Space periodic Jacobi elliptic solution for triad modified Schrödinger equations

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Abstract

We present an analytical solution for triad nonlinear evolution equations with modified Schrödinger terms. An example for application in compressible water waves is presented.

1 Introduction

We consider a two-dimensional problem of an interacting wave triad of the form

$$\Psi_{j,t}(x,t) = i\alpha_j [\Psi_{j,xx}(x,t) + \delta_j^2 \Psi_j(x,t)] + \gamma_j V_j(t) \Psi_j(x,t) \quad j = 1, 2, 3$$  \hspace{1cm} (1)

with $\alpha_j$, $\delta_j$, and $\gamma$ are parameters of the physical problem. We also assume that the relations

$$V_1 \Psi_1 = \Psi_2 \Psi_3, \quad V_2 \Psi_2 = \Psi_1 \Psi_3^*, \quad V_3 \Psi_3 = \Psi_1 \Psi_2^*$$ \hspace{1cm} (2)

are satisfied, $\delta_2 = \delta_3 = \delta_1/2 \equiv \delta/2$, and asterisks denote complex conjugates. The objective is to derive a periodic analytical solution, for application in long and short wave triad interactions.

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2 Solution

Redefine $\Psi_j(x,t) = g_j(x)f_j(t)$, so that equations (11) can be written as

\begin{align}
g_1f_{1,t} &= i\alpha_1 \left[ g_{1,xx} + \delta_1^2 g_1 \right] f_1 + \gamma_1 g_2 g_3 f_2 f_3 \\
g_2f_{2,t} &= i\alpha_2 \left[ g_{2,xx} + \delta_2^2 g_2 \right] f_2 + \gamma_2 g_1 g_3 f_1 f_3^* \\
g_3f_{3,t} &= i\alpha_3 \left[ g_{3,xx} + \delta_3^2 g_3 \right] f_3 + \gamma_3 g_1 g_2 f_1 f_2^* 
\end{align}

(3) (4) (5)

Based on (2), $g_1 = g_2 g_3$, $g_2 = g_1 g_3^*$, and $g_3 = g_1 g_2^*$, so that equations (3), (4), and (5), can be rewritten as

\begin{align}
f_{1,t} &= \gamma_1 f_2 f_3 + i\alpha_1 \left[ g_{1,xx} + \delta_1^2 g_1 \right] f_1 g_1^{-1} \\
f_{2,t} &= \gamma_2 f_1 f_3^* + i\alpha_2 \left[ g_{2,xx} + \delta_2^2 g_2 \right] f_2 g_2^{-1} \\
f_{3,t} &= \gamma_3 f_1 f_2^* + i\alpha_3 \left[ g_{3,xx} + \delta_3^2 g_3 \right] f_3 g_3^{-1} 
\end{align}

(6) (7) (8)

Now we can seek a solution in two parts. The first part requires that

$$g_{j,xx} + \delta_j^2 g_j = 0, \quad j = 1, 2, 3.$$  

(9)

A general solution of (9) is given by

$$g_j = a_j e^{ij \delta_j x} + b_j e^{-ij \delta_j x}. \quad (10)$$

For the second part of the solution we need to solve the following simplified system of three ordinary differential equations that amplitudes satisfy

$$f_{1,t} = \gamma_1 f_2 f_3, \quad f_{2,t} = \gamma_2 f_1 f_3^*, \quad f_{3,t} = \gamma_3 f_1 f_2^* \quad (11)$$

Multiplying (11) by $f_j^*$ and adding its conjugate multiplied by $f_j$, for $j=1,2,3$ respectively, we obtain the following set of equations

\begin{align}
|f_{1,t}|^2 &= \gamma_1 \Re\{f_1^* f_2 f_3\} + f_1 f_2 f_3 \\
|f_{2,t}|^2 &= \gamma_2 \Re\{f_1^* f_2 f_3\} + f_1 f_2 f_3 \\
|f_{3,t}|^2 &= \gamma_3 \Re\{f_1^* f_2 f_3\} + f_1 f_2 f_3 
\end{align}

(12) (13) (14)

More compactly we can write $|f_j|^2 = 2\gamma_j \Re\{f_1^* f_2 f_3\}$, where the Hamiltonian $\Re\{f_1^* f_2 f_3\}$ is a constant of the motion (Holm and Lynch (2002)). Now define $Z_t = \Re\{f_1^* f_2 f_3\}$ gives $|f_j|^2 = 2\gamma_j Z + \psi_j^2$. In order to carry on with the
solution the signs of $\gamma_j$ have to be determined. Note that for a resonating triad, $\gamma_1 + \gamma_2 + \gamma_3 = 0$, (Lynch et al. (2003)), thus one has a different sign than the others. Assume, with no loss of generality, that $\gamma_1$ is negative, that $\Psi_1(x, t = 0) = \psi_{01} = 0$, and that $|\psi_{03}| < |\psi_{02}|$ we obtain

$$Z_t = \sqrt{-8|\gamma_1|\gamma_2\gamma_3}Z \left( Z + \frac{\psi_{02}^2}{\gamma_2} \right) \left( Z + \frac{\psi_{03}^2}{\gamma_3} \right)$$

