Negative Real Interest Rates[1]

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Standard textbook general equilibrium term structure models such as that developed by Cox, Ingersoll and Ross (1985b), do not accommodate negative real interest rates. Given this, the Cox, Ingersoll and Ross (1985b) "technological uncertainty variable" is formulated in terms of the Pearson Type IV probability density. The Pearson Type IV encompasses mean reverting sample paths, time varying volatility and also allows for negative real interest rates. The Fokker-Planck (that is, the Chapman-Kolmogorov) equation is then used to determine the conditional moments of the instantaneous real rate of interest. These enable one to determine the mean and variance of the accumulated (that is, integrated) real rate of interest on a bank (or loan) account when interest accumulates at the instantaneous real rate of interest defined by the Pearson Type IV probability density. A pricing formula for pure discount bonds is also developed. Our empirical analysis of short dated Treasury bills shows that real interest rates in the U.K. and the U.S. are strongly compatible with a general equilibrium term structure model based on the Pearson Type IV probability density.

Key Words: Fokker-Planck equation; Mean reversion; Real interest rate; Pearson Type IV probability density.

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1. Introduction

The Cox, Ingersoll and Ross (1985b) model of the term structure of interest rates has been described as "... the premier textbook example of a continuous-time general equilibrium asset pricing model ..." and as "... one of the key breakthroughs of [its] decade ...." (Duffie, 2001, xiv). Here it will be recalled that Cox, Ingersoll and Ross (1985b) formulate a quasi-supply side model of the economy based on the weak aggregation criteria of Rubinstein (1974) and where the optimising behaviour of a representative economic agent centres on a "technological uncertainty" variable that evolves in terms of a continuous time branching process.[2] Bernoulli preferences are then invoked to determine the instantaneous prices of the Arrow securities for the economy and these in turn are used to form a portfolio of securities with an instantaneously certain real consumption pay-off. Adding the prices of the Arrow securities comprising this portfolio then allows one to determine the instantaneous real risk free rate of interest for the economy. This shows that the real risk free rate of interest develops in terms of the well known Cox, Ingersoll and Ross (1985b, 391) "square root" (or branching) process and that because of this, the real risk free rate of interest can never be negative. Whilst early empirical assessments of the Cox, Ingersoll Ross (1985b) term structure model were largely supportive, they were conducted before the onset of the Global Financial Crisis when the incidence of negative real interest rates was rare (Gibbons and Ramaswamy, 1993; Brown and Schaefer, 1994). This contrasts with the period following the Global Financial Crisis which has been characterised by a much greater incidence of negative real interest rates. The World Bank (2014), for example, reports that real interest rates were continuously negative in the United Kingdom over the period from 2009 until 2013. Other countries that have experienced negative real interest rates over all or part of this period

[2] Otherwise known as a Feller (1951a, 1951b) Diffusion.

include Algeria, Argentina, Bahrain, Belarus, China, Kuwait, Libya, Oman, Pakistan, Qatar, Russia and Venezuela to name but a few. Hence, given the increasing incidence of negative real interest rates since the onset of the Global Financial Crisis and the difficulties the Cox, Ingersoll and Ross (1985b) term structure model has in accommodating them, our purpose here is to propose a general stochastic process for the real rate of interest based on the Pearson Type IV probability density (Kendall and Stuart, 1977, 163-165). The Pearson Type IV is the limiting form of a skewed Student "*t*" probability density with mean reverting sample paths and time varying volatility and encompasses both the well known Uhlenbeck and Ornstein (1930) process and the scaled "*t*" process of Praetz (1972, 1978) and Blattberg and Gonedes (1974) as particular cases. More important, however, is the fact that the Pearson Type IV density can accommodate negative real interest rates.

We begin our analysis in section 2 by following Cox, Ingersoll and Ross (1985b, 390-391) in considering an economy in which variations in real output hinge on a state variable which summarises the level of "technological uncertainty" in the economy. The state variable is then used to develop a set of Arrow securities that lead to a real interest rate process whose steady state (that is, unconditional) statistical properties are compatible with the Pearson Type IV probability density function. Section 3 then invokes the Fokker-Planck (that is, the Chapman-Kolmogorov) equation in conjunction with the stochastic differential equation implied by the Pearson Type IV probability density to determine the conditional moments of the instantaneous real risk free rate of interest. In section 4 we employ the steady state interpretation of the Fokker-Planck equation in conjunction with real yields to maturity on short dated U.K. and U.S. Treasury bills to show that the Pearson Type IV probability density is strongly compatible with the way real interest rates evolve in practice. We then move on in section 5 to determine the mean and variance of the accumulated (that is, integrated) real rate of interest on a bank (or loan) account when interest accumulates at the instantaneous real rates of interest characterised by the Pearson Type IV probability density. In section 6 we determine the price of a pure discount bond when the real rate of interest evolves in terms of the stochastic differential equation which defines the Pearson Type IV probability density. Section 7 concludes the paper and identifies areas in which our analysis might be further developed.

2. The Stochastic Process

We begin our analysis by following Cox, Ingersoll and Ross (1985b, 390) in considering an economy in which variations in real output hinge on a state variable, *Y(t)*, which summarises the level of "technological uncertainty" in the economy.[3] The development of the technological uncertainty variable is described by the stochastic differential equation:[4]

$$dY(t) = (a + bY(t))dt + \sqrt{m_1^2 + m_2^2(-\frac{a+w}{b} - Y(t))^2}\,.dz(t) \qquad (1)$$

where $a > 0$, m_1, m_2 and $b < 0$ are parameters, *w* captures the skewness in the probability density for *Y(t)* and *dz(t)* is a white noise process with a unit variance parameter (Hoel, Port and Stone 1987, 142). This means that increments in technological uncertainty gravitate towards a long run mean of $-\frac{a}{b}$ with a variance that grows in magnitude the farther *Y(t)*

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[3] A formal mathematical statement of the role played by the technological uncertainty variable in the determination of the real rate of interest is to be found in Cox, Ingersoll and Ross (1985a, 364-368; 1985b, 390-391). Beyond this formal statement, however, Cox, Ingersoll and Ross (1985a, 1985b) have relatively little to say about the empirical meaning of the technological uncertainty variable. The context in which the technological uncertainty variable is introduced in the Cox, Ingersoll and Ross (1985a, 1985b) term structure model would suggest that it encapsulates factors such as the economy's natural endowments, the enterprise, ingenuity and industry of its people, the quality and effectiveness of its political institutions, the levels of and the neutrality (or otherwise) of its tax system, the political independence of its monetary authorities and so on.

[4] The specification of the state variable given here encompasses both positive and negative values. It therefore differs from the state variable employed for the technological uncertainty variable in the Cox, Ingersoll and Ross (1985b, 390) term structure model, which is based on a continuous time branching process. There are various interpretations of the branching process (Feller 1951a, 1951b) but all of them constrain the state variable to be non-negative and thus, they all differ from the state variable based on the Pearson Type IV probability density which can assume both positive and negative values.

departs from its skewness adjusted long run mean of $-(\frac{a+w}{b})$ (Cox, Ingersoll and Ross 1985b, 390; Black 1995, 1371-72). Moreover, real output in the economy, *e(t)*, is perfectly correlated with technological uncertainty (Cox, Ingersoll and Ross, 1985b, 390-391) in the sense that proportionate variations in real output evolve in terms of the stochastic differential equation:

$$\frac{de(t)}{e(t)} = hY(t)dt + \delta dz(t) \qquad (2)$$

where *h* is a constant of proportionality and δ is an intensity parameter defined on the white noise process *dz(t)*.[5] Standard optimising behaviour by a representative economic agent will then mean that the real risk free rate of interest, *r(t)*, over the instantaneous period from time *t* until time $(t+dt)$ can be determined from the identity (Rubinstein 1974, 232-233; Cox, Ingersoll and Ross 1985a, 367; Duffie 1988, 291-292):

$$e^{-r(t)dt} = E_t[\frac{v'(e(t+dt))}{v'(e(t))}] \qquad (3)$$

where *v(.)* represents the utility function over real consumption for the representative economic agent and *E(.)* is the expectation operator. Simple Taylor series expansions applied

