Auctions vs. Fixed Pricing: Competing for Budget Constrained Buyers

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Abstract: We investigate price mechanism selection in a setting where sellers compete for budget constrained buyers by adopting either fixed pricing or auctions (first or second price). We show that first and second price auctions are payoff equivalent when some bidders are financially constrained, so sellers are indifferent to adopt either format. We fully characterize possible equilibria and show that if the budget is high, then sellers compete via fixed pricing, if it is low then they compete via auctions, and if it is moderate then they mix, so both mechanisms coexist. The budget constraint becomes less binding if sellers use entry fees. Interestingly an improvement of the budget—e.g. letting customers pay in installments—may lead to fewer trades and a loss of efficiency.

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1 Introduction

We compare arguably the most popular selling rules—fixed pricing, first price and second price auctions—in a competitive setting where some customers are budget constrained. The adoption of a selling rule is a strategic decision in that it signals how the seller intends to share the surplus, which in turn influences the attractiveness of the store and pins down the expected demand. The selection becomes more interesting if potential customers have limited budgets. Indeed sellers often face customers who are willing to pay but have limited immediate financial resources to do so. Anecdotal evidence suggests that markets for houses, automobiles and other expensive durable goods (appliances, electronic equipment, furniture, business equipment, etc.) often exhibit this trait. Despite its practical importance, little attention has been paid to the relationship between buyers' limited purchasing power and the trading mechanism in place. To the best of our knowledge, this paper is the first attempt investigating trade mechanism selection in a competitive setting with budget constrained buyers.[1]

[1] We contribute to a segment of the directed search literature that studies trading mechanism selection in a competitive environment where the demand at a store endogenously depends on the trading mechanism in place; see, Kultti

A first result is the payoff equivalence of first and second price auctions. We show that, controlling for the expected demand and reserve price, both auction formats yield identical payoffs. Financially constrained buyers (low types) bid their budgets b under both formats and they have no chance of winning if a high budget type is present in the auction. High types, on the other hand, bid aggressively under second price auctions but shade their bids somewhat under first price auctions. In expected terms, however, they earn the same. Absent budget constraints, payoff equivalence is well established in the auction literature. With budget constrained bidders Che and Gale (1996) prove a similar result when the budget distribution is continuous; we show that the result holds with a discrete distribution as well. Payoff equivalence implies that in a competitive setting, sellers and buyers are indifferent to adopting or joining either auction format.

We fully characterize possible equilibria and show that the choice between fixed pricing and auctions depends on the size of b: if it is large then sellers pick fixed pricing, if it is small then they pick auctions and if it is moderate then both mechanisms coexist. To understand this result one needs to first look at the outcome with no budget constraints. In a setting with homogenous, financially unconstrained buyers, fixed pricing and auctions are payoff-equivalent and coexist in the same market (Kultti, 1999; Eeckhout and Kircher, 2010). Competition among sellers dictates that, irrespective of the mechanism they compete with, sellers must provide each customer with the same level of "market utility", which is endogenous and commensurate with the degree of competition in the market. Both fixed pricing as well as auctions are capable of doing this; as such they coexist.[2]

The introduction of low budget buyers into the homogenous model breaks the payoff equivalence between fixed pricing and auctions. With fixed pricing the fact that some buyers have lower budgets is immaterial so long as they can afford the equilibrium price in the homogenous model. This means that if b is high enough then the budget constraint is slack and fixed pricing is still capable of allocating all buyers the same market utility. With auctions, however, the budget constraint is never slack. Indeed, under both auction formats, no matter how large b is, low budget types have no chance of winning if a high budget type is present in the auction. Consequently, low types end up with a smaller market utility than high types (despite the fact that they have the same willingness to pay). This inequality is not compatible with profit maximization under competition and explains why sellers compete with fixed pricing, and not with auctions, when b is large.

If b falls below a threshold then, even with fixed pricing, the budget constraint starts to bind. In this region, serving customers indiscriminately is no longer feasible as low types are unable to afford the equilibrium posted price. So, sellers start to prioritize high types over low types by adopting auctions. The fraction of sellers switching to auctions rises as b decreases and if b falls below another

(1999), Eeckhout and Kircher (2010), Virag (2011) Geromichalos (2012), Selcuk (2012). We differ from these studies by considering budget constrained buyers. There are number of papers in the auction literature focusing on budget constrained bidders e.g. see Che and Gale (1998), Zheng (2001), Hafalir et al. (2012), Burkett (2015) and Kotowski (2015). These setups do not consider competition or the possibility of selling via a different rule. An exception is Che and Gale (2000) who study the optimal selling mechanism to budget-constrained buyers; however they consider a single seller rather than a competitive market.

[2] The result is in Eeckhout and Kircher (2010) is more general in that they show that payoff equivalence and coexistence is not restricted to fixed pricing or auctions, rather it holds for a range of mechanisms they label as "payoff-complete".

threshold then all sellers compete with auctions. Auction sellers use the reserve price as partial protection against the presence of low types. Indeed, the equilibrium reserve price rises if the budget decreases or if the percentage of the low types increases.

Sellers have the ability to screen out low types ex-ante by posting unaffordable prices; however, in equilibrium no seller employs such a tactic. The auction mechanism is already capable of screening customers ex-post as it makes sure that it is a high type who wins (and pays for) the item. Since this tool is available, screening customers ex-ante, i.e. point-blank refusing to deal with low types, is suboptimal.

