ON THE DISTANCE BETWEEN FROBENIUS NUMBERS

ISKANDER ALIEV AND DILPAK MOHAMMED

ABSTRACT. Let $n \geq 2$ and $k \geq 1$ be integers and $\boldsymbol{a} = (a_1, \dots, a_n)^t$ be an integer vector with positive coprime entries. The k-Frobenius number $\mathrm{F}_k(\boldsymbol{a})$ is the largest integer that cannot be represented as $\sum_{i=1}^{n} a_i x_i$ with $x_i \in \mathbb{Z}_{\geq 0}$ in at least k different ways. We study the quantity $(\mathrm{F}_k(\boldsymbol{a}) - \mathrm{F}_1(\boldsymbol{a}))(a_1 \cdots a_n)^{-1/(n-1)}$ and use obtained results to improve existing upper bounds for 2-Frobenius numbers. The proofs are based on packing and covering results from the geometry of numbers.

1. INTRODUCTION

Let $\boldsymbol{a} = (a_1, \dots, a_n)^t$, $n \geq 2$, be an integer vector with

$$0 < a_1 < \cdots < a_n \, , \ \gcd(a_1, \dots, a_n) = 1 \tag{1.1}$$

and let k be a positive integer. The *k-Frobenius number* $\mathrm{F}_k(\boldsymbol{a})$ is the largest integer that cannot be represented in at least k different ways as a non-negative integer combination of the a_i's, that is

$$\mathrm{F}_k(\boldsymbol{a}) = \max\{b \in \mathbb{Z} : \#\{\boldsymbol{x} \in \mathbb{Z}^n_{\geq 0} : \langle \boldsymbol{a}, \boldsymbol{x} \rangle = b\} < k\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^n$.

The classical *Frobenius number* $\mathrm{F}_1(\boldsymbol{a})$ has been extensively studied in the literature. For a comprehensive survey we refer the reader to the book of Ramirez Alfonsin [15]. For $k \geq 2$, the number $\mathrm{F}_k(\boldsymbol{a})$ was introduced and studied by Beck and Robins [6] who obtained for $n = 2$ the formula

$$\mathrm{F}_k(\boldsymbol{a}) = k \, a_1 \, a_2 - (a_1 + a_2) \, , \tag{1.2}$$

generalising a classical result on the Frobenius numbers, usually attributed to Sylvester [18]. In general setting, it was recently shown by Aliev, De Loera and Louveaux [1] that $\mathrm{F}_k(\boldsymbol{a})$ can be computed in polynomial time for fixed dimension n and parameter k, extending well-known results of Kannan [13] and Barvinok and Woods [5] for the Frobenius number $\mathrm{F}_1(\boldsymbol{a})$. When dimension n is a part of input, computing $\mathrm{F}_k(\boldsymbol{a})$ is NP-hard already for $k = 1$ due to a result of Ramirez Alfonsin [14].

Lower and upper bounds for the Frobenius number $\mathrm{F}_1(\boldsymbol{a})$ were given by many authors, see [15] and for more recent results Aliev and Gruber [3] and Fukshansky and Robins [8]. In recent years, some of these results have been

2010 *Mathematics Subject Classification.* 11D07, 11H06, 52C07, 11D45.

Key words and phrases. Frobenius number, successive minima, covering radius, difference body.

extended to the case $k \geq 2$. A sharp lower bound for $\mathrm{F}_k(\boldsymbol{a})$ was obtained by Aliev, Henk and Linke [4] as a generalisation of a bound from [3]. Upper bounds for the k-Frobenius number were established by Fukshansky and Schürmann [9] and Aliev, Fukshansky and Henk [2]. In particular, it was shown in [2] that

$$\mathrm{F}_k(\boldsymbol{a}) \leq \mathrm{F}_1(\boldsymbol{a}) + ((k-1)\,(n-1)!)^{\frac{1}{n-1}}\,\Pi(\boldsymbol{a})^{\frac{1}{n-1}}\,, \tag{1.3}$$

where $\Pi(\boldsymbol{a}) = a_1 \cdots a_n$. The inequality (1.3) allows us to use various upper bounds for the Frobenius number to estimate $\mathrm{F}_k(\boldsymbol{a})$.