[5] In the Cox, Ingersoll and Ross (1985b, 387) term structure model, changes in the magnitude of the technological uncertainty variable have exactly the same impact on the instantaneous mean and the instantaneous variance of the growth rate in the economy's real output (Rhys and Tippett 2001, 384-387). Thus, if the technological uncertainty variable declines in magnitude then the instantaneous mean growth rate and the instantaneous variance of the growth rate in the economy's real output will both decline by the same magnitude as the technological uncertainty variable (Cox, Ingersoll and Ross 1985b, 390). This contrasts with our modelling procedures where the initial impact of variations in the technological uncertainty variable is on the instantaneous mean proportionate growth rate in real output alone. Here the reader will be able to show by direct application of Itô's formula, that real output in the economy at time t will amount to:

$$e(t) = e(0).\exp\{h\int_0^t Y(s)ds - \tfrac{1}{2}\delta^2 t + \delta.z(t)$$

where *z(t)* possesses a normal density function with a mean of zero and a variance of t. In subsequent sections we demonstrate how this result implies that changes in the magnitude of the technological uncertainty variable will also have secondary effects on the *conditional* instantaneous variance of the future instantaneous growth rate in real output.

to both sides of the above identity will then show that the real risk free rate of interest has the alternative representation:

$$r(t) = -\{\frac{E_t[de(t)]}{dt}.\frac{v''(e(t))}{v'(e(t)} + \tfrac{1}{2}.\frac{E_t[(de(t))^2]}{dt}.\frac{v'''(e(t))}{v'(e(t))}\} + O(\sqrt{dt}) \qquad (4)$$

Moreover, one can follow Cox, Ingersoll and Ross (1985b, 390) in assuming that the representative economic agent possesses Bernoulli utility, in which case we have $v(e(t)) = \log(e(t))$. One can then substitute the relevant derivatives of the utility function into the above expression and then let $dt \to 0$ in which case it follows:

$$r(t) = hY(t) - \delta^2 \qquad (5)$$

will be the instantaneous real risk free rate of interest at time t in terms of the parameters which characterise the mean and variance of the instantaneous increment in aggregate output. It also follows from this that instantaneous changes in the real rate of interest will be governed by the differential equation $dr(t) = hdY(t)$, or upon substituting equation (1) for the technological uncertainty variable:

$$dr(t) = \beta\{\mu - r(t)\}dt + \sqrt{k_1^2 + k_2^2(\mu + \theta - r(t))^2}.dz(t) \qquad (6)$$

where $\beta = -b$, $\mu = -(\frac{ha}{b} + \delta^2)$, $k_1^2 = h^2 m_1^2$, $k_2^2 = m_2^2$ and $\theta = -\frac{hw}{b}$. This result shows that the expected instantaneous increment in the real rate of interest is given by:

$$E[dr(t)] = \beta\{\mu - r(t)\}dt \qquad (7)$$

This in turn will mean that the real rate of interest gravitates towards a long run mean of μ with an expected restoring force which is proportional to the difference between μ and the current instantaneous real rate of interest, *r(t)*. The constant of proportionality or "speed of adjustment coefficient" is defined by the parameter $\beta > 0$. Moreover, the variance of instantaneous increments in the real rate of interest is given by:

$$Var[dr(t)] = \{k_1^2 + k_2^2(\mu + \theta - r(t))^2\}dt \qquad (8)$$

This shows that the volatility of instantaneous changes in the real rate of interest grows in magnitude the farther the real rate of interest departs from its "skewness adjusted" long run mean of $(\mu + \theta)$ (Cox, Ingersoll and Ross 1985b, 390; Black 1995, 1371-72). Note also that setting $k_2^2 = 0$ leads to the Uhlenbeck and Ornstein (1930) process which is one of the most widely cited and applied stochastic processes in the financial economics literature (Gibson and Schwartz 1990, Barndorff-Nielsen and Shephard 2001, Hong and Satchell 2012). Moreover, setting $\theta = 0$ leads to the scaled "*t*" density function of Praetz (1972, 1978) and Blattberg and Gonedes (1974) which provides an early example of what has become another commonly applied stochastic process in the financial economics literature (Bollerslev 1987, Fernandez and Steel 1998, Aas and Ha 2006).

3. The Conditional Moments

Now, one can define the conditional expected centred instantaneous real rate of interest at time t as follows:

$$M(t) \equiv E(\mu - r) = \int_{-\infty}^{\infty} (\mu - r) g(r,t) dr \qquad (9)$$

where *g(r,t)* is the conditional probability density for the instantaneous real rate of interest. Moreover, one can differentiate through the above expression in which case it follows (Cox and Miller 1965, 217):

$$M'(t) = \int_{-\infty}^{\infty} (\mu - r) \frac{\partial g(r,t)}{\partial t} . dr \qquad (10)$$

Here, however, the Fokker-Planck (that is, the Chapman-Kolmogorov) equation shows that the conditional probability density bears the following relationship to the mean and variance of instantaneous changes in the real rate of interest (Cox and Miller 1965, 213-215):

$$\frac{\partial g(r,t)}{\partial t} = \tfrac{1}{2} . \frac{\partial^2}{\partial r^2} \{ \frac{Var[dr(t)]}{dt} . g(r,t) \} - \frac{\partial}{\partial r} \{ \frac{E[dr(t)]}{dt} . g(r,t) \} \qquad (11)$$

This in turn will mean that the derivative of the conditional expected centred instantaneous real rate of interest has the following representation:

$$M'(t) = \tfrac{1}{2} \int_{-\infty}^{\infty} (\mu - r) \frac{\partial^2}{\partial r^2} \{ \frac{Var[dr(t)]}{dt} . g(r,t) \} dr - \int_{-\infty}^{\infty} (\mu - r) \frac{\partial}{\partial r} \{ \frac{E[dr(t)]}{dt} . g(r,t) \} dr \qquad (12)$$

One can then use equation (7) to substitute the expected instantaneous increment in the real rate of interest into the second term on the right hand side of the above expression in which case we have:

$$\int_{-\infty}^{\infty} (\mu - r) \frac{\partial}{\partial r} \{ \frac{E[dr(t)]}{dt} . g(r,t) \} dr = \int_{-\infty}^{\infty} (\mu - r) \frac{\partial}{\partial r} \{ \beta (\mu - r) g(r,t) \} dr \qquad (13a)$$

Moreover, under appropriate high order contact conditions one can apply integration by parts to the right hand side of the above expression and thereby show (Ashton and Tippett 2006, 1590-1591):

$$\int_{-\infty}^{\infty} (\mu - r) \frac{\partial}{\partial r} \{ \beta (\mu - r) g(r,t) \} dr = \beta (\mu - r)^2 g(r,t) \Big|_{-\infty}^{\infty} + \beta \int_{-\infty}^{\infty} (\mu - r) g(r,t) dr = \beta M(t) \qquad (13b)$$

One can also use equation (8) in conjunction with a similar application of integration by parts in order to evaluate the first term on the right hand side of equation (12); namely:

$$\tfrac{1}{2}\int_{-\infty}^{\infty}(\mu-r)\frac{\partial^2}{\partial r^2}\{\frac{Var[dr(t)]}{dt}g(r,t)\}dr=\tfrac{1}{2}\int_{-\infty}^{\infty}(\mu-r)\frac{\partial^2}{\partial r^2}\{[k_1^2+k_2^2(\theta+\mu-r)^2]g(r,t)\}dr=0 \tag{14}$$

Bringing these latter two results together shows that the conditional expected centred instantaneous real rate of interest will satisfy the following differential equation:

$$M'(t)=-\beta M(t) \tag{15}$$

Solving the above differential equation under the initial condition $M(0)=(\mu-r(0))$ shows that the conditional expected centred instantaneous real rate of interest at time *t* amounts to (Boyce and DiPrima 2005, 32-33):

$$M(t)=E(\mu-r(t))=(\mu-r(0))e^{-\beta t} \tag{16a}$$

This in turn implies that the conditional expected instantaneous real rate of interest at time t is given by:

$$E[r(t)]=\mu+(r(0)-\mu)e^{-\beta t} \tag{16b}$$

Moreover, one can let $t\to\infty$ in which case it follows that the expected instantaneous real rate of interest in the “steady state” - that is, the unconditional expected instantaneous real rate of interest - will amount to $E[r(\infty)]=\mu$.