We extend the model by letting sellers to ask for an entry fee. With entry fees the nature of the equilibria remains the same but the thresholds are smaller than before; hence the budget constraint is less binding. Indeed, entry fees allow sellers to collect the revenue from all buyers present at the store—not just from the one who purchases the item—as such they shrink the parameter space in which the budget constraint kicks in.

2 Model

2.1 Environment

Consider an economy populated by a large number of risk-neutral buyers and sellers, where the aggregate buyer-seller ratio is λ. Each seller is endowed with one unit of a good and wants to sell it above his reservation price, zero. Similarly each buyer wants to purchase one unit of an indivisible good and is willing to pay up to his reservation price, one. Buyers are identical in terms of their valuation of the good but they differ in terms of their ability to pay. A fraction σ of buyers (low types) have limited budgets and can pay up to $b < 1$ whereas the rest (high types) can pay up to 1. A buyer's type is private information; however, the parameters λ, σ and b are common knowledge.

The game proceeds over the course of three stages. Here we give a brief overview and fill in the details later. In the first stage sellers simultaneously and independently choose a trading mechanism $m \in \mathbb{M}$ and a list price (reserve price in case of auctions) $r_m \in [0,1]$. The set of trading mechanisms $\mathbb{M}$ consists of fixed pricing, first price auctions and second price auctions. With fixed pricing the transaction necessarily occurs at the list price r_f. With auctions the reserve price is charged if a single customer is present at the store and bidding ensues if there are multiple customers.

In the second stage buyers observe sellers' selections and choose one store to visit; however once they reach a store they cannot move elsewhere. If the customer is alone at the store then he pays the reserve/list price and obtains the good for sure. If there are $n > 1$ buyers then with fixed pricing each buyer has an equal chance $1/n$ of obtaining the item. With auctions, however, bidding ensues and the winner as well as the sale price are determined based on the specifics of each auction format (more on this below). If trade takes place at price r then the seller realizes payoff r, the buyer realizes $1 - r$ whereas those who do not trade earn zero.

2.2 Demand Distribution

Following the directed search literature, we restrict attention to mixed visiting strategies that are symmetric and anonymous on and off the equilibrium path (Burdett et al., 2001; Shimer, 2005). Symmetry requires buyers of the same type to use the same visiting strategies, whereas anonymity means that sellers posting the same "package" (m, r_m) should be treated identically. Symmetry and anonymity in buyers' visiting strategies imply that the distribution of demand at any store is Poisson (Galenianos and Kircher, 2012). Hence, the probability that a seller with the terms (m, r_m) meets $n = 0, 1, 2...$ customers of type $i = h, l$ is given by $z_n(x_{i,m})$ where

$$z_n(x) = \frac{e^{-x}x^n}{n!}. \tag{1}$$

We refer to $x_{i,m}$ as the expected demand consisting of type i buyers. Since high types and low types arrive at independent Poisson rates $x_{h,m}$ and $x_{l,m}$, the distribution of the *total demand* is also Poisson with $x_{h,m} + x_{l,m}$ (Grimmett and Welsh, 1986). Expected demands $x_{h,m}$ and $x_{l,m}$ are endogenous and depend on the price package (m, r_m) and how it compares with the rest of the market (see below).[3]

Let $u_{i,m}(n)$ denote the conditional expected utility of a type i buyer at a store that trades via rule m and has n customers, including the buyer himself. Similarly let $\pi_m(n)$ denote a store's conditional expected profit (conditional on trading via rule m and having n customers). Below we pin down these payoffs for all mechanisms, starting with auctions.

2.3 Auctions

Under both auction formats if there is a single customer at the store, i.e. if $n = 1$, then the reserve price is charged but if $n \geq 2$ then bidding ensues. The outcome of the bidding process depends on the auction format as well as how many high types and how many low types are present in the auction. We know that low types and high types arrive at rates $x_{l,m}$ and $x_{h,m}$ (for now we drop the mechanism subscript m when understood). A buyer's type is his private information, so neither the seller nor the other buyers know whether a particular buyer is a high type or a low type, but they can work out this probability from the arrival rates. Specifically, given that are n customers present in the auction, the probability that exactly j of them are low types is equal to

$$\frac{\Pr(j \text{ low types } \& \ n-j \text{ high types})}{\Pr(n \text{ customers})} = \frac{z_j(x_l) \times z_{n-j}(x_h)}{z_n(x_l + x_h)} = \binom{n}{j}\theta^j(1-\theta)^{n-j},$$

where

$$\theta = \frac{x_l}{x_h + x_l}.$$

In words, the distribution of types is *binomial* (n, θ), where θ is the probability that a customer is a low type. The bidding strategies will depend on this probability. Note that θ is endogenous and

[3] Throughout the text, we refer to $x_{i,m}$ as "expected demand", "arrival rate" or "queue length", even though we realize that there are nuances across these terms.

it depends on the arrival rates x_h and x_l, which are also endogenous and they depend on what the seller posts and how it compares with the rest of the market. For instance, a change in the reserve price affects how many and what type of customers the seller gets, which, in turn, affects θ and thereby the outcome of the bidding process. So, market competition filters into the bidding process via the parameter θ.

