Note on the Complexity of the Mixed-Integer Hull of a Polyhedron

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Abstract

We study the complexity of computing the mixed-integer hull \( \text{conv}(P \cap \mathbb{Z}^n \times \mathbb{R}^d) \) of a polyhedron \( P \). Given an inequality description, with one integer variable, the mixed-integer hull can have exponentially many vertices and facets in \( d \). For \( n, d \) fixed, we give an algorithm to find the mixed integer hull in polynomial time. Given \( P = \text{conv}(V) \) and \( n \) fixed, we compute a vertex description of the mixed-integer hull in polynomial time and give bounds on the number of vertices of the mixed integer hull.

Keywords — Mixed-integer hull, polyhedron, mixed-integer concave minimization

1 Introduction

Given a polyhedron \( P \subseteq \mathbb{R}^n \times \mathbb{R}^d \), we focus on computing the mixed-integer hull \( P_{MI} = \text{conv}(P \cap \mathbb{Z}^n \times \mathbb{R}^d) \). The mixed-integer hull is a fundamental object in mixed-integer linear programming and is well known to be a polyhedron [12]. In 1992, Cook, Kannan, Hartman, and McDiarmid [5] showed that the integer hull of \( P_I = \text{conv}(P \cap \mathbb{Z}^n) \) has at most \( 2m^3(6m^2\varphi)^{n-1} \) many vertices, where \( m \) is the number of facets of \( P \) and \( \varphi \) is the maximum binary encoding size of a facet. Hartman [8] gave a polynomial time algorithm in fixed-dimension to enumerate the vertices of the integer hull of a polyhedron. See [14] for a survey of these results, improvements, and lower bounds and also [4] for a discussion of implementations. As far as we know, no similar results have been shown for the mixed-integer hull.

We will give two algorithms to compute the mixed-integer hull; each algorithm produces a bound for the number of vertices of the mixed-integer hull. In Section 2, we consider \( P \) given as an inequality description. We show that even with one integer variable and the restriction of small subdeterminants of the inequality description, the mixed-integer hull can have exponentially many facets and vertices if \( d \) varies. Hence, we fix both the number of continuous and integer variables. Through a simple scaling or decertization technique, we can apply the results of [5] and [8] to compute the integer hull of a scaled polyhedron. By scaling back, we obtain the mixed-integer hull. This leads to a bound on the number of vertices that is exponential in \( n + d \).

In Section 3, we consider \( P \) given by a list of vertices and extreme rays. In this setting, we can allow \( d \) to vary. We reduce the original task of computing the mixed-integer hull of a polyhedron to the special case of polytopes by writing an extended formulation using a Minkowski-Weyl type decomposition of \( P \). Hence we can assume that \( P \) is bounded and given the vertices of \( P_{MI} \) that runs in polynomial time in the encoding size of the vertices provided that the number of integer variables is fixed. This algorithm implies a better bound on the number of vertices of the mixed-integer hull that depends on the number of vertices of the original polytope and does not depend on the number of continuous variables \( d \). This algorithm also implies an algorithm for concave minimization over the mixed-integer points.
in a polytope since the solution lies at an extreme point.

**Theorem 1** (Mixed-integer concave minimization). Let $V \subseteq \mathbb{Q}^{n+d}$, $P = \text{conv}(V)$, and $f: \mathbb{R}^{n+d} \to \mathbb{R}$ be concave. When $n$ is fixed, the problem $\min \{ f(x, y) : (x, y) \in P \cap \mathbb{Z}^n \times \mathbb{R}^d \}$ can be solved in polynomial time in the evaluation time of $f$, $d, |V|$ and $\nu$, where $\nu$ is the maximum binary encoding size of a point in $V$.

Concave minimization over polyhedra presented by an inequality description is NP-Hard when the dimension varies, even for the case of minimizing a concave quadratic function over the continuous points in a cube. This is because every extreme point can be a local minimum. Most exact algorithms require in the worst case to enumerate all extreme points of the feasible region \[11\] which can be of exponential size in the maximum binary encoding size of a point in $V$. Notice that in this setting, it may be impossible to give a compact facet or vertex representation of the mixed-integer hull if $d$ is allowed to vary. This is demonstrated in the following example that has only one integer variable. Notice that in the example, the maximum subdeterminant in absolute value of $A$ is two.