Similar procedures show that the conditional second moment of the centred instantaneous real rate of interest may be defined as follows:

$$V(t) \equiv E[(\mu - r)^2] = \int_{-\infty}^{\infty} (\mu - r)^2 g(r,t) dr \tag{17}$$

Differentiating through the above expression and substituting the Fokker-Planck equation will then show:

$$V'(t) = \tfrac{1}{2} \int_{-\infty}^{\infty} (\mu - r)^2 \frac{\partial^2}{\partial r^2} \{[k_1^2 + k_2^2 (\theta + \mu - r)^2] g(r,t)\} dr - \int_{-\infty}^{\infty} (\mu - r)^2 \frac{\partial}{\partial r} \{\beta (\mu - r) g(r,t)\} dr \tag{18}$$

Moreover, under appropriate high order contact conditions one can again apply integration by parts to both terms on the right hand side of the above equation and thereby show that the expression for the conditional second moment of the centred instantaneous real rate of interest will satisfy the following differential equation:

$$V'(t) + (2\beta - k_2^2) V(t) = (k_1^2 + k_2^2 \theta^2) + 2 k_2^2 \theta (\mu - r(0)) e^{-\beta t} \tag{19}$$

Standard methods will then show that the general solution of the above differential equation takes the form (Boyce and DiPrima 2005, 32-33):

$$V(t) = \frac{k_1^2 + k_2^2 \theta^2}{2\beta - k_2^2} + \frac{2 k_2^2 \theta (\mu - r(0)) e^{-\beta t}}{\beta - k_2^2} + c e^{(k_2^2 - 2\beta) t} \tag{20}$$

where *c* is a constant of integration. One can use this result in conjunction with equation (16) to show that the conditional variance of the centred instantaneous real rate of interest will take the general form:[6]

$$\sigma^2(t) = E[(\mu - r)^2] - \{E(\mu - r)\}^2 =$$

[6] The variance of the instantaneous centred real rate of interest can be re-expressed as:

$$\sigma^2(t) = E[(\mu - r)^2] - \{E(\mu - r)\}^2 = E[\{(\mu - r) - E(\mu - r)\}^2 = E[(r - E(r))^2]$$

Thus, the variance of the centred instantaneous real rate of interest is equivalent to the variance of the instantaneous real rate of interest itself.

$$\frac{k_1^2 + k_2^2\theta^2}{2\beta - k_2^2} + \frac{2k_2^2\theta(\mu - r(0))e^{-\beta t}}{\beta - k_2^2} + ce^{(k_2^2 - 2\beta)t} - \{(\mu - r(0))e^{-\beta t}\}^2 \tag{21}$$

Now at $t = 0$ the conditional probability density for the instantaneous real rate of interest, *g(r,0)*, will take the form of a Dirac delta function with a probability density which is completely concentrated at *r(0)* (Sneddon 1961, 51-53; Cox and Miller 1965, 209). This in turn will mean that the variance of the centred instantaneous real rate of interest must satisfy the initial condition $\sigma^2(0) = 0$. Using this initial condition in conjunction with equation (21) enables one to show that:

$$c = (\mu - r(0))^2 - \frac{k_1^2 + k_2^2\theta^2}{2\beta - k_2^2} - \frac{2k_2^2\theta(\mu - r(0))}{\beta - k_2^2}$$

Substituting this latter result into equation (21) will then show that the conditional variance of the centred instantaneous real rate of interest is given by:[7]

$$\sigma^2(t) = \frac{k_1^2 + k_2^2\theta^2}{2\beta - k_2^2}\{1 - e^{-(2\beta - k_2^2)t}\} + \frac{2k_2^2\theta[(\mu - r(0))e^{-\beta t}]}{\beta - k_2^2}\{1 - e^{-(\beta - k_2^2)t}\} + [(\mu - r(0))e^{-\beta t}]^2\{e^{k_2^2 t} - 1\} \tag{22}$$

Note that setting $k_2^2 = 0$ in the above expression leads to the conditional variance associated with the Uhlenbeck and Ornstein (1930, 828) process; namely:

$$\sigma^2(t) = \frac{k_1^2}{2\beta}(1 - e^{-2\beta t}) \tag{23}$$

[7] One can state this result on a per unit time basis and then apply L'Hôpital's Rule on a term by term basis in order to determine the variance of the centred instantaneous real rate of interest; namely:

$$\underset{t \to 0}{Limit}\frac{\sigma^2(t)}{t} = (k_1^2 + k_2^2\theta^2) + 2k_2^2\theta(\mu - r(0)) + k_2^2(\mu - r(0))^2 = k_1^2 + k_2^2(\theta + \mu - r(0))^2$$

This shows that the expression for the conditional variance of the centred instantaneous real rate of interest as given by equation (22) is compatible with the expression for the variance of instantaneous changes in the real rate of interest as given by equation (8).

Moreover, setting $\theta = 0$ leads to the conditional variance associated with the scaled "*t*" density function:

$$\sigma^2(t) = \frac{k_1^2}{2\beta - k_2^2}\{1 - e^{-(2\beta - k_2^2)t}\} + [(\mu - r(0))e^{-\beta t}]^2\{e^{k_2^2 t} - 1\} \tag{24}$$

Finally, one can let $t \to \infty$ in which case the steady state (that is, unconditional) variance of the centred instantaneous real rate of interest will be:

$$\sigma^2(\infty) = \frac{k_1^2 + k_2^2\theta^2}{2\beta - k_2^2} = \frac{\nu_1 + \theta^2}{2\nu_2 - 1} \tag{25}$$

where $\nu_1 = \frac{k_1^2}{k_2^2}$ and $\nu_2 = \frac{\beta}{k_2^2}$, a result previously developed by Ashton and Tippett (2006, 1591). One can also use the Fokker-Planck equation and similar procedures to those employed in this section to determine the third and higher conditional moments of the instantaneous real rate of interest. However, it facilitates the empirical application of our model if we now demonstrate how one determines the unconditional probability density function for the instantaneous real rate of interest.

4. Unconditional Probability Density for the Instantaneous Real Rate of Interest

We begin with the assumption that the instantaneous real rate of interest, *r(t),* possesses a steady-state (that is, unconditional) probability density which is independent of its initial condition, *r(0).* It then follows that one can substitute the requirement (Merton 1975, 389-390; Karlin and Taylor 1981, 220):

$$\underset{t \to \infty}{Limit} \frac{\partial g(r,t)}{\partial t} = 0 \qquad (26)$$

into the Fokker-Planck equation (11) in which case the unconditional probability (that is, steady state) density function for the instantaneous real rate of interest, *g(r)*, will satisfy the ordinary differential equation:

$$\tfrac{1}{2}\frac{d^2}{dr^2}[\{k_1^2 + k_2^2(\theta+\mu-r)^2\}g(r)] = \frac{d}{dr}[\beta\{\mu - r(t)\}g(r)] \qquad (27)$$

Solving this differential equation subject to the normalising condition $\int_{-\infty}^{\infty} g(r)dr = 1$ leads to the Pearson Type IV probability density:

$$g(x) = c[1 + \frac{(\theta+x)^2}{\nu_1}]^{-(1+\nu_2)} \exp[\frac{-2\nu_2\theta}{\sqrt{\nu_1}} \tan^{-1}(\frac{\theta+x}{\sqrt{\nu_1}})] \qquad (28a)$$

where, as previously, $x = (\mu - r)$ is the centred instantaneous real rate of interest and (Jeffreys 1961, 75; Yan 2005, 6):

$$c = \frac{\Gamma(\nu_2+1)}{\sqrt{\pi}\Gamma(\nu_2+\frac{1}{2})} \left| \frac{\Gamma(\nu_2 + 1 + \frac{i\nu_2\theta}{\sqrt{\nu_1}})}{\Gamma(\nu_2+1)} \right|^2 \qquad (28b)$$

is the normalising constant. Moreover, $i = \sqrt{-1}$ is the pure imaginary number, $\Gamma(.)$ is the gamma function, and | . | is the modulus of a complex number.

Now, a cursory inspection of equation (25) shows that $\nu_2 > \frac{1}{2}$ is a necessary condition for the variance of the Pearson Type IV probability density to be a convergent statistic.[8] The violation of this condition will mean that neither the variance nor any of the higher moments will be well defined and in these circumstances techniques like the Generalised Method of Moments (GMM) will constitute an inefficient means of parameter estimation. Moreover, Yan (2005, 6) and Kendall and Stuart (1977, 163) note how the transcendental nature of the normalising constant, *c*, and the slow rate at which its various series representations converge will be a significant "obstacle" in the application of maximum likelihood parameter estimation procedures. Given this, parameter estimation for the Pearson Type IV was conducted using the "χ^2 minimum method" (Avni 1976, Berkson 1980) based on the Cramér-von Mises goodness-of-fit statistic as summarised by Cramér (1946, 426-427).