2.3.1 Second Price Auctions

Consider a second price auction, where the winner pays the second highest bid. To prevent buyers from bidding above their budgets we assume that a bid must be accompanied by a deposit of equal value. If the bidder wins, then he gets back the difference between the deposit and the sale price. Otherwise he gets back the entire deposit. Given that low types cannot overbid, it is straightforward to verify that in the unique pure strategy equilibrium low types bid b and high types bid 1. Therefore equilibrium payoffs are

$$u_h(n) = \theta^{n-1}(1-b) \quad \text{and} \quad u_l(n) = \frac{\theta^{n-1}}{n}(1-b) \ \text{ for } n \geq 2. \tag{2}$$

Note that $u_h > u_l$ because a low type has no chance of winning if there is a high type present in the auction, no matter how small the budget difference is. Given the bidding strategies, the seller's expected profit is given by

$$\pi(n) = \left[\theta^n + n\theta^{n-1}(1-\theta)\right] b + 1 - \theta^n - n\theta^{n-1}(1-\theta) \ \text{ for } n \geq 2. \tag{3}$$

The expression in square brackets is the probability of having n low types and zero high types or having $n-1$ low types and one high type. In either case the winning bid is b. The remainder of the expression is the probability of having two or more high types, in which case the winning bid is 1.

The expressions above encompass corner scenarios where one type may be absent from bidding. To see why, suppose that low types stay away from the auction, i.e. suppose $x_l = 0$. Then $\theta = 0$ and therefore $u_h(n) = 0$ and $\pi(n) = 1$. Indeed, if all bidders are high types then they all bid 1, the seller earns 1 and all buyers earn 0. Similarly if high types stay away, i.e. if $x_h = 0$, then $\theta = 1$ and therefore $u_l(n) = \frac{1-b}{n}$ and $\pi(n) = b$. Indeed, if all bidders are low types then they all bid b, the seller earns b and, due to the tie, each buyer has an equal chance $1/n$ of winning the item.

2.3.2 First Price Auctions

With first price auctions the winner pays his own bid rather than the second highest bid.[4] It is easy to verify that the game does not have a pure strategy equilibrium. In what follows we focus on a symmetric mixed strategy equilibrium where players of the same type purse the same bidding strategies.

[4] Overbidding can be ruled out either by requiring a cash bond or, alternatively, by not giving the item to the bidder who reneges on his bid and by imposing a (small) penalty on him. The fact that the winner pays his own bid eliminates the appeal of overbidding.

Proposition 1 *Consider a first price auction where some buyers have limited budgets. In the unique symmetric equilibrium low types bid b whereas high types continuously randomize in the interval $\left[b, 1-\theta^{n-1}(1-b)\right]$ according to cdf*

$$G_h(p) = \frac{\theta}{1-\theta}\left[\left(\frac{1-b}{1-p}\right)^{\frac{1}{n-1}} - 1\right].$$

Equilibrium payoffs are given by

$$u_h(n) = \theta^{n-1}(1-b) \quad \textit{and} \quad u_l(n) = \frac{\theta^{n-1}}{n}(1-b) \ \textit{for } n \geq 2. \tag{4}$$

Comparing (2) and (4), it is clear that, from a buyer's perspective, both auction formats are payoff equivalent and deliver the same expected utilities. Low types, under both formats, bid their budget b. High types, on the other hand, bid less aggressively with the first price auction than they do in the second price auction; however, in expected terms, they end up earning the same. Note that high types are sure to bid more than b; hence, low types have no chance of winning if a high type is present in the auction.

The equivalence of buyers' payoffs implies that, from a seller's point of view, both auction formats are revenue equivalent because no surplus is left on the table. In the rest of this section we will formally prove this claim. To start, let $P(j;n)$ denote the expected value of the winning bid in a first price auction when there n buyers present in the auction and j of them are high types. If $j=0$ then all buyers are low types and the winning bid is b i.e. $P(0;n)=b$. If $j \geq 1$ then the winner will be a high type as they outbid low types for sure, however the distribution of the winning bid needs to be determined. Recall that each high type randomizes in $\left[b, 1-\theta^{n-1}(1-b)\right]$ according to cdf G_h. So, each bid is an independent random variable drawn from the same interval via the same cdf and the winning bid is the maximum of these j bids. It follows that the winning bid is distributed in the same interval but according to the cdf G_h^j (Grimmett and Welsh, 1986). Given the distribution, the expected value of the winning bid is given by

$$P(j;n) = \int_b^{\bar{p}_n} p dG_h^j(p) = pG_h^j(p)\Big|_b^{\bar{p}_n} - \int_b^{\bar{p}_n} G_h^j(p)\, dp,$$

where $\bar{p}_n \equiv 1-\theta^{n-1}(1-b)$. In the second step we used integration by parts. Since $G_h(\bar{p}_n)=1$ and $G_h(b)=0$ we have

$$P(j;n) = \bar{p}_n - \int_b^{\bar{p}_n} G_h^j(p)\, dp.$$

The expected payoff of the seller is given by

$$\pi(n) = \theta^n b + \sum_{j=1}^{n} \binom{n}{j} \theta^{n-j}(1-\theta)^j P(j;n).$$