**Notation:** For a set $Q \subseteq \mathbb{R}^n \times \mathbb{R}^d$, we define $Q_\hat{x} = \{(x, y) \in Q : x = \hat{x}\} \subseteq \mathbb{R}^n \times \mathbb{R}^d$ and $\text{proj}_\hat{x}(Q) \subseteq \mathbb{R}^n$ as the projection of $Q$ onto the first $n$ variables.

2 Mixed-integer hull from inequalities

We study the problem of computing the mixed-integer hull of a polyhedron $P$ when we are given an inequality description of $P$. Unfortunately, in this setting, it may be impossible to give a compact facet or vertex representation of the mixed-integer hull if $d$ is allowed to vary. This is demonstrated in the following example that has only one integer variable. Notice that in the example, the maximum subdeterminant in absolute value of $A$ is two.

**Example 2.** Let $P = R \prod_{i=1}^{d+1} [-b_i, b_i] = \{ x \in \mathbb{R}^{d+1} : -b_i \leq A_i x \leq b_i, \ \forall \ i \}$ be the linear transformation of the hypercube with

\[
R = \begin{bmatrix}
\frac{1}{2} & \frac{1}{2} & \ldots & \frac{1}{2} \\
0 & I_d \\
\end{bmatrix}, \quad A = \begin{bmatrix}
2 & -1 & \ldots & -1 \\
0 & I_d \\
\end{bmatrix},
\]

where $I_d$ is the $d \times d$ identity matrix. Choosing odd numbers $b_i = 2^i + 1$, the first coordinates of neighboring vertices of $P$ are contained in the interiors of distinct and separated unit intervals. It follows that the mixed-integer hull $P_{MI} = \text{conv}(P \cap \mathbb{Z} \times \mathbb{R}^d)$ does not contain any of the vertices of $P$. In particular, it can be shown that every vertex of $P$ violates a distinct facet defining inequality of $P_{MI}$.

**Remark 3.** In Example 2, if we instead choose $b_i = 3$, the mixed integer hull $P_{MI}$ will still have exponentially many vertices and facets. In this case though, the first coordinate of all vertices is polynomially bounded by $d$, which allows one to write an extended formulation of $P_{MI}$ as the convex hull of the union of polynomially many integer fibers. By [2], this leads to a polynomial size representation. Note that it could still be possible that for the case of $b_i = 2^d + 1$ there is a polynomial size extended formulation for $P_{MI}$.

In view of Example 2, we first restrict ourselves to the setting where $n, d$ are fixed. We present a scaling algorithm that can be used to apply previous results on...
computing the integer hull. This technique is similar to [6] where they show that the mixed-integer split closure of a polyhedron is again a polyhedron by a scaling argument to transform the mixed-integer linear program to an integer linear program.

Let \( P = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^d : (A_1, A_2)(x, y) \leq b\} \) be a polyhedron with \( A_1 \in \mathbb{Z}^{mn \times n}, A_2 \in \mathbb{Z}^{md \times d} \) and \( b \in \mathbb{Z}^m \). We compute the mixed-integer hull \( P_{MI} = \text{conv}(P \cap \mathbb{Z}^n \times \mathbb{R}^d) \) in time polynomial in the binary encoding size of \( A_1, A_2 \) and \( b \). The output is either a facet or a vertex description of \( P_{MI} \). Since \( n, d \) are both fixed, the facet and vertex descriptions are polynomial time equivalently computable (see, for instance, [12]).