Our data are comprised of the yields to maturity on U.K. (Datastream code TRUK1MT) and U.S. (Datastream code TRUS1MT) Treasury bills issued over the period from 1 August, 2001 until 1 May, 2015. Our sample is based on the maximum period for which data is available on U.S. Treasury Bill yields at the time of writing. Moreover, given the instantaneous nature of our modelling procedures our focus is with the yield to maturity on Treasury bills with the shortest maturity period of one month. This in turn means our data is comprised of the continuously compounded yields to maturity on one month Treasury bills issued at the beginning of each month over the period from 1 August, 2001 until 1 May, 2015. The real yield to maturity is calculated by subtracting the continuously compounded rate of inflation as measured by the Consumer Price Index (CPI) for the given month and country from the

[8] More generally, a necessary condition for the nth moment of the Pearson Type IV probability density to be a convergent statistic is that $\nu_2 > \frac{n-1}{2}$. (Ashton and Tippett 2006, 1591).

continuously compounded yield to maturity for Treasury bills issued at the beginning of that month and which have one month to maturity.[9]

Table 1 summarises basic distributional properties across the $N = 166$ real yields to maturity on U.K. and U.S. one month Treasury bills over the period from 1 August, 2001 until 1 May, 2015. Note how the average real yield to maturity for U.K. treasury bills is slightly positive at *0.38%* (per annum) with a standard deviation of *4.65%*. In contrast, the average real yield to maturity for U.S. treasury bills is negative at *-0.81%* (per annum) with a standard deviation of *5.10%*. The median real yield to maturity for U.K. Treasury bills is slightly negative at

INSERT TABLE ONE ABOUT HERE

-0.09% (per annum) but much more negative for U.S. Treasury bills at *-1.18%* (per annum). Moreover, the standardised skewness and standardised excess kurtosis measures for U.K. Treasury bills are not significantly different from zero at all conventional levels. In contrast, whilst the standardised skewness measure for U.S. Treasury bills is not significantly different from zero, the standardised excess kurtosis measure for U.S. real yields is significantly different from zero at all conventional levels.

Table 2 summarises the results from implementing the χ^2 minimum method to estimate the parameters of the Pearson Type IV probability density using our sample of real yields on U.K. and U.S. Treasury bills with one month to maturity.[10] Thus, the estimate of the long

[9] As an example, the one month Treasury bill issued by the U.K. Debt Management Office on 1 May, 2002 had an average yield to maturity of 3.908% (per annum). This is equivalent to a continuously compounded yield to maturity of $100\log(1+\frac{3.908}{100}) = 3.834\%$ (per annum). On 30 April, 2002 the U.K. Consumer Price Index (CPI) stood at 95.3. By 31 May, 2002 the CPI had risen to 95.5. This means the continuously compounded rate of inflation over the month of May, 2002 amounted to $1200\log(\frac{95.5}{95.3}) = 2.515\%$ (per annum). Hence, the continuously compounded real yield to maturity on Treasury bills with one month to maturity as issued on 1 May, 2002 amounts to 3.834 - 2.515 = 1.319% (per annum).

INSERT TABLE TWO ABOUT HERE

run expected real yield to maturity for one month U.K. Treasury bills is $\mu = 0.21\%$ (per annum) with the remaining parameter estimates being $\theta = 0.3717$, $\nu_1 = 0.1126$ and $\nu_2 = 73.6103$. This contrasts with a negative estimated long run expected real yield to maturity of $\mu = -0.81\%$ (per annum) for one month U.S. Treasury bills with the remaining parameter estimates being $\theta = 0.1611$, $\nu_1 = 0.0353$ and $\nu_2 = 13.7863$. Figure 1 provides a graphical representation of the estimated distribution function for the real annual yield to maturity on U.K. Treasury bills over the period from 1 August, 2001 until 1 May, 2015. Thus, the first panel of this figure summarises the difference between the actual distribution function and

INSERT FIGURE ONE AND FIGURE TWO ABOUT HERE

the empirically estimated distribution function of the real yield on U.K. Treasury bills with parameter values of $\mu = 0.21\%$ (per annum), $\theta = 0.3717$, $\nu_1 = 0.1126$ and $\nu_2 = 73.6103$. The second panel in Figure 1 is a graph of the Pearson Type IV probability density with the above parameter values. Similarly, the first panel of Figure 2 summarises the difference between the actual distribution function and the empirically estimated distribution function of the real yield on U.S. Treasury bills with parameter values of $\mu = -0.81\%$ (per annum), $\theta = 0.1611$, $\nu_1 = 0.0353$ and $\nu_2 = 13.7863$. The second panel in Figure 2 is a graph of the Pearson Type IV probability density with the above parameter values.

Here it is important to note how Anderson and Darling (1952, 203) show that if the real yields on which our empirical analysis is based are drawn from the hypothesised Pearson

[10] See Guo et al. (2015) for a more detailed exposition of how the χ^2 minimum method is implemented empirically.

Type IV probability density there is only a 5% chance of the Cramér-von Mises T_3 statistic exceeding 0.4614. Given this, the minimised test statistics summarised in Table 2 of $T_3 = 0.0312$ for U.K. real yields and $T_3 = 0.0300$ for U.S. real yields would appear to confirm that the Pearson Type IV probability density with the given parameter values provides a very good description of the way real yields on one month U.K. and U.S. Treasury bills evolve over time. It needs to be emphasised, however, that Anderson and Darling (1952) determine the distributional properties of the T_3 statistic on the assumption that none of the parameters of the hypothesised Pearson Type IV probability density have had to be estimated. When, as in the present instance, the parameters of the Pearson Type IV have had to be estimated, Anderson (2010, 6) notes that the significance scores for the T_3 statistic will be both distribution specific and "much smaller than those … for the case where [the] parameters are known." Hence, one cannot use the Cramér-von Mises T_3 statistics as we have calculated them to make a formal assessment about the adequacy or otherwise of the fitted Type IV probability densities. Fortunately, Cramér (1946, 506) has shown that when the parameters of the hypothesised Type IV probability density are estimated by minimising the T_3 statistic, one can still assess the adequacy or otherwise of the fitted probability density by applying the χ^2 goodness of fit test but with the loss of one degree of freedom for each parameter that has had to be estimated.

We thus ordered the $N = 166$ real one month Treasury bill yields comprising our sample from the lowest real yield up to the highest real yield and then divided the ordered real yields into eleven groups of approximately equal size. The χ^2 goodness of fit test was then applied using the estimates summarised in Table 2 for the parameters μ, θ, ν_1, and ν_2 obtained by minimising the Cramér-von Mises goodness-of-fit statistic, T_3. The final column of Table 2 summarises the calculated χ^2 goodness of fit test statistics which are both distributed with $11 - 4 = 7$ degrees of freedom. Both the $\chi^2 = 4.6886$ score for the U.K. and the $\chi^2 = 3.0376$

score for the U.S. are not significant at conventional levels thereby indicating that real yields to maturity for one month U.K. and U.S. Treasury bills are strongly compatible with the Pearson Type IV probability density with the parameter values summarised in Table 2.

5. Accumulated Interest

The focus of our analysis to date is with determining the properties of the instantaneous real rate of interest. We now develop the properties of the accumulated (that is, integrated) real rate of interest by using the conditional moments developed in section 3 to determine the conditional mean and variance of the accumulated real rate of interest on a bank (or loan) account when interest accumulates at the instantaneous real rates of interest defined by the Pearson Type IV probability density. We begin by integrating through the expression for the conditional expected instantaneous real rate of interest as given by equation (16b) in which case it follows that the expected conditional accumulated real rate of interest over the period from time zero until time t will be:

$$E[\int_0^t r(s)ds] = \mu t + \frac{(r(0)-\mu)(1-e^{-\beta t})}{\beta} \qquad (29)$$

Note how this result shows that the speed of adjustment parameter, β, plays a crucial role in determining the magnitude of the expected accumulated real rate of interest to be earned on the bank (or loan) account. This is particularly so if the initial condition (that is, the opening instantaneous real rate of interest), *r(0)*, differs significantly from its long run mean, μ, and the speed of adjustment coefficient, β, is relatively small.

Of course expectations are seldom realised and so, the conventional practice is to summarise the uncertainty associated with the interest paid on the bank (or loan) account in terms of the variance of the accumulated real rate of interest. Hence, in the Appendix we demonstrate how one can use the Law of Iterated (or Double) Expectations to show that the covariance function associated with the instantaneous real rates of interest, *r(s)* and *r(t)*, that evolve in terms of the stochastic differential equation (6) will be of the form (Freeman, 1963, 54-57):

$$Cov[r(t),r(s)] = \sigma^2(s)e^{-\beta(t-s)} \tag{30}$$

for $t \geq s \geq 0$ and where $\sigma^2(s)$ is defined by equation (22). Here, one can integrate through equation (30) and thereby show that the conditional variance of the accumulated real rate of interest over the period from time zero until time *t* will be given by (Cox and Miller 1965, 227-228):

$$Var[\int_0^t r(s)ds] = \int_0^t \int_0^t Cov[r(s),r(u)]dsdu = 2\int_0^t \sigma^2(s)\int_s^t e^{-\beta(u-s)}duds = \frac{2}{\beta}\int_0^t \sigma^2(s)\{1-e^{-\beta(t-s)}\}ds \tag{31}$$