With probability θ^n all buyers are low types, so the winning bid is b. With probability $\binom{n}{j}\theta^{n-j}(1-\theta)^j$ there are $j \geq 1$ high types and $n-j$ low types present in the auction, in which case the winner is a high type and the expected winning bid is $P(j;n)$. Substituting for $P(j;n)$ yields

$$\pi(n) = \theta^n b + \bar{p}_n \sum_{j=1}^{n} \binom{n}{j} \theta^{n-j}(1-\theta)^j - \int_b^{\bar{p}_n} \sum_{j=1}^{n} \binom{n}{j} \theta^{n-j}(1-\theta)^j G_h^j(p)\,dp.$$

After applying the binomial theorem to the expressions with summations we have

$$\pi(n) = \theta^n b + \bar{p}_n(1-\theta^n) - \int_b^{\bar{p}_n} \{[\theta + (1-\theta) G_h(p)]^n - \theta^n\} dp.$$

Substituting for $G_h(p)$ and evaluating the integral yields

$$\pi(n) = \bar{p}_n - \theta^n (n-1)(1-b)^{\frac{n}{n-1}} (1-p)^{-\frac{1}{n-1}} \Big|_b^{\bar{p}_n}.$$

Substituting for $\bar{p}_n$ and re-arranging we have

$$\pi(n) = \left[\theta^n + n\theta^{n-1}(1-\theta)\right] b + 1 - \theta^n - n\theta^{n-1}(1-\theta) \text{ for } n \geq 2. \tag{5}$$

Comparing (3) and (5) term by term reveals that, ceteris paribus, both auction formats raise the same expected revenue for the seller. Revenue equivalence between fixed and second price auctions is well known in the literature. Here we show that it remains valid with financially constrained bidders. Che and Gale (1996) prove a similar result when the budget distribution is continuous. We show that it holds with a discrete distribution as well.

The discussion so far establishes that, controlling for the expected demand and the reserve price, both auction formats are payoff equivalent, which means that in a competitive setting sellers and buyers are indifferent to adopting or joining either auction format. So from now on we make no distinction between first price or second price auctions, and use the generic term "auctions" instead.

2.4 Buyers

A type i buyer's expected utility from visiting a store competing with mechanism m is given by

$$U_{i,m}(r_m, x_{h,m}, x_{l,m}) = \sum_{n=0}^{\infty} z_n (x_{h,m} + x_{l,m})\, u_{i,m}(n+1). \tag{6}$$

Note that $U_{i,m}$ is a weighted sum of all conditional utilities $u_{i,m}$: With probability $z_n(\cdot)$ the buyer finds $n = 0, 1, ..$ *other* customers at the same store; so, in total there are $n+1$ customers (including himself) and the conditional expected utility corresponding to this scenario is $u_{i,m}(n+1)$.

Start with auctions (denoted with subscript a) and for now suppose that the reserve price is affordable i.e. $r_a \leq b$. If a buyer is alone at an auction store then he obtains the item by paying the reserve price i.e. $u_{h,a}(1) = u_{l,a}(1) = 1 - r_a$. If $n \geq 2$ then we know that under both auction formats

$$u_{h,a}(n) = \theta^{n-1}(1-b) \text{ and } u_{l,a}(n) = \frac{\theta^{n-1}}{n}(1-b), \text{ where } \theta = \frac{x_{l,a}}{x_{h,a} + x_{l,a}}.$$

Substituting these expressions into (6) yields

$$\begin{aligned} U_{h,a} &= z_0(x_{h,a} + x_{l,a})(1 - r_a) + \sum_{n=1}^{\infty} z_n(x_{h,a} + x_{l,a})\theta^n(1-b) \text{ and} \\ U_{l,a} &= z_0(x_{h,a} + x_{l,a})(1 - r_a) + \sum_{n=1}^{\infty} z_n(x_{h,a} + x_{l,a})\frac{\theta^n}{n+1}(1-b). \end{aligned}$$

After substituting for θ and re-arranging we have

$$U_{h,a} = z_0(x_{h,a} + x_{l,a})(1 - r_a) + z_0(x_{h,a})(1 - z_0(x_{l,a}))(1-b) \text{ and} \tag{7}$$

$$U_{l,a} = z_0(x_{h,a} + x_{l,a})(1 - r_a) + z_0(x_{h,a})\frac{1 - z_0(x_{l,a}) - z_1(x_{l,a})}{x_{l,a}}(1-b). \tag{8}$$

Observe that if $b < 1$ then $U_{h,a} > U_{l,a}$ i.e. at an auction store, a low type obtains a strictly lower expected utility than a high type. The reason is that under both auction formats low types are always second to high types at the point of service; they have no chance of winning the item if a high type is present in the auction. This is true no matter how large b is. At fixed price stores things are different. Assuming $r_f \leq b$ we have

$$u_{h,f}(n) = u_{l,f}(n) = \frac{1 - r_f}{n} \text{ for all } n \geq 1.$$

Substituting this into (6) yields

$$U_{h,f} = U_{l,f} = \sum_{n=0}^{\infty} z_n(x_{h,f} + x_{l,f})\frac{1 - r_f}{n+1} = \frac{1 - z_0(x_{h,f} + x_{l,f})}{x_{h,f} + x_{l,f}}(1 - r_f). \tag{9}$$

In a fixed price store both types earn the same expected utility because fixed pricing is egalitarian at the point of service. The mechanism does not screen out customers ex-post (i.e. at the point of transaction): If the list price is affordable then all customers have the same chance $1/n$ of acquiring the good. The auction mechanism, on the other hand, screens out customers ex-post by prioritizing who obtains the good according to their budgets.