(15)

This is an elliptic function with a solution given by (see Byrd and Friedman (1971), equation 236.00, p.79)

$$Z = -\frac{\psi_{03}^2}{\gamma_3} \text{sn}^2(u, k)$$

(16)

where $\text{sn}(u, k)$ is the sine amplitude Jacobian elliptic function of argument $u$, and modulus $k$ given by

$$u = \sqrt{2|\gamma_1|\gamma_3|\psi_{02}|}t, \quad k = \frac{|\psi_{03}|}{|\psi_{02}|} \sqrt{\frac{\gamma_2}{\gamma_3}}$$

(17)

and the expression for $|f_j|$ are

$$|f_j|^2 = |\psi_{0j}|^2 - 2\gamma_j|\psi_{03}|^2 \text{sn}^2(u, k)$$

(18)

Finally, the analytical solution is given by

$$|\Psi_1(x, t)|^2 = -2\gamma_1 \frac{|\psi_{03}|^2}{\gamma_3} \text{sn}^2(u, k) \left[ \exp(2i\delta x) + \exp(-2i\delta x) \right]$$

(19)

$$|\Psi_2(x, t)|^2 = |\psi_{02}|^2 - 2\gamma_2 \frac{|\psi_{03}|^2}{\gamma_3} \text{sn}^2(u, k) \left[ \exp(i\delta x) + \exp(-i\delta x) \right]$$

(20)

$$|\Psi_3(x, t)|^2 = |\psi_{03}|^2 \left[ 1 - 2\text{sn}^2(u, k) \right] \left[ \exp(i\delta x) + \exp(-i\delta x) \right]$$

(21)

3 Application

The solution presented here can be applied in various long-short wave triad interactions, such as Rossby-type waves (see Pedlosky (1987); Charney (1948)), or wave motion in an inhomogeneous plasma (Hasegawa and Mima (1977)). Nevertheless, the following example considers the interaction of two surface gravity waves with an acoustic wave in a mechanism similar to that proposed by Longuet-Higgins (1950), and more recently by Kadri and Stiassnie (2013).
Given an acoustic wave and two gravity waves with potential amplitudes $\phi_a$, $\phi_{g1}$, and $\phi_{g2}$, satisfying the following evolution equations

$$\phi_{a,t} = -\frac{ic^2\delta}{2\omega h} \left( \phi_{a,xx} + \frac{4\omega^2}{c^2} \psi_a \right) - \frac{2\omega}{hc} \phi_{g1}\phi_{g2}$$  \hspace{1cm} (22)

$$\phi_{g1,t} = \frac{2\omega^3}{gc} \phi_a \phi^*_{g2}; \quad \phi_{g2,t} = \frac{2\omega^3}{gc} \phi_a \phi^*_{g1}$$  \hspace{1cm} (23)

where $c = 1500 \text{ m/s}$, is the speed of sound in water, $\omega$ is the frequency of the gravity waves, $h$ is the water depth. The solution of evolution equations is then given by

$$|\Phi_a(x,t)|^2 \hspace{0.1cm} = \hspace{0.1cm} \frac{2g|\phi_{0(g2)}|^2}{\omega h} \text{sn}^2(u,k) \left( e^{4i\omega x/c} + e^{-4i\omega x/c} \right)$$  \hspace{1cm} (24)

$$|\Phi_{g1}(x,t)|^2 \hspace{0.1cm} = \hspace{0.1cm} \left[ |\phi_{0(g1)}|^2 - 2|\phi_{0(g2)}|^2 \text{sn}^2(u,k) \right] \left( e^{2i\omega x/c} + e^{-2i\omega x/c} \right)$$  \hspace{1cm} (25)

$$|\Phi_{g2}(x,t)|^2 \hspace{0.1cm} = \hspace{0.1cm} |\phi_{0(g2)}|^2 \left[ 1 - 2\text{sn}^2(u,k) \right] \left( e^{2i\omega x/c} + e^{-2i\omega x/c} \right)$$  \hspace{1cm} (26)

with $u = 2\sqrt{2/gh}\omega^2/c$, and $k = |\phi_{0(g2)}|/|\phi_{0(g1)}|$.

**References**