For a polytope \( Q \subseteq \mathbb{R}^n \times \mathbb{R}^d \), define \( Q^f := \{(x, ty) \in \mathbb{R}^n \times \mathbb{R}^d : (x, y) \in Q\} \). Therefore, \( Q^1 = Q \). Notice that we can write

\[
P^t = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^d : (A_1, A_2)(x, \frac{t}{d}) \leq b\}
= \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^d : (A_1, \frac{1}{t} A_2)(x, y) \leq b\}.
\]

**Theorem 4 (Mixed-integer hull from inequality description).** The mixed-integer hull can be computed in time polynomial in the binary encoding size of \( A_1, A_2, b \), provided that \( n \) and \( d \) are fixed. Furthermore, let \( \phi \) be the maximum binary encoding size of a row of \( (A_1, A_2, b) \). Then \( |\text{vert}(P_{MI})| \leq 2m^{n+d}(6(n+d)^2 \varphi)^{n+d-1} \) where \( \varphi = \phi + n\phi(m+n)^{n+d} \).

**Proof.** We first show that we can compute in polynomial time a vertex description of \( P_{MI} \). Let \( (\hat{x}, \hat{y}) \) be a vertex of \( P_{MI} \). Notice that \( (\hat{x}, \hat{y}) \) is a vertex of the \( d \)-dimensional polyhedron \( P_\hat{x} = \text{conv}(P \cap \{(x, y) : x = \hat{x}\}) \). We analyze the vertices of \( P_\hat{x} \) by considering its inequality description as \( P_\hat{x} = \{(x, y) : A(x, y) \leq \hat{b}\} \) where

\[
A = \begin{bmatrix}
A_1 & A_2 \\
I_n & O_d \\
[-I_n] & O_d
\end{bmatrix} \in \mathbb{Z}^{(m+2n) \times (n+d)}
\]

and \( \hat{b} = (b, \hat{x}, -\hat{x}) \). Here \( O_d \) is the \( d \times d \) matrix of all \( 0 \)'s. Therefore, there exists a basis \( B \) of the rows of \( A \) of size \( n + d \) such that \( (\hat{x}, \hat{y}) = (A_B)^{-1}(b, \hat{x}, -\hat{x})_B \). By Cramer’s rule, for any \( i \in B \), we have

\[
(\hat{x}, \hat{y})_i = \frac{\text{det}(A^i_B)}{\text{det}(A_B)}
\]

where \( A^i_B \) is the matrix \( A_B \) where the \( i \)th column is replaced by the vector \( (b, \hat{x}, -\hat{x})_B \). Therefore \( \text{det}(A^i_B) \in \mathbb{Z} \). Hence \( (\hat{x}, \hat{y})_i \cdot \text{det}(A_B) \in \mathbb{Z}^d \) is integral. Note that \( \text{det}(A_B) \) is completely independent of \( (\hat{x}, \hat{y}) \). Now if we let \( t := \prod_{B \text{ basis}} \text{det}(A_B) \), then \( (x, ty) \) is integral for any basis \( B \) of \( A \). There are at most \( \binom{m+n}{n+d} \leq (m+n)^{n+d} \) many bases \( B \). By Hadamard’s inequality, \( \text{det}(A_B) \leq \prod_{i=1}^n 2^\phi = 2^{n\phi} \). Therefore, \( t \leq (2^{n\phi})^{(m+n)^{n+d}} = 2^{n\phi(m+n)^{n+d}} \).

Since \( P_{MI}^t \) is the convex hull of all polyhedra \( P_\hat{x}^t \) for \( \hat{x} \in \mathbb{Z}^n \), and \( P_\hat{x}^t \) has integer vertices by our choice of \( t \), we have that \( P_{MI}^t = P_{MI}^t \). By computing the integer hull \( P_{MI}^t \) using [8], and then scaling back by computing \( (P_{MI}^t)^{1/t} = (P_{MI}^{t})^{1/t} = P_{MI} \), we find a description of the mixed-integer hull. Note that [8] yields a vertex description of the integer hull. This can be converted to an inequality description in polynomial time since the dimension is fixed. By [5], \( |\text{vert}(P_{MI})| = |\text{vert}(P_{MI}^t)| \leq 2m^{n+d}(6(n+d)^2 \varphi)^{n+d-1} \) where \( \varphi = \phi + n\phi(m+n)^{n+d} \). \( \square \)