In view of (1.3), to estimate $\mathrm{F}_k(\boldsymbol{a})$ from above it is natural to study the *(normalised) distance*

$$\tau_k(\boldsymbol{a}) = \frac{\mathrm{F}_k(\boldsymbol{a}) - \mathrm{F}_1(\boldsymbol{a})}{\Pi(\boldsymbol{a})^{\frac{1}{n-1}}}$$

between $\mathrm{F}_k(\boldsymbol{a})$ and $\mathrm{F}_1(\boldsymbol{a})$ and the constant

$$c(n,k) = \sup_{\boldsymbol{a}} \tau_k(\boldsymbol{a})\,, \tag{1.4}$$

where the supremum in (1.4) is taken over all integer vectors satisfying (1.1). Clearly, (1.3) implies the bound

$$c(n,k) \leq ((k-1)\,(n-1)!)^{\frac{1}{n-1}}\,. \tag{1.5}$$

In view of (1.2), in this paper we will focus on the case $n \geq 3$. The first result shows that, roughly speaking, cutting off special families of input vectors cannot make the order of magnitude of $\mathrm{F}_k(\boldsymbol{a}) - \mathrm{F}_1(\boldsymbol{a})$ smaller than $\Pi(\boldsymbol{a})^{\frac{1}{n-1}}$.

Theorem 1.1. *Let $n \geq 3$ and $k \geq 2$. For any direction vector $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_{n-1})^t \in \mathbb{Q}^{n-1}$ with $0 < \alpha_1 < \cdots < \alpha_{n-1} < 1$ there exists an infinite sequence of distinct integer vectors $\boldsymbol{a}(t) = (a_1(t), \ldots, a_n(t))^t$ satisfying (1.1) such that*

(i) $\lim_{t\to\infty} \frac{a_i(t)}{a_n(t)} = \alpha_i$, $1 \leq i \leq n-1$,
(ii) $\lim_{t\to\infty} \tau_k(\boldsymbol{a}(t)) = p(n-1,k)$, *where* $p(d,k) = \min\{m \in \mathbb{Z}_{\geq 0} : \binom{m+d}{d} \geq k\}$.

It follows that $c(n,k) \geq p(n-1,k)$. Since for a fixed dimension $n \geq 3$ we have $p(n-1,k)((k-1)\,(n-1)!)^{-\frac{1}{n-1}} \to 1$ as $k \to \infty$, Theorem 1.1 also implies that for large k the upper bound (1.5) (and hence (1.3)) cannot be significantly improved.

The exact values of the constants $c(n,k)$ remain unknown apart of the case $c(2,k) = k-1$, which follows from (1.2). In this paper we give a new upper bound for the case $k = 2$:

Theorem 1.2. *Let $n \geq 3$. Then*

$$c(n,2) \leq 2\left(\frac{(n-1)!}{\binom{2(n-1)}{n-1}}\right)^{\frac{1}{n-1}}\,. \tag{1.6}$$

Theorem 1.2 improves (1.5) with the factor $f(n) = 2\binom{2(n-1)}{n-1}^{-\frac{1}{n-1}}$. The asymptotic behavior and bounds for $f(n)$ can be easily derived from results on extensively studied *Catalan numbers* $C_d = (d+1)^{-1}\binom{2d}{d}$. In particular, $f(n) < \frac{1}{2}(4\pi(n-1)^2/(4(n-1)-1))^{1/(2(n-1))} < 0.82$ and $f(n) \sim \frac{1}{2}(\pi(n-1))^{1/(2(n-1))}$ (see [7]). The proof of Theorem 1.2 is based on the geometric approach used in [2], combined with results on the difference bodies dated back to works of Minkowski (see e.g. Gruber [10], Section 30.1) and Rogers and Shephard [16].

2. Covering radii and Frobenius numbers

In what follows, $\mathcal{K}^d$ will denote the space of all d-dimensional *convex bodies*, i.e., closed bounded convex sets with non-empty interior in the d-dimensional Euclidean space $\mathbb{R}^d$. Recall that the *Minkowski sum* $X+Y$ of the sets $X, Y \subset \mathbb{R}^d$ consists of all points $\boldsymbol{x}+\boldsymbol{y}$ with $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in Y$. For $K \in \mathcal{K}^d$, the *difference body* D_K of K is the origin-symmetric convex body defined as $D_K = K - K = K + (-K)$.