Hence, substituting equation (22) into the above expression and evaluating the indicated integral shows that the conditional variance of the accumulated real rate of interest is given by:

$$\begin{aligned} Var[\int_0^t r(s)ds] &= \frac{2k_1^2 + k_2^2\theta^2}{\beta^2(2\beta-k_2^2)}\{\beta t - \frac{3\beta-k_2^2}{2\beta-k_2^2} + \frac{2\beta-k_2^2}{\beta-k_2^2}e^{-\beta t} - \frac{\beta^2}{(2\beta-k_2^2)(\beta-k_2^2)}e^{-(2\beta-k_2^2)t}\} + \\ &\frac{4k_2^2\theta(\mu-r(0))}{\beta^2(\beta-k_2^2)}\{\frac{\beta-k_2^2}{2\beta-k_2^2} + (\frac{k_2^2}{\beta-k_2^2} - \beta t)e^{-\beta t} - \frac{\beta^2}{(2\beta-k_2^2)(\beta-k_2^2)}e^{-(2\beta-k_2^2)t}\} + \frac{(\mu-r(0))^2}{\beta^2}\{\frac{k_2^2}{2\beta-k_2^2} - \frac{2k_2^2}{\beta-k_2^2}e^{-\beta t} - e^{-2\beta t} + \frac{2\beta^2}{(2\beta-k_2^2)(\beta-k_2^2)}e^{-(2\beta-k_2^2)t}\} \end{aligned} \tag{32}$$

Now, setting $k_2^2 = 0$ in the above expression allows one to determine the conditional variance of the accumulated real rate of interest on the bank (or loan) account when the instantaneous real rate of interest evolves in terms of an Uhlenbeck and Ornstein (1930, 832) process:

$$Var[\int_0^t r(s)ds] = \frac{k_1^2}{\beta^3}(\beta t - \tfrac{3}{2} + 2e^{-\beta t} - \tfrac{1}{2}e^{-2\beta t}) \tag{33}$$

Similarly, setting θ = 0 in equation (32) shows that when the instantaneous real rate of interest evolves in terms of the scaled "*t*" probability density then the conditional variance of the accumulated real rate of interest will be:

$$Var[\int_0^t r(s)ds] = \frac{2k_1^2}{\beta^2(2\beta - k_2^2)}\{\beta t - \frac{3\beta - k_2^2}{2\beta - k_2^2} + \frac{2\beta - k_2^2}{\beta - k_2^2}e^{-\beta t} - \frac{\beta^2}{(2\beta - k_2^2)(\beta - k_2^2)}e^{-(2\beta - k_2^2)t}\} +$$

$$\frac{(\mu - r(0))^2}{\beta^2}\{\frac{k_2^2}{2\beta - k_2^2} - \frac{2k_2^2}{\beta - k_2^2}e^{-\beta t} - e^{-2\beta t} + \frac{2\beta^2}{(2\beta - k_2^2)(\beta - k_2^2)}e^{-(2\beta - k_2^2)t}\} \tag{34}$$

Finally, note how for the general case involving $k_2^2, \theta \neq 0$, the conditional variance of the accumulated real rate of interest is comprised of three elements - the first of which hinges purely on the investment horizon, *t*, whilst the second and third depend on a combination of the investment horizon and the difference between the long run mean instantaneous real rate of interest and the current instantaneous real rate of interest, or $(\mu - r(0))$.

6. Pure Discount Bonds

We now determine the price of a pure discount bond, *B(r,t)*, that pays one unit of real output at some future point in time, $T > t$ (Cox, Ingersoll and Ross 1985b, 392). We begin by noting that equation (6) taken in conjunction with Itô's formula shows that the instantaneous real return on the bond will evolve in accordance with the following stochastic differential equation:

$$\frac{dB(r,t)}{B(r,t)} =$$

$$[\frac{\beta(\mu - r)}{B}\frac{\partial B}{\partial r} + \frac{1}{B}\frac{\partial B}{\partial t} + \frac{\{k_1^2 + k_2^2(\mu + \theta - r)^2\}}{2B}\frac{\partial^2 B}{\partial r^2}]dt + \frac{\sqrt{k_1^2 + k_2^2(\mu + \theta - r)^2}}{B}\frac{\partial B}{\partial r}dz(t) \quad (35)$$

Now, suppose one follows Cox, Ingersoll and Ross (1985a, 366-367) in forming a self financing portfolio comprised of an investment of W_1 units of real output in the pure discount bond, W_2 in a security which costs one unit of real output at time t and returns $\frac{e(t+dt)}{e(t)} = \{1 + \frac{de(t)}{e(t)}\}$ units of real output at time $(t + dt)$ and $W_3 = -(W_1 + W_2)$ in instantaneous borrowing or lending at the real risk free rate of interest. It then follows that the instantaneous increment in the real value of this portfolio will be:

$$dW_1 + dW_2 + dW_3 = W_1\frac{dW_1}{W_1} + W_2\frac{dW_2}{W_2} + W_3\frac{dW_3}{W_3} = W_1\frac{dB(r,t)}{B(r,t)} + W_2\frac{de(t)}{e(t)} - (W_1 + W_2)r(t)dt \quad (36)$$

Substituting equation (35) and equation (2) into the above expression will then show that the instantaneous increment in the real value of the portfolio will have a deterministic component which amounts to:

$$\{[\frac{\beta(\mu-r)}{B}\frac{\partial B}{\partial r}+\frac{1}{B}\frac{\partial B}{\partial t}+\frac{\{k_1^2+k_2^2(\mu+\theta-r)^2\}}{2B}\frac{\partial^2 B}{\partial r^2}]W_1+hYW_2-(W_1+W_2)r\}dt$$

(37a)

as well as a stochastic component given by:

$$\{\frac{\sqrt{k_1^2+k_2^2(\mu+\theta-r)^2}}{B}\frac{\partial B}{\partial r}W_1+\delta W_2\}dz(t)$$

(37b)

Hence, if one fixes the proportionate investments in the pure discount bond and real output security so that:

$$W_2=-\frac{\sqrt{k_1^2+k_2^2(\mu+\theta-r)^2}}{\delta B}\frac{\partial B}{\partial r}W_1 \qquad (38)$$

then the stochastic component of the instantaneous increment in the real value of the portfolio will be zero. Ruling out potential arbitrage profits will then require that the deterministic component of the portfolio must also be zero, or:

$$[\frac{\beta(\mu-r)}{B}\frac{\partial B}{\partial r}+\frac{1}{B}\frac{\partial B}{\partial t}+\frac{\{k_1^2+k_2^2(\mu+\theta-r)^2\}}{2B}\frac{\partial^2 B}{\partial r^2}]W_1+hYW_2-(W_1+W_2)r=0$$

Substituting equation (5) and equation (38) into the above expression will then show that the price of the pure discount bond will have to satisfy the following partial differential equation:

$$\tfrac{1}{2}\{k_1^2+k_2^2(\mu+\theta-r)^2\}\frac{\partial^2 B}{\partial r^2}+\beta(\mu-r)\frac{\partial B}{\partial r}+\frac{\partial B}{\partial t}-\delta\sqrt{k_1^2+k_2^2(\mu+\theta-r)^2}\frac{\partial B}{\partial r}-rB=0$$

(39)

with the boundary condition being $B(r,T)=1$ and where time T is the date of the bond's maturity.[11] An important canonical interpretation of the above partial differential equation sets $k_2^2=0$ in which case the instantaneous real rate of interest evolves in terms of an Uhlenbeck and Ornstein (1930) process. One may then use the method of separation of variables (Sneddon, 1961, 48) to show that the unique solution of the above boundary value problem and hence, the price of the pure discount bond will be:

$$B(r,t)=H(t)\exp\{\frac{-r}{\beta}[1-e^{-\beta(T-t)}]\} \tag{40}$$

where:

$$H(t)=\exp\{\frac{k_1^2}{2\beta^3}[\beta(T-t)-\tfrac{3}{2}+2e^{-\beta(t-t)}-\tfrac{1}{2}e^{-2\beta(T-t)}]+\frac{\delta k_1-\beta\mu}{\beta^2}[\beta(T-t)-1+e^{-\beta(t-t)}]\}$$

Note that the first term in the argument for *H(t)* is proportional to the variance of the accumulated real rate of interest as summarised by equation (33) and therefore, will be non-negative. The second term is proportional to the variance of the accumulated real rate of interest when r is in statistical equilibrium[12] and will be positive or negative according to whether δk_1 exceeds or is less than $\beta\mu$. It thus follows that whilst *H(t)* will be strictly

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[11] Taking expectations through equation (35) and substituting the result into equation (39) shows that the instantaneous expected return on the bond is given by:

$$E(\frac{1}{B}\frac{dB}{dt})=r+\delta\sqrt{k_1^2+k_2^2(\mu+\theta-r)^2}\,\frac{\partial B}{\partial r}$$

where the second term on the right hand side of the above expression is the covariance between the instantaneous real rate of interest and instantaneous proportionate changes in aggregate output. Since $\frac{\partial B}{\partial r}<0$ it follows that this covariance term will have a negative (positive) impact on the bond's expected return when the real rate of interest increases (decreases) in magnitude - although the impact will be asymmetric according to whether the real rate of interest is positive or negative.