### 3 Mixed-integer hull from vertices

We now consider the case where we are given \( P \) as a list of vertices and extreme rays and we show how to compute a vertex and extreme ray description of the mixed-integer hull. In this setting, we show that the number of continuous variables \( d \) may vary, and we still obtain a polynomial time algorithm to compute the mixed-integer hull. We begin by reducing the problem to finding the mixed-integer hull of a polytope.
3.1 Reduction to bounded polyhedra

We adapt a result of Nemhauser and Wolsey [10, Section I.4, Theorem 6.1] to the mixed-integer case. The proof is essentially the same, but we provide it here to obtain complexity bounds. Recall that by the Minkowski-Weyl theorem, a polyhedron \( P \) can be represented as an inequality description or a vertex and extreme ray description, that is, \( P = \{ x \in \mathbb{R}^{n+d} : Ax \leq b \} = \text{conv}(V) + \text{cone}(W) \) for some sets \( V \subseteq \mathbb{Q}^{n+d} \), \( W \subseteq \mathbb{Z}^{n+d} \). Here, \( \text{rec}(P) = \{ x \in \mathbb{R}^{n+d} : Ax \leq 0 \} = \text{cone}(W) \). In the following, we request that \( P \cap \mathbb{Z}^n \times \mathbb{R}^d \) is non-empty, which can be tested in polynomial time provided that \( n \) is fixed using Lenstra’s algorithm [9].

**Lemma 5** (Relevant mixed-integer points). Let \( P \) be a rational polyhedron given by a vertex/ray representation \( \text{conv}(V) + \text{cone}(W) \) for \( V \subseteq \mathbb{Q}^{n+d} \) and \( W \subseteq \mathbb{Z}^{n+d} \) (resp. an inequality description \( \{ x \in \mathbb{R}^{n+d} : Ax \leq b \} \) for \( A \in \mathbb{Q}^{m \times (n+d)} \) and \( b \in \mathbb{Q}^m \)). Suppose that \( P \cap \mathbb{Z}^n \times \mathbb{R}^d \neq \emptyset \). In polynomial time, we can compute a vertex description (resp. an inequality description) of a rational polytope \( Q \) of polynomial size such that \( P = Q + \text{rec}(P) \) and

\[
P_{MI} = \text{conv}(Q \cap \mathbb{Z}^n \times \mathbb{R}^d) + \text{rec}(P).
\]

**Proof.** By the Minkowski-Weyl theorem, we can decompose \( P \) as \( P = \text{conv}(V) + \text{cone}(W) \) where \( V \) is the set of vertices of \( P \) and \( W \) is the set of minimal integral extreme rays of \( P \), and hence \( \text{cone}(W) = \text{rec}(P) \).

For any \( x \in P \cap \mathbb{Z}^n \times \mathbb{R}^d \), by this decomposition and Carathéodory’s theorem we can write

\[
x = \sum_{i \in I} \lambda_i v_i + \sum_{j \in J} \mu_j w_j
\]

where \( \lambda_i, \mu_j \geq 0 \), \( \sum_{i \in I} \lambda_i = 1 \), \( v_i \in V \), \( w_j \in W \), \( |I| \leq n + d + 1 \) and \( |J| \leq n + d \). Since \( x \in \mathbb{Z}^n \times \mathbb{R}^d \) and \( \sum_{j \in J} [\mu_j] w_j \in \mathbb{Z}^{n \times d} \), we have that \( x - \sum_{j \in J} [\mu_j] w_j = \sum_{i \in I} \lambda_i v_i + \sum_{j \in J} (\mu_j - [\mu_j]) w_j \in \mathbb{Z}^n \times \mathbb{R}^d \).

Therefore, if we define \( T := \text{conv}(V) + (n + d) \text{conv}(W \cup \{0\}) \), then \( x - \sum_{j \in J} [\mu_j] w_j \in T \cap \mathbb{Z}^n \times \mathbb{R}^d \). It follows that

\[
\text{conv}(P \cap \mathbb{Z}^n \times \mathbb{R}^d) = \text{conv}(T \cap \mathbb{Z}^n \times \mathbb{R}^d) + \text{rec}(P).
\]

If \( V \) and \( W \) are given as input, then we are done by setting \( Q = T \) for which we obtain a vertex description by taking the Minkowski sum \( V + (n + d)W \). On the other hand, if we are given as input an inequality description of \( P \), then the descriptions of \( V \) and \( W \) may be exponential in the input. In this case, we instead determine a box that contains \( T \) and intersect that box with \( P \) to obtain our choice of \( Q \).