By $\mathcal{L}^d$ we denote the set of all d-dimensional lattices in $\mathbb{R}^d$. Given a matrix $B \in \mathbb{R}^{d\times d}$ with $\det B \neq 0$ and a set $Q \subset \mathbb{R}^d$ let $BQ := \{B\boldsymbol{x} : \boldsymbol{x} \in Q\}$ be the image of Q under linear map defined by B. Then we can write $\mathcal{L}^d = \{B\,\mathbb{Z}^d : B \in \mathbb{R}^{d\times d}, \det B \neq 0\}$. For $\Lambda = B\,\mathbb{Z}^d \in \mathcal{L}^d$, $\det(\Lambda) = |\det B|$ is called the *determinant* of the lattice Λ.

For $K \in \mathcal{K}^d$ and $\Lambda \in \mathcal{L}^d$ the *k-covering radius* of K with respect to Λ is the smallest positive number μ such that any point $\boldsymbol{x} \in \mathbb{R}^d$ is *covered with multiplicity at least k by $\mu K + \Lambda$*, that is

$$\begin{aligned}\mu_k(K,\Lambda) = \min\{\mu > 0 : &\text{ for all } \boldsymbol{x} \in \mathbb{R}^d \text{ there exist } \boldsymbol{b}_1, \dots, \boldsymbol{b}_k \in \Lambda \\ &\text{ such that } \boldsymbol{x} \in \boldsymbol{b}_i + \mu K\,,\ 1 \le i \le k\}\,.\end{aligned}$$

For $k=1$ we get the well-known covering radius, see e.g. Gruber [10] and Gruber and Lekkerkerker [12]. These books also serve as excellent sources for futher information on lattices and convex bodies in the context of the geometry of numbers.

The k-covering radii appear to be closely related to the k-Frobenius numbers. Given integer vector $\boldsymbol{a}$ satisfying (1.1), define the $(n-1)$-dimensional simplex

$$S_{\boldsymbol{a}} = \left\{\boldsymbol{x} \in \mathbb{R}^{n-1}_{\ge 0} : a_1\,x_1 + \cdots + a_{n-1}\,x_{n-1} \le 1\right\}$$

and the $(n-1)$-dimensional lattice

$$\Lambda_{\boldsymbol{a}} = \left\{\boldsymbol{x} \in \mathbb{Z}^{n-1} : a_1\,x_1 + \cdots + a_{n-1}\,x_{n-1} \equiv 0 \bmod a_n\right\}.$$

Kannan [13] established the identity

$$\mu_1(S_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}}) = \mathrm{F}_1(\boldsymbol{a}) + a_1 + \cdots + a_n.$$

The result of Kannan was further generalised in [2] as follows.

Lemma 2.1 (Theorem 3.2 in [2]). *Let $n \geq 2$, $k \geq 1$. Then*
$$\mu_k(S_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}}) = \mathrm{F}_k(\boldsymbol{a}) + a_1 + \cdots + a_n.$$

The following three lemmas will be used in the proof of Theorem 1.1. Let $S^d = \left\{\boldsymbol{x} \in \mathbb{R}^d_{\geq 0} : x_1 + \cdots + x_d \leq 1\right\}$ be the standard simplex in $\mathbb{R}^d$.

Lemma 2.2. *Let $d \geq 2$, $k \geq 1$. Then*
$$\mu_k(S^d, \mathbb{Z}^d) = p(d,k) + d\,. \tag{2.1}$$

Proof. Let $F = [0,1)^d$ be the fundamental cell of the lattice $\mathbb{Z}^d$ with respect to the standard basis. It is easy to see that
$$\begin{aligned}\mu_k(S^d, \mathbb{Z}^d) = \min\{\mu > 0 :& \text{ there exist } \boldsymbol{b}_1, \dots, \boldsymbol{b}_k \in \mathbb{Z}^d \\ &\text{such that } F \subset (\boldsymbol{b}_i + \mu S^d)\,,\ 1 \leq i \leq k\}\,.\end{aligned} \tag{2.2}$$
This implies, in particular, that $\mu_k(S^d, \mathbb{Z}^d)$ is an integer number $\geq d$.