[12] That is, when $(T-t)\to\infty$ (Cox and Miller 1965, 228). It then follows from equation (23) that the variance of the instantaneous real rate of interest in statistical equilibrium is given by $\sigma^2(\infty)=\frac{k_1^2}{2\beta}$.

positive, it will assume values which exceed or fall below unity according to whether the second term is negative and exceeds the first term in absolute magnitude. Moreover, differentiating through equation (40) will also show that the bond's price is a decreasing convex function of the real rate of interest, r, and the long run mean real rate of interest, μ. In contrast, the bond's price is an increasing concave function of the variance parameters k_1^2 and δ^2. The intuition behind this latter result is that larger values of k_1^2 and δ^2 signify more uncertainty about future productive opportunities in the economy in which case a risk-averse investor will value the certain income claims afforded by bonds more highly (Cox, Ingersoll and Ross 1985b, 394). Finally, variations in the speed of adjustment coefficient, β, can have either a positive or negative impact on the bond's price depending on the time remaining until the bond's maturity and the relative magnitude of the instantaneous real rate of interest, r.

The principal advantage of simple canonical solutions like the one developed here is that they enable clear assessments to be made within the model about the impact which variations in technological uncertainty will have on real interest rates, expected changes in real output and the uncertainty associated with future productive opportunities. There are, however, significant limitations associated with simple canonical models of the interest rate process based on the Uhlenbeck and Ornstein (1930) process - whether they be developed within a general equilibrium framework (Hull and White 1990) or as is more commonly the case, through the application of standard no-arbitrage pricing conditions (Vasicek 1977). The first of these arises from the requirement $k_2^2 = 0$ or that the variance of instantaneous increments in the real rate of interest is independent of the current level of the instantaneous real rate of interest. We have previously noted how this requirement violates the commonly held belief that the uncertainty associated with changes in most economic time series becomes larger as the affected variable itself, grows in magnitude (Cox, Ingersoll and Ross 1985b, Black 1995).

Moreover, the requirement that $k_2^2 = 0$ will also mean that the conditional probability density of the bond price at any given time must be lognormal - something which is of doubtful empirical validity (Hull and White 1990, 579).

More general solutions of the partial differential equation (39) will, unfortunately, have to be determined numerically.[13] A particularly important case, given the empirical evidence summarised in section 4, relates to the scaled "*t*" interpretation of the Pearson Type IV probability density which is defined by equation (28) with $\theta = 0$. We thus consider the following trial solution for the differential equation (39):

$$B(r,t) = H(t)\exp\{G(t)\sinh^{-1}[\frac{k_2}{k_1}(\mu - r)]\} \qquad (41)$$

where $H(t)$ and $G(t)$ are continuously differentiable functions of time. Substitution will then show that the above expression satisfies equation (39) when:

$$-(\tfrac{1}{2}k_2^2 + \beta)G(t)\tanh\{\sinh^{-1}[\frac{k_2}{k_1}(\mu - r)]\} + G'(t)\sinh^{-1}[\frac{k_2}{k_1}(\mu - r)] + (\mu - r) +$$

$$\tfrac{1}{2}k_2^2[G(t)]^2 + \delta k_2 G(t) + \frac{H'(t)}{H(t)} - \mu = 0 \qquad (42)$$

Here one can apply a first order Taylor series approximation to the hyperbolic functions appearing in the above expression and thereby show:

$$(\mu - r)\{\frac{k_2}{k_1}G'(t) - [\frac{k_2^3 + 2\beta k_2}{2k_1}]G(t) + 1\} + \tfrac{1}{2}k_2^2[G(t)]^2 + \delta k_2 G(t) + \frac{H'(t)}{H(t)} - \mu + O[(\mu - r)^3] = 0 \qquad (43)$$

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[13] See Carnahan, Luther and Wilkes (1969) and Crank (1975) for a more detailed exposition of the numerical solution procedures which may be applied in such circumstances.

Moreover, if one lets:

$$G(t)=\frac{2k_1}{(k_2^3+2\beta k_2)}\{1-\exp[-\tfrac{1}{2}(k_2^2+2\beta)(T-t)]\} \tag{44}$$

and ignores terms of $O[(\mu-r)^3]$, it then follows:

$$\frac{H'(t)}{H(t)}=\mu-\tfrac{1}{2}k_2^2[G(t)]^2-\delta k_2 G(t)$$

or, equivalently:

$$\frac{H'(t)}{H(t)}=\mu-\frac{2k_1^2}{(k_2^2+2\beta)^2}\{1-\exp[-\tfrac{1}{2}(k_2^2+2\beta)(T-t)]\}^2-\frac{2\delta k_1}{(k_2^2+2\beta)}\{1-\exp[-\tfrac{1}{2}(k_2^2+2\beta)(T-t)]\} \tag{45}$$

Solving the above differential equation under the boundary condition *H(T) = 1* will then show:

$$H(t)=\exp\langle\{\frac{2k_1^2}{(k_2^2+2\beta)^2}+\frac{2\delta k_1}{(k_2^2+2\beta)}-\mu\}(T-t)-\{\frac{8k_1^2}{(k_2^2+2\beta)^3}+\frac{4\delta k_1}{(k_2^2+2\beta)^2}\}\{1-\exp[-\tfrac{1}{2}(k_2^2+2\beta)(T-t)]\}+\frac{2k_1^2}{(k_2^2+2\beta)^3}\{1-\exp[-(k_2^2+2\beta)(T-t)]\}\rangle \tag{46}$$

One can also take logarithms across equation (41) and thereby show that the yield to maturity when the bond has *(T - t)* years remaining to maturity will be:

$$-\frac{\log\{B(r,t)\}}{(T-t)} = -\frac{\log\{H(t)\}}{(T-t)} - \frac{2\{1-\exp[-\tfrac{1}{2}(k_2^2+2\beta)(T-t)]\}(\mu-r)}{(k_2^2+2\beta)(T-t)} + O[(\mu-r)^3] \tag{47}$$

Now, if one ignores terms of $O[(\mu-r)^3]$ and takes expectations across the above expression at time zero, then by substituting equation (16) it follows that the conditional expected yield to maturity when the bond has *(T - t)* years remaining until maturity will be:

$$-E[\frac{\log\{B(r,t)\}}{(T-t)} = -\frac{\log\{H(t)\}}{(T-t)} - \frac{2\{1-\exp[-\tfrac{1}{2}(k_2^2+2\beta)(T-t)]\}\{\mu-r(0)\}e^{-\beta t}}{(k_2^2+2\beta)(T-t)} \tag{48}$$

Similar calculations taken in conjunction with equation (24) will also show that the conditional variance of the yield to maturity is be given by:

$$Var[-\frac{\log\{B(r,t\}}{(T-t)}] = \frac{4\{1-\exp[-\tfrac{1}{2}(k_2^2+2\beta)(T-t)]\}^2\{\frac{k_1^2}{(2\beta-k_2^2)}[1-e^{-(2\beta-k_2^2)t}]+[(\mu-r(0))e^{-\beta t}]^2[e^{k_2^2 t}-1]\}}{(k_2^2+2\beta)^2(T-t)^2} \tag{49}$$

Finally, one can let $(T - t) \to \infty$ in equation (48) (the instantaneous real interest rate, r, is in statistical equilibrium) and thereby show that the yield to maturity on the bond has a limiting value of:

$$\underset{(T-t)\to\infty}{Limit} \frac{-\log\{B(r,t)\}}{(T-t)} = \mu - \frac{2k_1^2}{(k_2^2+2\beta)^2} - \frac{2\delta k_1}{(k_2^2+2\beta)} \qquad (50)$$

which is independent of the current instantaneous real rate of interest, r.