More precisely, let \( R \geq 0 \) be a bound on the infinity norm of the vertices \( V \) and an integral representation of the extreme rays \( W \). By, for instance [12], we can choose \( R \) of polynomial encoding size. Setting \( R'(n + d + 1)R \), we have \( T \subseteq B := [-R', R']^{n \times d} \). Setting \( Q = P \cap B \) finishes the argument. \( \square \)

3.2 Mixed-integer hull for polytopes

We begin by showing how to compute the vertices of the integer hull of a polytope \( Q = \text{conv}(V) \) presented by its vertex set \( V \subseteq \mathbb{Q}^n \). To do so, we employ the algorithm from [8] to compute integer hulls from an inequality description, but this requires some care. If we apply a standard transformation of \( Q \) into an inequality description, this description could have many facets, causing the algorithm to require a double exponential complexity in terms of \( n \). Therefore, we instead find a triangulation of \( V \) and then compute the integer hull of simplices, which have exactly \( n + 1 \) facets, allowing us to procure a single exponential complexity for the number of vertices of \( Q_I \).

**Lemma 6** (Integer hull from vertex description). Let \( V \subseteq \mathbb{Q}^n \) and let \( Q = \text{conv}(V) \). When \( n \) is fixed, we can compute a vertex description of the integer hull \( Q_I \) in polynomial time in \( n \) and \( |V| \) where \( n \) is the maximum binary encoding size of a vector in \( V \). Furthermore, \( |\text{vert}(Q_I)| \leq \frac{1}{3} 12^n n^{3n-2} \varphi^{n-1} |V|^{n+1} \) where \( \varphi = 2(n +
1^2(n + \log(n + 1))$. If $|V| = n + 1$, we obtain the tighter bound of $|\text{vert}(C_I)| \leq \frac{1}{2} \cdot 2^n n^{3n-2} \varphi^{n-1}$.

**Proof.** Let $n' = \dim(\text{conv}(V))$. We compute a Delaunay triangulation of the points $V$, yielding a list of $n'$-dimensional simplices. As is well known, this can be done by computing the convex hull of the extended point set $V' = \{(v, \|v\|^2) : v \in V\} \subseteq \mathbb{R}^{n+1}$. Then facets of $\text{conv}(V')$ corresponds to a $n'$-dimensional cells of the triangulation. The convex hull can be computed in polynomial time using [1]. See for instance, [7] for a discussion of algorithms to compute the Delaunay triangulation. Moreover, there are at most $\binom{|V|}{n+1} \leq |V|^{n+1}$ cells. If $|V| = n + 1$, we can instead use the bound $2^{n+1}$ for the number of cells. For general $|V|$, a tighter asymptotic bound on the number of simplices is $O(|V|^{n/2})$ [13]. For each simplex $C$ in the triangulation, we compute an inequality description. Since it may be lower dimensional, at most $2n$ inequalities are needed to describe it. These inequalities can be computed in polynomial time using Gaussian elimination and have binary encoding size bounded by $\varphi = 2(n+1)^2(n + \log(n + 1))$ which follows from Cramer’s rule and Hadamard’s inequality.

Finally, we can apply the result of [5] to see $|\text{vert}(C_I)| \leq 2m^n (6n^2 \varphi)^{n-1}$ where $m$ is the number of facets. Here $m \leq 2n$, so we have $|\text{vert}(C_I)| \leq 2(2n)^n (6n^2 \varphi)^{n-1}$. Finally, applying this to each cell $C$, we have at most $|V|^{n+1} 2(2n)^n (6n^2 \varphi)^{n-1}$ many vertices of the integer hull, or only $\frac{1}{3} \cdot 2^{n} n^{3n-2} \varphi^{n-1} |V|^{n+1}$ many vertices of the integer hull; or only $\frac{1}{3} \cdot 2^{n} n^{3n-2} \varphi^{n-1}$ if we choose $|V| = n + 1$ and use the improved bound mentioned above. By [8], it follows that these can be computed in polynomial time when $n$ is fixed. This creates a superset of the vertices of the integer hull; points that are not vertices can be discovered using linear programming, which can be done in polynomial time. \hfill \square