Suppose that F is covered by $\boldsymbol{u} + \bar{t}S^d$ with $\boldsymbol{u} \in \mathbb{Z}^d$. Then, clearly, $\boldsymbol{u} \in \mathbb{Z}^d_{\leq 0}$ and $\bar{t} \geq d$. Observe that
$$F \subset (\boldsymbol{u} + \bar{t}S^d) \iff \boldsymbol{0} \in (\boldsymbol{u} + (\bar{t} - d)S^d) \iff -\boldsymbol{u} \in (\bar{t} - d)S^d\,.$$
Hence, F is covered with multiplicity at least k by $(m+d)S^d + \mathbb{Z}^d$ if and only if mS^d contains at least k integer points. Therefore, by (2.2),
$$\mu_k(S^d, \mathbb{Z}^d) = \min\{m \in \mathbb{Z}_{\geq 0} : \#(mS^d \cap \mathbb{Z}^d) \geq k\} + d\,.$$
Noting that $\#(mS^d \cap \mathbb{Z}^d) = \binom{m+d}{d}$, we obtain (2.1).

□

Following Gruber [11], we say that a sequence S_t of convex bodies in $\mathbb{R}^d$ converges to a convex body S if the sequence of *distance functions* of S_t converges uniformly on the unit ball in $\mathbb{R}^d$ to the distance function of S. For the notion of convergence of a sequence of lattices to a given lattice we refer the reader to p. 178 of [12].

Lemma 2.3 (see Satz 1 in [11]). *Let S_t be a sequence of convex bodies in $\mathbb{R}^d$ which converges to a convex body S and let Λ_t be a sequence of lattices in $\mathbb{R}^d$ convergent to a lattice Λ. Then*
$$\lim_{t\to\infty} \mu_k(S_t, \Lambda_t) = \mu_k(S, \Lambda)\,.$$

The last ingredients of the proof of Theorem 1.1 is the following result from [3] which is also implicit in Schinzel [17].

Lemma 2.4 (Theorem 1.2 in [3]). *For any lattice Λ with basis $\boldsymbol{b}_1, \dots, \boldsymbol{b}_d$, $\boldsymbol{b}_i \in \mathbb{Q}^d$, $i = 1, \dots, d$, and for all rationals $\alpha_1, \dots, \alpha_d$ with $0 < \alpha_1 < \cdots < \alpha_d < 1$, there exists a sequence*
$$\boldsymbol{a}(t) = (a_1(t), \dots, a_d(t), a_{d+1}(t))^t \in \mathbb{Z}^{d+1}\,, t = 1, 2, \dots\,,$$

such that $\gcd(a_1(t),\dots,a_d(t),a_{d+1}(t))=1$ *and the lattice* $\Lambda_{\boldsymbol{a}(t)}$ *has a basis* $\boldsymbol{b}_1(t),\dots,\boldsymbol{b}_d(t)$ *with*

$$\frac{b_{ij}(t)}{s\,t}=b_{ij}+O\left(\frac{1}{t}\right)\,,\quad i,j=1,\dots,d\,, \tag{2.3}$$

where $s\in\mathbb{N}$ *is such that* $s\,b_{ij}, s\,\alpha_j\,b_{ij}\in\mathbb{Z}$ *for all* $i,j=1,\dots,d$. *Moreover,*

$$a_{d+1}(t)=\det(\Lambda)s^dt^d+O(t^{d-1}) \tag{2.4}$$

and

$$\alpha_i(t):=\frac{a_i(t)}{a_{d+1}(t)}=\alpha_i+O\left(\frac{1}{t}\right). \tag{2.5}$$

Recall that successive minima $\lambda_i(K,\Lambda)$ of an origin-symmetric convex body $K\in\mathcal{K}^d$ with respect to a lattice $\Lambda\in\mathcal{L}^d$ are defined as

$$\lambda_i(K,\Lambda)=\min\left\{\lambda>0:\dim\left(\lambda K\cap\Lambda\right)\geq i\right\},1\leq i\leq d.$$

The proof of Theorem 1.2 is based on a link between lattice coverings with multiplicity at least two with usual lattice coverings and packings of convex bodies. Following the classical approach of Minkowski, we will use difference bodies and successive minima in our work with lattice packings.

