2}} \\
&\le C\|w_n\|_{2,2,\rho_1}\left(\int_0^\infty \rho^{-1}(s)s^2\left||u''(s)|^{2-p}-|\tilde{u}_n''(\tau,s)|^{2-p}\right|\,\mathrm{d}s\right)^{\frac{1}{2}}.
\end{aligned}$$

Using Corollary 3.6 and Lemma 3.7 we get that the last integral converges to zero as $n \to \infty$.

Consequently we get the following identity

$$\int_0^\infty \rho(t)|u''(t)|^{p-2}w''(t)v''(t)\,\mathrm{d}t = \lambda \int_0^\infty \sigma(t)|u(t)|^{p-2}w(t)v(t)\,\mathrm{d}t \tag{3.41}$$

holds for all $v \in W^{2,2}_\infty(\rho_1)$, i.e., w is an eigenfunction and λ is the corresponding eigenvalue.
The eigenfunction u corresponding to the eigenvalue λ of nonlinear problem (1.5) also solves (3.41). Then by Lemma 3.12 there exists a constant c such that $w(t) = cu(t)$. Let us consider the following functional

$$\Psi(v) = \int_0^\infty \rho(t)u''(t)\varphi(v''(t))\,\mathrm{d}t.$$

Then

$$\begin{aligned}
0 &= \lim_{n\to\infty} \big(\Psi(u_n) - \Psi(u)\big) \\
&= \lim_{n\to\infty} \int_0^1 D\Psi(u + \tau(u_n - u), u_n - u)\,\mathrm{d}\tau \\
&= \lim_{n\to\infty} \int_0^1 D\Psi(u + \tau(u_n - u), u_n - u)\,\mathrm{d}\tau \\
&= \lim_{n\to\infty} (p-1) \int_0^\infty \rho(t)u''(t) \left(\int_0^1 |\tilde{u}''_n(\tau,t)|^{p-2}\,\mathrm{d}\tau\right) (u''_n(t) - u''(t))\,\mathrm{d}t \\
&= \lim_{n\to\infty} (p-1) \int_0^\infty \rho|u''|^{\frac{p-2}{2}} \left(\int_0^1 |\tilde{u}''_n(\tau,t)|^{p-2}\,\mathrm{d}\tau\right)^{\frac{1}{2}} u''w''_n\,\mathrm{d}t = D\Psi(u)w.
\end{aligned}$$

Hence

$$0 = D\Psi(u)w = \int_0^\infty \rho(t)|u''(t)|^{p-2}u''(t)w''(t)\,\mathrm{d}t = c\int_0^\infty \rho(t)|u''(t)|^p\,\mathrm{d}t = c\lambda \neq 0$$

which is a contradiction.

The proof of Theorem 3.1 is complete. □

4 Proof of Theorem 1.1

Necessity. We prove the necessity of (1.2) by contradiction, i.e., suppose that $D-$*property* for (1.1) is satisfied, but (1.2) does not hold. Hence at least one of the limsup in (1.2) is strictly positive. For simplicity we suppose tat

$$\lim_{t\to\infty} A_1(0;\ t) > 0.$$

Then there exists $n_0 > 0$ such that for all $n \geq n_0$ the following estimate

$$\lim_{t \to \infty} A_1(0;\ t) > \frac{1}{\lambda_n}$$

holds. But then by Theorem 2.1, we have that every eigenfunction u_n associated with λ_n has infinitely many zeros in $(0, \infty)$, which contradicts to the $D-property$.

The second case

$$\lim_{t \to \infty} A_2(0;\ t) > 0$$

is treated analogously.

Sufficiency. Let conditions (1.2) be satisfied. Then by Theorem 3.1 we get that the set of all eigenvalues of (1.1) can be written as a monotone increasing sequence $0 < \lambda_1 < ... < \lambda_n < ...$ diverging to infinity. Then using Theorem 2.5 we have that each eigenfunction has finite number of zeros, which implies $D-property$ for (1.1).

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