Although this approach may not produce a tight bound on the number of vertices of the integer hull, the bound in Lemma 6 is in the right order of magnitude in terms of $\varphi$. Indeed, in [3], they show that for every dimension $n \geq 2$, there exists a simplex $P \subseteq \mathbb{R}^n$ given as an inequality description with encoding size $\varphi$ such that $P_{I}$ has at least $c_n \varphi^{n-1}$ many vertices, where $c_n$ is a constant depending only on $n$. This shows that even if $P$ has a small number of vertices, the number of vertices of the integer hull can be large.

Our next goal is to make use of Lemma 6 in order to compute the mixed-integer hull. This requires one more ingredient.

**Lemma 7.** Every vertex of $P_{MI}$ lies in a face $F$ of $P$ with $\dim(F) \leq n$. Furthermore, let $n' = \min(n, \dim(P))$ and let $F_{n'}$ denote the set of faces of $P$ of dimension $n'$. Then

$$\text{vert}(P_{MI}) \subseteq \bigcup_{F \in F_{n'}} \bigcup_{\hat{x} \in \text{vert}(\text{proj}_x(F))} \text{vert}(F_{\hat{x}}).$$

**Proof.** Since $P_{MI} = \text{conv}(P_{\hat{x}} : \hat{x} \in \mathbb{Z}^n)$, every vertex $(\hat{x}, \hat{y})$ of $P_{MI}$ is a vertex of $P_{\hat{x}}$. Since $P_{\hat{x}} = \{(x, y) : A(x, y) \leq b, Ix = \hat{x}\}$ and any vertex of $P_{\hat{x}}$ is defined uniquely by $n + d$ tight inequality constraints, at least $d$ of those tight constraints come from the inequalities $A(x, y) \leq b$. Hence, $(\hat{x}, \hat{y})$ satisfies at least $d$ affine independent tight constraints from $P$, i.e., it is contained in a face $F$ that is $n + d - d = n$ dimensional at most. Since we choosing a maximal such face $F$, we only consider $F$ of dimension $n' = \min(n, \dim(P))$. Furthermore, there exist $d$ tight inequalities $(A_1, A_2)(x, y) \leq \hat{b}$ such that $A_2$ is invertible.

Then, for any $\hat{x} \in \text{proj}_x(F)$, the corresponding $\hat{y}$ such that $(\hat{x}, \hat{y}) \in F$ is given uniquely as

$$\hat{y} = A_2^{-1}\hat{b} - A_2^{-1}A_1\hat{x}.$$

Now let $(\hat{x}, \hat{y}) \in \text{vert}(P_{MI})$ and suppose that $\hat{x} \notin \text{vert}(\text{proj}_x(F))$. Then $\hat{x} = \sum \mu_i x^i$ for some $\mu_i > 0$, $\sum \mu_i = 1$, and $x^i \in \text{vert}(\text{proj}_x(F))$. Then setting, $y^i := A_2^{-1}\hat{b} - A_2^{-1}A_1x^i$ it follows that $y^i \in F \cap \mathbb{Z}^n \times \mathbb{R}^d$ and $(\hat{x}, \hat{y}) = \sum \mu_i (x^i, y^i)$; therefore, $(\hat{x}, \hat{y})$ is not a vertex of $P_{MI}$. This proves (1). \hfill \square

**Theorem 8** (Mixed-integer hull from vertex description). Let $V \subseteq \mathbb{Q}^{n+d}$ and let $P = \text{conv}(V)$. For fixed
$n \geq 1$ there exists an algorithm to compute $\text{vert}(P_{M1})$ that runs in polynomial time in $d, \nu, |V|$ where $\nu$ is the maximum binary encoding size of a point in $V$. Furthermore,

$$|\text{vert}(P_{M1})| \leq \frac{4}{3} 48^n n^{3n-2} \varphi^{n-1} |V|^{n+1}$$

where $\varphi = 2(n+1)^2(\nu + \log(n+1))$.