7. Summary Conclusions

In the period following the onset of the Global Financial Crisis a significant number of countries have experienced negative real rates of interest. Unfortunately, the Cox, Ingersoll and Ross (1985b) square root process - one of the most commonly applied stochastic processes for modelling term structure phenomena - cannot accommodate negative real rates of interest. Given this, we modify the Cox, Ingersoll and Ross (1985b) term structure model by proposing a general stochastic process for the real rate of interest based on the Pearson Type IV probability density. The Pearson Type IV is the limiting form of skewed Student "*t*" probability density with mean reverting sample paths and time varying volatility that encompasses both the Uhlenbeck and Ornstein (1930) and scaled "*t*" processes as particular cases. More important, however, is the fact that the Pearson Type IV probability density can accommodate negative real interest rates. We also use the Fokker-Planck (that is, the Chapman-Kolmogorov) equation in conjunction with the stochastic differential equation implied by the Pearson Type IV probability density to determine the conditional moments of the instantaneous real rate of interest. The conditional moments are then used to determine the mean and variance of the accumulated real rate of interest on a bank (or loan) account when interest accumulates at the instantaneous real rates of interest defined by the Pearson

Type IV probability density. We conclude the paper by determining the price of a pure discount bond when the real rate of interest evolves in terms of the stochastic differential equation that characterises the Pearson Type IV probability density. Our empirical analysis of short dated Treasury Bills shows that real interest rates in the U.K. and the U.S. are strongly compatible with a general equilibrium term structure model based on the Pearson Type IV probability density.

Appendix Covariance Function for Instantaneous Real Rate of Interest

One can generalise equation (16b) and thereby show that the conditional expected instantaneous real rate of interest can be stated as:

$$E[r(t)\,|\,r(s)] = \mu + (r(s)-\mu)e^{-\beta(t-s)} = r(s)e^{-\beta(t-s)} + \mu(1-e^{-\beta(t-s)}) \tag{A1}$$

for $t > s > 0$. Now, by the Law of Iterated (or Double) Expectations we have (Freeman, 1963, 54-57):

$$E[r(t).r(s)] = E\{r(s)E[r(t)\,|\,r(s)]\}$$

Moreover, one can substitute equation (A1) into the above expression in which case we have:

$$E[r(t).r(s)] = E\{r(s)[\mu + (r(s)-\mu)e^{-\beta(t-s)}]\}$$

or equivalently:

$$E[r(t).r(s)] = E[r^2(s)e^{-\beta(t-s)}] + \mu(1-e^{-\beta(t-s)})E[r(s)]$$

Here one can use the fact that variance of the instantaneous real rate of interest at time s is given by $\sigma^2(s) = E[r^2(s)] - \langle E[r(s)]\rangle^2$ to re-state the above expression as:

$$E[r(t).r(s)] = \{\sigma^2(s) + \langle E[r(s)]\rangle^2\}e^{-\beta(t-s)} + \mu(1-e^{-\beta(t-s)})E[r(s)]$$

Simple algebraic manipulation will then show that the above result may be re-stated as:

$$E[r(t).r(s)] = \sigma^2(s)e^{-\beta(t-s)} + E[r(s)]\{E[r(s)]e^{-\beta(t-s)} + \mu(1-e^{-\beta(t-s)})\}$$

However, taking expectations across (A1) shows that this latter result may be re-stated as:

$$E[r(t).r(s)] = \sigma^2(s)e^{-\beta(t-s)} + E[r(t)]E[r(s)]$$

This may be equivalently stated as:

$$Cov[r(s), r(t)] = E[r(t).r(s)] - E[r(t)]E[r(s)] = \sigma^2(s)e^{-\beta(t-s)}$$

which is the covariance between the instantaneous real rate of interest at time *s*, and the instantaneous real rate of interest at time *t*.

Table 1. Distributional properties of the real yield to maturity on one month U.K. and U.S. Treasury bills covering the period from 1 August, 2001 until 1 May, 2015.

	U.K. *N = 166*	U.S. *N = 166*
Average (per annum)	0.38%	-0.81%
Standard Deviation	4.65%	5.10%
Standardised Skewness	0.39	0.99
Standardised Kurtosis	0.60	3.02
Median	-0.09%	-1.18%
Maximum	14.23%	23.84%
Minimum	-11.83%	-11.50%

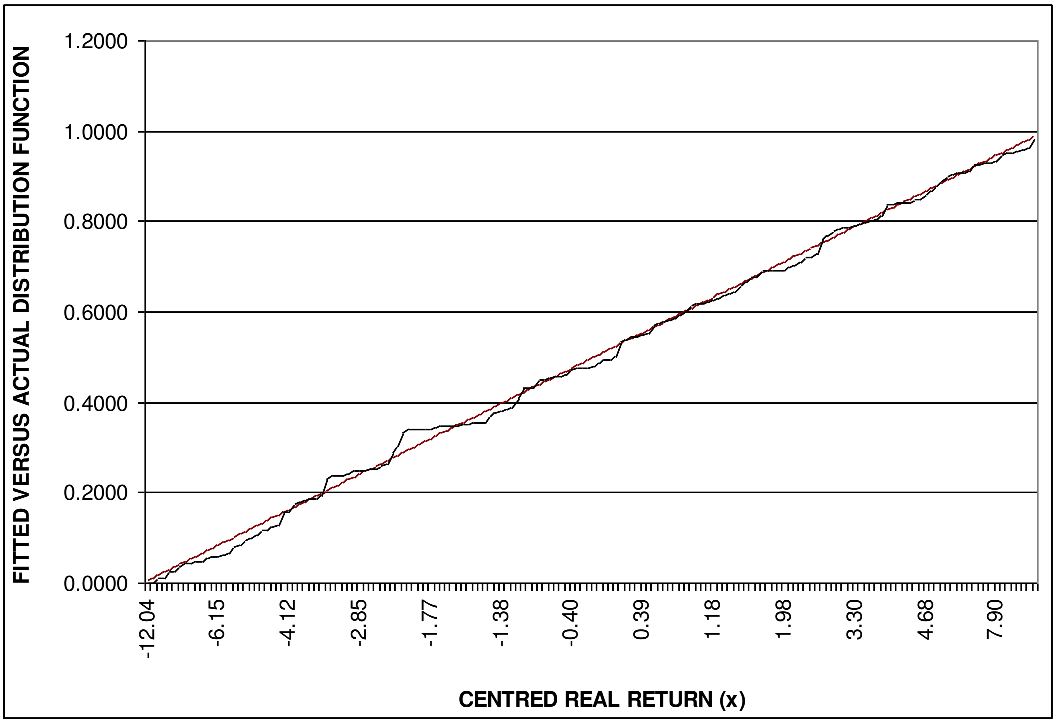
Notes: The above table is based on the *N = 166* real yields to maturity for one month U.K and U.S. Treasury bills issued over the period from 1 August, 2001 until 1 May, 2015. The real yield to maturity is calculated by subtracting the continuously compounded rate of inflation as measured by the Consumer Price Index (CPI) for the given month from the continuously compounded yield to maturity for Treasury bills with one month to maturity and which were issued at the beginning of that month.

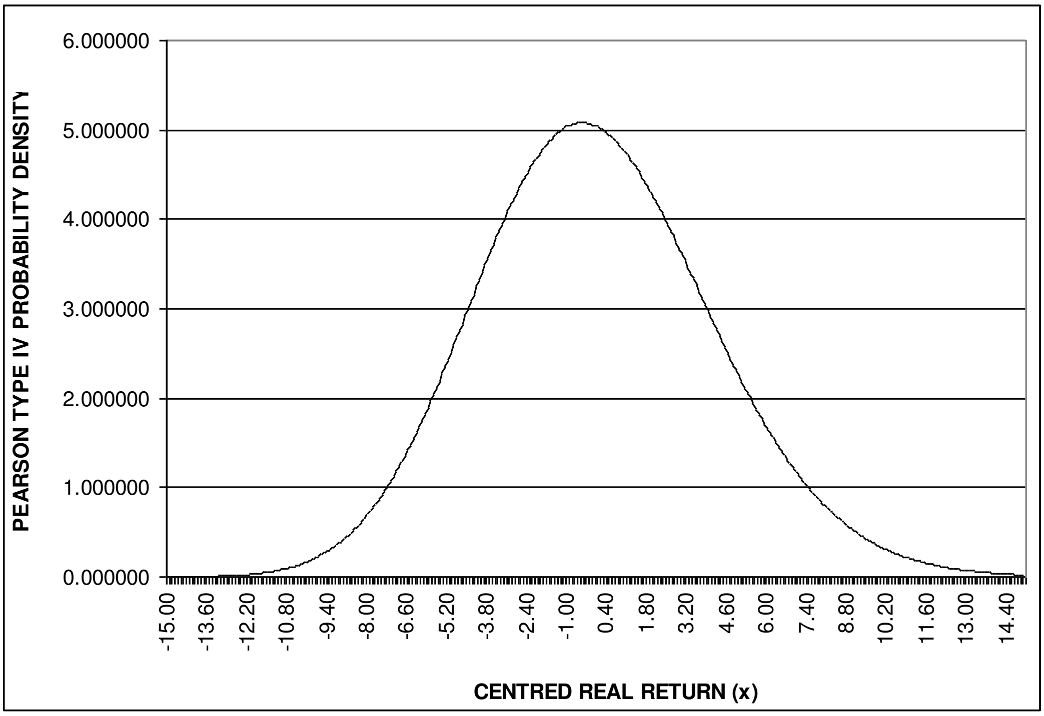
Table 2. Estimated parameters using χ^2 minimum method for the real yield to maturity on one month U.K. and U.S. Treasury bills covering the period from 1 August, 2001 until 1 May, 2015.