Lemma 2.5. *Let* $\Lambda\in\mathcal{L}^d$ *and* $K\in\mathcal{K}^d$. *Then*

$$\mu_2(K,\Lambda)\leq\mu_1(K,\Lambda)+\lambda_1(D_K,\Lambda).$$

Proof. There exists a nonzero point $\boldsymbol{u}\in\Lambda$ in the set λ_1D_K, where $\lambda_1=\lambda_1(D_K,\Lambda)$. Then, by the definition of difference body, there exists a point $\boldsymbol{v}$ in the intersection $\lambda_1K\cap(\boldsymbol{u}+\lambda_1K)$. Indeed, $\boldsymbol{u}=\boldsymbol{u}_1-\boldsymbol{u}_2$ with $\boldsymbol{u}_1,\boldsymbol{u}_2\in\lambda_1K$ and hence we can take $\boldsymbol{v}:=\boldsymbol{u}_1=\boldsymbol{u}+\boldsymbol{u}_2\in\lambda_1K\cap(\boldsymbol{u}+\lambda_1K)$.

Next, given an arbitrary point $\boldsymbol{x}\in\mathbb{R}^d$ we know by the definition of the covering radius $\mu_1=\mu_1(K,\Lambda)$ that there exists a point $\boldsymbol{z}\in\Lambda$ such that $\boldsymbol{x}-\boldsymbol{v}\in\boldsymbol{z}+\mu_1\,K$. Hence

$$\begin{aligned}\boldsymbol{x}&\in\boldsymbol{z}+(\mu_1+\lambda_1)\,K\ \text{ and}\\ \boldsymbol{x}&\in\boldsymbol{z}+\boldsymbol{u}+(\mu_1+\lambda_1)\,K,\end{aligned}$$

so that $\boldsymbol{x}$ is covered with multiplicity at least two by $(\mu_1+\lambda_1)\,K+\Lambda$. Therefore

$$\mu_2(K,\Lambda)\leq\mu_1+\lambda_1$$

and the lemma is proved.

$\square$

3. Proof of Theorem 1.1

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{n-1})^t$ be any rational vector in $\mathbb{Q}^{n-1}$ satisfying

$$0 < \alpha_1 < \dots < \alpha_{n-1} < 1 \tag{3.1}$$

and let $D(\boldsymbol{\alpha}) = \mathrm{diag}(\alpha_1^{-1}, \dots, \alpha_{n-1}^{-1})$. Then $\Lambda(\boldsymbol{\alpha}) = D(\boldsymbol{\alpha})\mathbb{Z}^{n-1}$ is the lattice of determinant $\det(L(\boldsymbol{\alpha})) = (\Pi(\boldsymbol{\alpha}))^{-1}$ and $S(\boldsymbol{\alpha}) = D(\boldsymbol{\alpha})S^{n-1}$ is the simplex of volume $\mathrm{vol}\,(S(\boldsymbol{\alpha})) = (\Pi(\boldsymbol{\alpha})(n-1)!)^{-1}$.

Applying Lemma 2.4 to the lattice $\Lambda = \Lambda(\boldsymbol{\alpha})$ and the numbers $\alpha_1, \dots, \alpha_{n-1}$, we get a sequence $\boldsymbol{a}(t)$, satisfying (2.3), (2.4) and (2.5). Furthermore, by (3.1) and (2.5),

$$0 < a_1(t) < a_2(t) < \dots < a_n(t)$$

for sufficiently large t.

Define the simplex S_t and the lattice Λ_t as

$$S_t = a_n(t)S_{\boldsymbol{a}(t)} = \{(x_1, \dots, x_{n-1})^t \in \mathbb{R}^{n-1}_{\geq 0} : \sum_{i=1}^{n-1} \alpha_i(t)x_i \leq 1\},$$

$$\Lambda_t = (\Pi(\boldsymbol{\alpha})a_n(t))^{-1/(n-1)}\Lambda_{\boldsymbol{a}(t)}.$$

Then, in particular,

$$\mu_k(S_{\boldsymbol{a}(t)}, \Lambda_{\boldsymbol{a}(t)}) = \Pi(\boldsymbol{\alpha})^{1/(n-1)}a_n(t)^{n/(n-1)}\mu_k(S_t, \Lambda_t). \tag{3.2}$$