Proof. Let $n' = \min(n, \dim(P)) \leq |V|$ and let $(\hat{x}, \hat{y})$ be a vertex of $P_{M1}$. By Lemma 7, there exists a face $F$ of $P$ of dimension $n'$ such that $(\hat{x}, \hat{y}) \in \text{vert}(F)$ and $\hat{y} \in \text{vert}(\text{proj}_{x}(F)_{I})$. By Carathéodory’s theorem, there exist $n'+1$ vertices of $F$ such that $(\hat{x}, \hat{y}) \in \hat{F} = \text{conv}(v^1, \ldots, v^{n'+1})$. Since $\hat{F} \subseteq F$ and $(\hat{x}, \hat{y}) \in \hat{F}$, it follows that $\hat{x} \in \text{vert}(\text{proj}_{x}(\hat{F})_{I})$ and $(\hat{x}, \hat{y}) \in \text{vert}(\hat{F})$. Indeed, if $\hat{x} \notin \text{vert}(\text{proj}_{x}(\hat{F})_{I})$, then it can be written as a strict convex combination of points in $\text{vert}(\text{proj}_{x}(\hat{F})_{I}) \subseteq \text{conv}(\text{vert}(\text{proj}_{x}(\hat{F})_{I}))$, which shows that $\hat{x} \notin \text{vert}(\text{proj}_{x}(\hat{F})_{I})$.

Therefore, we can compute a superset of the vertices of $P_{M1}$ by enumerating every $(n'+1)$-elementary subset $\{v^1, \ldots, v^{n'+1}\}$ of $V$, yielding at most $|V|^{n'+1}$ sets to consider. Fix a subset $\{v^1, \ldots, v^{n'+1}\}$ and set $\hat{F} = \text{conv}(v^1, \ldots, v^{n'+1})$. We will next find a vertex description of $\text{conv}(\hat{F} \cap \mathbb{Z}^n \times \mathbb{R}^d)$.

To this end, let $\hat{v}^i = \text{proj}_{x}(v^i)$. Applying Lemma 6 with $Q = \text{conv}(\hat{v}^1, \ldots, \hat{v}^{n'+1})$, we have a vertex description of $Q_I$ with at most $2^{n'+1} \varphi^{n'-1} n^{3n'-2}$ many vertices. For each vertex $\hat{x}$ of $Q_I$, we compute the vertices of $\hat{F}$, which can be written as the set of $(x, y) \in \mathbb{R}^n \times \mathbb{R}^d$ satisfying

$$(x, y) = \sum_{i=1}^{n'+1} \lambda_i v^i, x = \hat{x}, \sum_{i=1}^{n'+1} \lambda_i = 1, \lambda_i \geq 0.$$

Every vertex of this set corresponds to a unique face of the $(n'+1)$-dimensional standard simplex. There are $2^{n'+1}$ many faces of the $n'$-dimensional simplex. Hence, this set has at most $2^{n'+1}$ vertices, which can be enumerated, for instance by [1] in time polynomial in the encoding size $\varphi$ provided that $n$ is fixed. This number of enumerated points provides us with an upper bound on the number of vertices of $P_{M1}$. Since we can test whether points are vertices of $P_{M1}$ using linear programming, the proof is complete. \hfill $\Box$

Combined with Lemma 5, the above theorem gives a similar result for polyhedra.

Remark 9. Consider a polyhedron $P = \{(x, y) \in \mathbb{R}^{n+d} : A(x, y) \leq b, (x, y) \geq 0\}$, where $A \in \mathbb{Q}^{m \times (n+d)}$, $b \in \mathbb{Q}^m$. It follows that $|\text{vert}(P)| \leq (n + d)^m$. Therefore, for fixed $m$ and $n$, using Theorem 8, we can find a vertex description of the mixed-integer hull in polynomial time. This applies, for instance, to a mixed-integer knapsack problem with no upper bounds on the variables.

References


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