In addition to ex-post screening, the list/reserve price may be used as an ex-ante screening device to prevent low types from shopping at a particular store. A seller who wishes to trade with high types only can do so by advertising a price above b.[5] If this is the case, then high type buyers'

[5] We assume that sellers can use cash bonds or financial disclosure requirements to implement ex-ante screening. Buyers pay up-front a sum equal to the reserve price to a third party. In case the buyer obtains the good, the deposit is transferred to the seller; otherwise it is returned to its owner at no cost. Such a practice prevents low types from showing up at unaffordable stores. A financial disclosure requirement is also effective.

expected utilities at such stores can be obtained by plugging $x_{l,a} = 0$ into (7) or $x_{l,f} = 0$ into (9).

Lemma 1 *Assuming $r_m \leq b$ we have $\frac{\partial U_{i,m}}{\partial r_m} < 0$, $\frac{\partial U_{i,m}}{\partial x_{h,m}} < 0$ and $\frac{\partial U_{i,m}}{\partial x_{l,m}} < 0$ for $m = a, f$ and $i = h, l$. If $r_m > b$ then, again, $\frac{\partial U_{h,m}}{\partial r_m}$ and $\frac{\partial U_{h,m}}{\partial x_{h,m}}$ are both negative.*

The Lemma says that buyers dislike expensive and crowded stores. The signs of the partial derivatives wrt r_m are obvious. For the ones wrt $x_{h,m}$ and $x_{l,m}$ note that a larger $x_{h,m}$ or $x_{l,m}$ shifts the probability mass from small to large demand realizations. Such a shift causes the expected utility to decline because customer are less likely to be served at stores with a large demand. For a formal proof see Camera and Selcuk (2009).

Let Ω_i denote the maximum expected utility ("market utility") a type i customer can obtain in the entire market.[6] For now we treat Ω_i as given, subsequently it will be determined endogenously. Consider an individual seller who advertises (m, r_m) and suppose that high and low type buyers respond to this advertisement with arrival rates $x_{h,m} \geq 0$ and $x_{l,m} \geq 0$. These rates satisfy

$$x_{i,m} \begin{cases} > 0 & \text{if} \quad U_{i,m}(r_m, x_{h,m}, x_{l,m}) = \Omega_i \\ = 0 & \text{if} \quad U_{i,m}(r_m, x_{h,m}, x_{l,m}) < \Omega_i \end{cases} . \tag{10}$$

In words, the tuple $(r_m, x_{h,m}, x_{l,m})$ must generate an expected utility of at least Ω_h for high type customers, else they will stay away ($x_{h,m} = 0$) and at least Ω_l for low type customers, else they will stay away ($x_{l,m} = 0$). The indifference condition holds on *and* off the equilibrium path, i.e. if a seller posts something no one else posts, his queue lengths are still determined by (10). Notice, however, Ω_i is not affected by unilateral deviations. The reason is that in a large economy the covariance of demand across stores vanishes; hence a change in the probability of visiting a particular store does not affect the distribution of demand at other stores. Peters (2000) provides micro-foundations of this argument.

Note that by definition $\Omega_i \geq U_{i,m}$. Furthermore, recall that $U_{h,a} > U_{l,a}$ and $U_{h,f} = U_{l,f}$; thus $\Omega_h \geq \Omega_l$. The indifference condition reveals a "law of demand" in that the expected demand $x_{i,m}$ decreases as the price r_m increases. In words, cheaper stores attract more customers and expensive stores attract fewer customers. To see why, note that $U_{i,m}(r_m, x_{h,m}, x_{l,m}) = \Omega_i$ implies

$$\frac{dx_{i,m}}{dr_m} = -\frac{\partial U_{i,m}/\partial r_m}{\partial U_{i,m}/\partial x_{i,m}} < 0.$$

The numerator and the denominator are both negative (Lemma 1); hence $dx_{i,m}/dr_m$ is negative, indicating that if the seller raises r then buyers respond by decreasing x. From a seller's point of view, raising the price brings in more revenue; however it lowers the expected demand. The seller's problem involves finding a balance between these two opposing effects, which we study next.

[6] The market utility approach greatly facilitates the characterization of equilibrium and, therefore, is standard in the directed search literature. For an extended discussion see Galenianos and Kircher (2012).