By (2.3) and (2.4), the sequence Λ_t converges to the lattice $\Lambda(\boldsymbol{\alpha})$. Next, the point $\boldsymbol{p} = (1/(2n), \dots, 1/(2n))$ is an inner point of the simplex $S(\boldsymbol{\alpha})$ and all the simplicies S_t for sufficiently large t. By (2.5) and Lemma 2.3, the sequence $\mu_k(S_t - \boldsymbol{p}, \Lambda_t)$ converges to $\mu_k(S(\boldsymbol{\alpha}) - \boldsymbol{p}, \Lambda_{\boldsymbol{\alpha}})$. Here we consider the sequence $\mu_k(S_t - \boldsymbol{p}, \Lambda_t)$ instead of $\mu_k(S_t, \Lambda_t)$ because the distance functions of the family of convex bodies in Lemma 2.3 need to converge on the unit ball. Now, since k-covering radii are independent of translation, the sequence $\mu_k(S_t, \Lambda_t)$ converges to $\mu_k(S(\boldsymbol{\alpha}), \Lambda(\boldsymbol{\alpha}))$. Clearly,

$$\mu_k(S(\boldsymbol{\alpha}), \Lambda(\boldsymbol{\alpha})) = \mu_k(D(\boldsymbol{\alpha})^{-1}S(\boldsymbol{\alpha}), D(\boldsymbol{\alpha})^{-1}\Lambda(\boldsymbol{\alpha})) = \mu_k(S^{n-1}, \mathbb{Z}^{n-1}).$$

Therefore, using Lemma 2.2, we have

$$\begin{aligned} \mu_k(S_t, \Lambda_t) - \mu_1(S_t, \Lambda_t) \to \mu_k(S^{n-1}, \mathbb{Z}^{n-1}) &- \mu_1(S^{n-1}, \mathbb{Z}^{n-1}) \\ &= p(n-1, k), \end{aligned}$$

as $t \to \infty$. Therefore, by Lemma 2.1, (3.2) and (2.5), we obtain

$$\begin{aligned} \tau_k(\boldsymbol{a}(t)) = \frac{F_k(\boldsymbol{a}(t)) - F_1(\boldsymbol{a}(t))}{\Pi(\boldsymbol{a}(t))^{\frac{1}{n-1}}} &= \frac{\mu_k(S_{\boldsymbol{a}(t)}, \Lambda_{\boldsymbol{a}(t)}) - \mu_1(S_{\boldsymbol{a}(t)}, \Lambda_{\boldsymbol{a}(t)})}{\Pi(\boldsymbol{a}(t))^{\frac{1}{n-1}}} \\ &= \frac{\Pi(\boldsymbol{\alpha})^{1/(n-1)}a_n(t)^{n/(n-1)}(\mu_k(S_t, \Lambda_t) - \mu_1(S_t, \Lambda_t))}{\Pi(\boldsymbol{a}(t))^{\frac{1}{n-1}}} \\ &= \frac{\Pi(\boldsymbol{\alpha})^{1/(n-1)}(\mu_k(S_t, \Lambda_t) - \mu_1(S_t, \Lambda_t))}{(\prod_{i=1}^{n-1} \alpha_i(t))^{\frac{1}{n-1}}} \to p(n-1, k) \end{aligned}$$

as $t \to \infty$. Together with (2.5) this concludes the proof of Theorem 1.1.

4. Proof of Theorem 1.2

Let $\boldsymbol{\alpha} = (1/a_1, \ldots, 1/a_{n-1})$ and let $\Gamma_{\boldsymbol{a}} = D(\boldsymbol{\alpha})\Lambda_{\boldsymbol{a}}$, where in notation of Section 3 we set $D(\boldsymbol{\alpha}) = \operatorname{diag}(\alpha_1^{-1}, \ldots, \alpha_{n-1}^{-1})$. Then $\Gamma_{\boldsymbol{a}}$ is the lattice of determinant $\det(\Gamma_{\boldsymbol{a}}) = \Pi(\boldsymbol{\alpha})^{-1}\det(\Lambda_{\boldsymbol{a}}) = \Pi(\boldsymbol{a})$ and since $S^{n-1} = D(\boldsymbol{\alpha})S_{\boldsymbol{a}}$, we have

$$\mu_k(S_{\boldsymbol{a}}, \Lambda_{\boldsymbol{a}}) = \mu_k(S^{n-1}, \Gamma_{\boldsymbol{a}})\,. \tag{4.1}$$