	μ	θ	ν_1	ν_2	Cramér-von Mises Statistic T_3	χ^2 Goodness of fit Statistic
U.K.	0.21	0.3717	0.1126	73.6103	0.0312	4.6886
U.S.	-0.81%	0.1611	0.0353	13.7863	0.0300	3.0376

Notes: The χ^2 minimum method for estimating the parameters of the Pearson Type IV probability density was implemented by minimising the Cramér-von Mises goodness-of-fit statistic (Conover 1980, 306) across the $N = 166$ real yields to maturity and then determining the adequacy of the fitting procedure by calculating the Chi-square goodness of fit statistic (Conover 1980, 186). The Chi-square goodness of fit statistic summarised in the above Table possesses 7 degrees of freedom.

Figure 1. (a) Difference between actual distribution function and empirically estimated distribution function for the $N = 166$ real yields to maturity on one month U.K. Treasury bills. (b) Estimated Pearson Type IV probability density for real yields to maturity on one month U.K. Treasury bills.





The above graphs are based on the probability density:

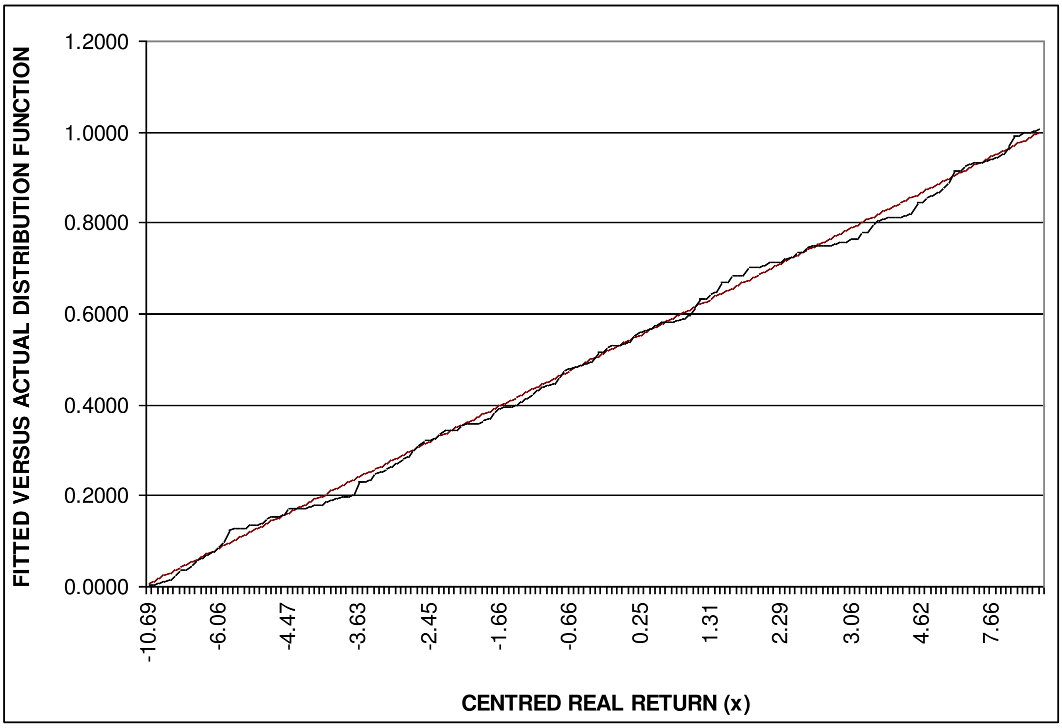
$$g(x) = c[1+\frac{(\theta+x)^2}{\nu_1}]^{-(1+\nu_2)} \exp[\frac{-2\nu_2\theta}{\sqrt{\nu_1}} \tan^{-1}(\frac{\theta+x}{\sqrt{\nu_1}})]$$

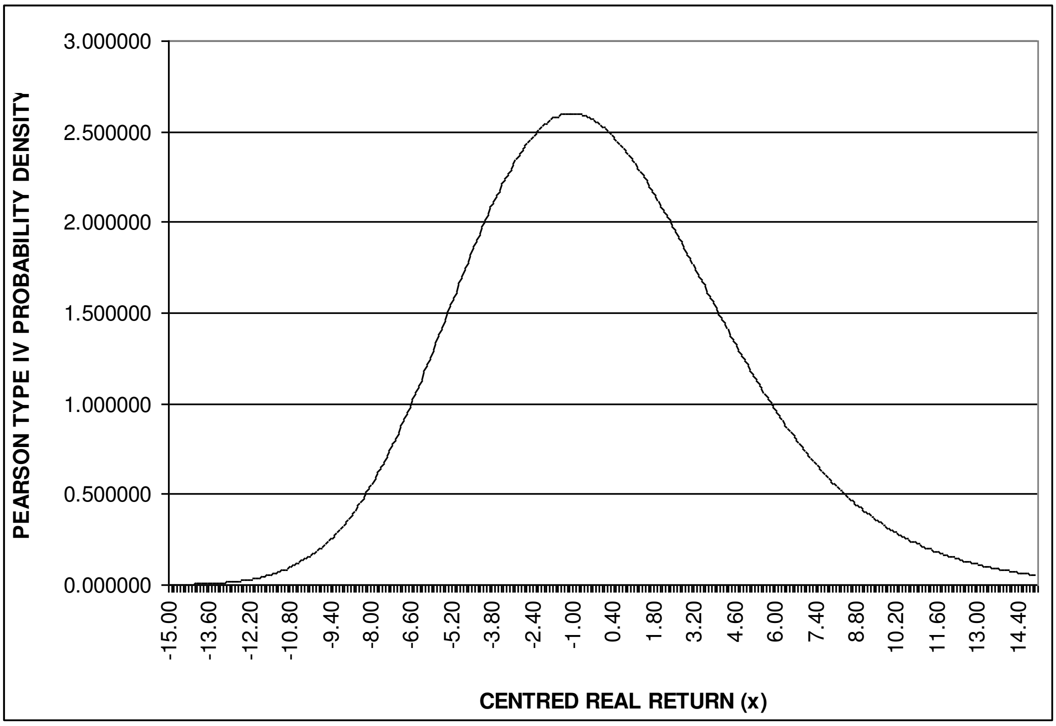
where:

$$c = \frac{\Gamma(\nu_2+1)}{\sqrt{\pi}\Gamma(\nu_2+\frac{1}{2})}\left|\frac{\Gamma(\nu_2+1+\frac{i\nu_2\theta}{\sqrt{\nu_1}})}{\Gamma(\nu_2+1)}\right|^2$$

is the normalising constant, $x = (\mu - r)$ is the centred instantaneous real rate of interest, $i = \sqrt{-1}$ is the pure imaginary number, $|\ .\ |$ is the modulus of a complex number, $\Gamma(.)$ is the gamma function, $\mu = 0.21\%$ (per annum), $\theta = 0.3717$, $\nu_1 = 0.1126$ and $\nu_2 = 73.6103$.

Figure 2. (a) Difference between actual distribution function and empirically estimated distribution function for the $N = 166$ real yields to maturity on one month U.S. Treasury bills. (b) Estimated Pearson Type IV probability density for real yields to maturity on one month U.S. Treasury bills.





The above graphs are based on the probability density:

$$g(x) = c[1 + \frac{(\theta + x)^2}{\nu_1}]^{-(1+\nu_2)} \exp[\frac{-2\nu_2\theta}{\sqrt{\nu_1}} \tan^{-1}(\frac{\theta + x}{\sqrt{\nu_1}})]$$

where:

$$c = \frac{\Gamma(\nu_2 + 1)}{\sqrt{\pi}\Gamma(\nu_2 + \frac{1}{2})} \left| \frac{\Gamma(\nu_2 + 1 + \frac{i\nu_2\theta}{\sqrt{\nu_1}})}{\Gamma(\nu_2 + 1)} \right|^2$$

is the normalising constant, $x = (\mu - r)$ is the centred instantaneous real rate of interest, $i = \sqrt{-1}$ is the pure imaginary number, | . | is the modulus of a complex number, $\Gamma(.)$ is the gamma function, μ = *-0.81%* (per annum), θ = *0.1611*, ν_1 = *0.0353* and ν_2 = *13.7863*.

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