2.5 Sellers

Consider a seller who competes with mechanism m. His expected profit is given by

$$\Pi_m \left(r_m, x_{h,m}, x_{l,m}\right) = \sum_{n=1}^{\infty} z_n \left(x_{h,m} + x_{l,m}\right) \pi_m \left(n\right).$$

Clearly Π_m is a weighted sum of conditional payoffs π_m: with probability $z_n(\cdot)$ the seller gets n customers and the corresponding payoff associated with this scenario is $\pi_m(n)$. Again, start with auctions. If a single customer is present then the reserve price is charged, i.e. $\pi_a(1) = r_a$. If $n \geq 2$ then both auction formats yield the same $\pi_a(n)$, given by (5); hence

$$\Pi_a = z_1 \left(x_{h,a} + x_{l,a}\right) r_a + \sum_{n=2}^{\infty} z_n \left(x_{h,a} + x_{l,a}\right) \{[\theta^n + n\theta^{n-1}(1-\theta)]b + 1 - \theta^n - n\theta^{n-1}(1-\theta)\}.$$

After substituting for z_n and θ and simplifying we have

$$\begin{aligned} \Pi_a = \;& z_1 \left(x_{h,a} + x_{l,a}\right) r_a + 1 - z_0 \left(x_{h,a}\right) - z_1 \left(x_{h,a}\right) \\ & + b\{z_0 \left(x_{h,a}\right) + z_1 \left(x_{h,a}\right) - z_0 \left(x_{h,a} + x_{l,a}\right) - z_1 \left(x_{h,a} + x_{l,a}\right)\}. \end{aligned} \tag{11}$$

If the seller sets $r_a > b$, then his expected profit can be found by substituting $x_{l,a} = 0$ into the expression above. The expected profit of a fixed price seller is easier to calculate. We have

$$\pi_f(n) = r_f \text{ for } n \geq 1$$

hence

$$\Pi_f = \sum_{n=1}^{\infty} z_n \left(x_{h,f} + x_{l,f}\right) r_f = \{1 - z_0(x_{h,f} + x_{l,f})\} r_f. \tag{12}$$

The expression inside the curly brackets is the probability of getting at least one customer, high type or low type. In either case the seller charges the posted price r_f. Again if $r_f > b$, then his expected profit is obtained by letting $x_{l,f} = 0$. The equation below reveals the connection between a seller's expected profit and his customers' expected utilities.

Lemma 2 *The following relationship holds both for auctions as well as for fixed pricing:*

$$\Pi_m = 1 - z_0 \left(x_{h,m} + x_{l,m}\right) - x_{h,m} U_{h,m} - x_{l,m} U_{l,m}, \text{ where } m = a, f. \tag{13}$$

The expression $1 - z_0 \left(x_{h,m} + x_{l,m}\right)$ can be interpreted as the expected revenue. It is the value created by a sale (one), multiplied by the probability of trading. One can interpret $x_{h,m}U_{h,m} + x_{l,m}U_{l,m}$ as the expected cost. The seller promises a payoff $U_{h,m}$ to each high type and $U_{l,m}$ to each low type customer. On average he gets $x_{h,m}$ high type and $x_{l.m}$ low type customers; so the total cost equals to $x_{h,m}U_{h,m} + x_{l,m}U_{l,m}$. The profit Π_m is simply the difference between the revenue and the cost.

Each seller chooses a mechanism $m \in \mathbb{M}$ and a price $r_m \in [0,1]$ but realizes that expected demands $x_{h,m}$ and $x_{l,m}$ are determined via (10). So, a seller's problem is

$$\max_{m \in \mathbb{M},\ r_m \in [0,1],\ (x_{h,m}, x_{l,m}) \in \mathbb{R}_+^2} \Pi_m \left(r_m, x_{h,m}, x_{l,m}\right) \text{ subject to (10)} \tag{14}$$

Indifference conditions in (10) determine expected demands $x_{h,m}$ and $x_{l,m}$ as functions of the pricing rule m and the price r_m. Note that the seller faces two constraints—one for high type customers and one for low type customers—one of which must hold with equality. If both constraints bind, then the seller is able to attract both types of customers. If a single constraint binds then he attracts one type only. (The case where neither constraint binds, of course, is ruled out as it implies that the seller does not get any customer at all).

To close down the model, we need a feasibility condition to ensure that the weighted sum of expected demands across all sellers equals to the aggregate buyer-seller ratio. Sellers are free to pick between fixed pricing and auctions and similarly they are free to post any list/reserve price within the interval $[0,1]$. So let α_m denote the fraction of sellers opting for rule $m \in \mathbb{M}$ and let $\varphi_m(r_m)$ denote the fraction of sellers, among the ones who adopted rule m, posting $r_m \in [0,1]$. Recall that λ is the aggregate buyer-seller ratio and that σ is the fraction of low type buyers. Letting $\lambda_l \equiv \sigma\lambda$ and $\lambda_h \equiv (1-\sigma)\lambda$ we have

$$\sum_{m \in \mathbb{M}} \alpha_m \int_0^1 \varphi_m(r_m) x_{i,m}(r_m) dr_m = \lambda_i \text{ for } i = h, l. \tag{15}$$

Note that there are two conditions in (15), one for high types and one for low types.

3 Results

For reference, we record the outcome with no financial constraints.

Remark 1 *Suppose $b = 1$ i.e. suppose that buyers are homogenous. There exists a continuum of payoff equivalent equilibria where both mechanisms coexist. In any given equilibrium the expected demand at a store is λ and sellers earn*

$$\mu(\lambda) = 1 - z_0(\lambda) - z_1(\lambda) \tag{16}$$

no matter which rule they compete with whereas buyers earn $z_0(\lambda)$ no matter which seller's rule they join in. Sellers competing with fixed pricing post $r_f^* = \rho(\lambda)$, where

$$\rho(\lambda) = \frac{\mu(\lambda)}{1 - z_0(\lambda)}, \tag{17}$$

whereas the ones competing with auctions post $r_a^* = 0$.