By Lemmas 2.1 and 2.5, together with (4.1), we obtain

$$\begin{aligned}\frac{F_2(\boldsymbol{a}) - F_1(\boldsymbol{a})}{\Pi(\boldsymbol{a})^{\frac{1}{n-1}}} &= \frac{\mu_2(S^{n-1}, \Gamma_{\boldsymbol{a}}) - \mu_1(S^{n-1}, \Gamma_{\boldsymbol{a}})}{\Pi(\boldsymbol{a})^{\frac{1}{n-1}}} \\ &\le \frac{\lambda_1(D_{S^{n-1}}, \Gamma_{\boldsymbol{a}})}{\Pi(\boldsymbol{a})^{\frac{1}{n-1}}}\,.\end{aligned} \tag{4.2}$$

As was shown by Rogers and Shephard [16], $\operatorname{vol}(D_{S^d}) = \binom{2d}{d}\operatorname{vol}(S^d) = \binom{2d}{d}/d!$. Hence, by Minkowski's second fundamental theorem, we get the inequality

$$\lambda_1(D_{S^{n-1}}, \Gamma_{\boldsymbol{a}}) \le 2\left(\frac{\det(\Gamma_{\boldsymbol{a}})}{\operatorname{vol}(D_{S^{n-1}})}\right)^{\frac{1}{n-1}} = 2\left(\frac{(n-1)!}{\binom{2(n-1)}{n-1}}\right)^{\frac{1}{n-1}}\Pi(\boldsymbol{a})^{\frac{1}{n-1}}\,. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain the bound (1.6).

References

[1] I. Aliev, J. A. De Loera and Q. Louveaux, *Semigroups of polyhedra with prescribed number of lattice points and the k-Frobenius problem*, http://arxiv.org/abs/1409.5259.

[2] I. Aliev, L. Fukshansky and M. Henk, *Generalized Frobenius numbers: bounds and average behavior*, Acta Arith., **155** (2012), 53-62.

[3] I. Aliev and P. M. Gruber, *An optimal lower bound for the Frobenius problem*, Journal of Number Theory **123** (2007), no. 1, 71–79.

[4] I. Aliev, M. Henk and E. Linke, *Integer points in knapsack polytopes and s-covering radius*, Electron. J. Combin., **20** (2013), no. 2, Paper 42, 17 pp.

[5] A. I. Barvinok and K. Woods, *Short rational generating functions for lattice point problems*, J. Amer. Math. Soc. **16** (2003), 957–979.

[6] M. Beck and S. Robins, *A formula related to the Frobenius problem in two dimensions*, Number theory (New York, 2003), 17-23, Springer, New York, 2004.

[7] R. D. Dutton and R. C. Brigham, *Computationally efficient bounds for the Catalan numbers*, European J. Combin. **7** (1986), 211–213.

[8] L. Fukshansky and S. Robins, *Frobenius problem and the covering radius of a lattice*, Discrete Comput. Geom. **37** (2007), no. 3, 471–483.

[9] L. Fukshansky and A. Schürmann, *Bounds on generalized Frobenius numbers*, European J. Combin., **32** (2011), 361–368.

[10] P. M. Gruber, *Convex and discrete geometry*, Springer, Berlin, 2007.

[11] P. M. Gruber, *Zur Gitterüberdeckung des $\mathbb{R}^n$ durch Sternkörper*, Österreich. Akad. Wiss. Math.-Natur. Kl. S.-B. II, **176** (1967), 1–7.

[12] P. M. Gruber and C.G. Lekkerkerker, *Geometry of numbers*, 2nd ed., North-Holland, Amsterdam, 1987.

[13] R. Kannan, *Lattice translates of a polytope and the Frobenius problem*, Combinatorica, **12**(2)(1992), 161–177.

[14] J. L. Ramírez Alfonsín, *Complexity of the Frobenius problem*, Combinatorica, **16** (1996), 143–147.

[15] J. L. Ramírez Alfonsín, *The Diophantine Frobenius problem*, Oxford Lecture Series in Mathematics and its Applications **30** (2005), xvi+243.

[16] C. A. Rogers and G. C. Shephard, *The difference body of a convex body*, Arch. Math. (Basel) **8** (1957), 220-233.

[17] A. Schinzel, *A property of polynomials with an application to Siegel's lemma*, Monatsh. Math., **137** (2002), 239–251.

[18] J. J. Sylvester, Problem 7382, *Educational Times*, **37** (1884), 26.

School of Mathematics and Wales Institute of Mathematical and Computational Sciences, Cardiff University, Senghennydd Road, Cardiff, Wales, UK

E-mail address: alievi@cardiff.ac.uk

E-mail address: mohammeddh@cardiff.ac.uk