For a formal proof see Eeckhout and Kircher (2010). Competition among sellers dictates that, irrespective of the mechanism a seller competes with, he must provide all customers with the same market utility, $z_0(\lambda)$, which is commensurate with the degree of competition in the market.[7] Any mechanism that is capable of such a surplus allocation may be adopted in equilibrium. Eeckhout and Kircher (2010) show that payoff equivalence and coexistence is not restricted to fixed pricing or auctions, rather it holds for a range of mechanisms they label as "payoff-complete". In an earlier paper Kultti (1999) proves a similar result for fixed pricing and second price auctions.

3.1 Outcomes when $\mathbb{M} = \{$fixed pricing$\}$ and $\mathbb{M} = \{$auctions$\}$

For a moment ignore auctions. Below we characterize possible outcomes when sellers compete via fixed pricing only.

Proposition 2 *Suppose $\mathbb{M} = \{$fixed pricing$\}$. If $b \geq \rho(\lambda)$ then the budget constraint is slack. All sellers advertise $\rho(\lambda)$ and serve both types of customers indiscriminately. If $\mu(\lambda_h) < b < \rho_f(\lambda)$ then a fraction of stores ("affordable stores") post b and serve low types, whereas remaining stores ("expensive stores") post above b and serve high types. High types avoid affordable stores as they are too crowded. Finally, if $b \leq \mu(\lambda_h)$ then low types are screened out completely: all sellers advertise $\rho(\lambda_h) > b$ and serve high types only. Each outcome described above is the unique equilibrium within its parameter region.*

Recall that $\rho(\lambda)$ is the equilibrium fixed price in the homogenous model. If $b \geq \rho(\lambda)$ then this price is affordable even to low types; hence the budget constraint is slack and the fixed price equilibrium with homogenous buyers remains intact. In the other extreme where b is severely low, sellers ignore low types altogether and target high types only. Doing so effectively reduces the buyer-seller ratio to λ_h, and per Remark 1, in such an outcome sellers earn $\mu(\lambda_h)$. If $b \leq \mu(\lambda_h)$ then no seller would deviate from this outcome by catering to low types, because doing so can at most bring in a revenue b which is less than $\mu(\lambda_h)$ anyway.

If b is moderate, i.e. if it is neither large enough to slacken the budget constraint nor small enough to justify ignoring low types altogether, then there exists a unique separating equilibrium where some stores are expensive and cater to high types while others are affordable and cater to low types. Affordable stores are too crowded and possess too much trade risk—the risk of not being able to purchase—hence high types avoid shopping at such stores. Instead they shop at expensive stores where the price is high but customers can relatively rest assured of being able to purchase. Now, we turn to auctions.

Lemma 3 *If $\mathbb{M} = \{$auctions$\}$ then each seller must set $r_a \leq b$ and cater to both types of customers.*

The Lemma says that with auctions, unlike with fixed pricing, low types are never screened out ex-ante; neither partially nor completely. This is true no matter how small the budget is or how

[7] The buyer-seller ratio λ, inversely, proxies the degree of market competition: the lower this ratio, the more competitive (from a seller's point of view) the market. Note that $z_0(\lambda)$ rises if λ falls, i.e. if the market is highly competitive then buyers expect to be rewarded with a high level of utility.

few low types are. The intuition is this. Practicing ex-ante screening (i.e. setting a reserve price above b) and catering exclusively to high types is viable only if doing so has some distinct appeal for high budget shoppers. With fixed pricing there is such an appeal: expensive stores are less crowded so they offer a much better prospect of buying the item. With auctions, though, this advantage disappears. In a bidding contest high types are not deterred by the presence of low types; they can outbid them (no matter how many low types may be present in the contest). So, from such a customer's point of view, whether or not an auction store is unaffordable to low types is immaterial, which is why no seller sets $r_a > b$.

Proposition 3 *If $\mathbb{M} = \{auctions\}$ then there exists a unique equilibrium where all sellers set the same reserve price $r_a^* = \min\{b, \hat{r}(\lambda_l)\}$, where $\hat{r}$ is given by (23). The expected demand at each store equals to $\lambda_h + \lambda_l = \lambda$.*

The equilibrium reserve price r_a^* is either interior or corner depending on the severity of the budget constraint. If b is sufficiently large then $\hat{r} < b$, so we have the interior solution where $r_a^* = \hat{r}$. Else we have the corner solution where $r_a^* = b$. One can verify that $d\hat{r}/db < 0$ and $d\hat{r}/d\lambda_l > 0$, i.e. the worse the budget constraint (low b and/or high λ_l) the higher the reserve price. In words, sellers offset the shortfall in profits caused by budget constraints through raising the reserve price. Observe that if $b = 1$ then $\hat{r} = 0$, which indeed corresponds to the equilibrium reserve price in a model with homogenous buyers (see Remark 1; see also Julien et al. (2000)). However, as soon as b falls below 1, the reserve price starts to grow beyond zero. The implication is that with auctions, unlike with fixed pricing, the budget constraint is never slack; as long as b is below 1 the outcome with homogenous customers ceases to exist.

3.2 Fixed Pricing or Auctions?

We now turn to the full-fledged model where sellers are free to choose between fixed pricing and auctions.

Lemma 4 *If $\mathbb{M} = \{fixed\ pricing,\ auctions\}$ then auction stores set the same reserve price $r_a = \min\{\hat{r}(x_{l,a}), b\}$ and cater to both types of customers. Fixed price stores, on the other hand, advertise $r_f = \min\{\rho(x_{h,f} + x_{l,f}), b\}$.*

The lemma does not prove if an auction equilibrium or a fixed price equilibrium exists; however it clarifies what list/reserve price sellers would post and what type of customers they would get if such equilibria were to exist. Furthermore, it establishes a symmetry result by showing that sellers trading with the same rule will post the same price. These results greatly facilitate the characterization of the equilibria.

Note that both r_a and r_f ought to be less than or equal to b. Recall that if $\mathbb{M} = \{\text{fixed pricing}\}$ then sellers would post $r_f > b$ and screen out low types if b was severely low. However, if $\mathbb{M} = \{\text{fixed pricing, auctions}\}$ then this is no longer the case. Indeed, the auction mechanism is already capable of screening customers ex-post and making sure that a high type wins the item. Since this tool is available, refusing to deal with low types up-front is suboptimal.

Proposition 4 *Suppose* $\mathbb{M} = \{$*fixed pricing, auctions*$\}$. *If* b *is sufficiently large then an equilibrium where all sellers compete with auctions, fails to exist.*

In an auction equilibrium with financially constrained buyers the market utilities are such that

$$\Omega_h > z_0(\lambda) > \Omega_l,$$

whereas in a homogenous setting with financially unconstrained buyers the market utilities satisfy

$$\Omega_h = \Omega_l = z_0(\lambda).$$

Absent budget constraints, competition among sellers dictates that both types of buyers, who have the same willingness to pay, ought to earn the same market utility $z_0(\lambda)$. This allocation cannot be achieved via auctions because the auction mechanism prioritizes high types over low types even when $b = 1 - \varepsilon$. This is why in an auction equilibrium high types' market utility Ω_h exceeds $z_0(\lambda)$, which in turn exceeds low types' market utility Ω_l. This inequality presents an opportunity to deviate to fixed pricing and attract a disproportionate number of low types. We show that such a deviation is feasible if low types have sufficiently large budgets.

In a similar setting, but with no budget constraints, McAfee (1993) shows that the unique equilibrium entails all sellers holding second price auctions with a suitable reserve price and buyers randomizing across stores. We show that this result is not robust to the presence of budget constrained buyers; indeed, merely lowering the budgets of a few buyers is enough to invalidate an (unconstrained) auction equilibrium.[8] Results pertaining payoff equivalence in Eeckhout and Kircher (2010) and Kultti (1999) disappear similarly in the face of budget constraints. We can now state the main result of the paper.

Proposition 5 *Suppose* $\mathbb{M} = \{$*fixed pricing, auctions*$\}$. *There are three possible outcomes taking place in mutually exclusive parameter regions:*
1. If $b \geq \rho(\lambda)$ *then all stores adopt fixed pricing and post the same list price* $r_f^* = \rho(\lambda)$. *The expected demand at each store is* $\lambda_h + \lambda_l = \lambda$.
2. If $b^{\#} < b < \rho(\lambda)$, *where the threshold* $b^{\#} > 0$ *is defined in the Appendix, fixed pricing and auctions coexist. Fixed price stores advertise* $r_f^* = b$ *and serve low types only. Auction stores set* $r_a^* = \min\{b, \hat{r}(x_{l,a}^*)\}$ *and serve both types of customers. High types avoid fixed price stores as they are too crowded, specifically* $x_{h,f}^* = 0$ *while* $x_{l,f}^* > \lambda > x_{l,a}^* + x_{h,a}^*$.
3. If $b \leq b^{\#}$ *then all stores adopt auctions and set the same reserve price* $r_a^* = \min\{b, \hat{r}(\lambda_l)\}$. *Each seller expects to get* $\lambda_h + \lambda_l = \lambda$ *customers.*

[8] The setup in McAfee (1993) has buyers who differ in their valuations. In our model, however, buyers have identical valuations; so to verify that our result is not a knife-edge case, we have carried out the following robustness check (available upon request). Suppose that a fraction of customers have valuation $v < 1$ and that some of the high value customers have low budgets. If buyers are sufficiently similar in terms of their valuations *and* budgets, then, again, an auction equilibrium fails to exist.

If $b \geq \rho(\lambda)$ then the budget constraint is slack and the fixed price equilibrium with homogenous buyers remains intact. Despite the presence of low budget types, sellers are still capable of providing all buyers the same market utility by adopting fixed pricing. In this parameter region auctions are competed away because, as mentioned above, the auction mechanism rewards buyers with different market utilities, which is not compatible with profit maximization in the absence of budget constraints (the budget constraint can be avoided via fixed pricing).

If, however, b falls below $\rho(\lambda)$ then the budget constraint starts to bind. In this region, serving customers indiscriminately (via fixed pricing) is no longer feasible as low types are unable to afford the equilibrium list price. So, sellers start to prioritize high types over low types by switching to auctions. The fraction of sellers adopting auctions rises as b decreases and if b falls below the threshold $b^{\#}$ then all sellers compete with auctions (The threshold $b^{\#}$ is the unique value of b satisfying equation (25) in the Appendix). See Figure 1a for an illustration.

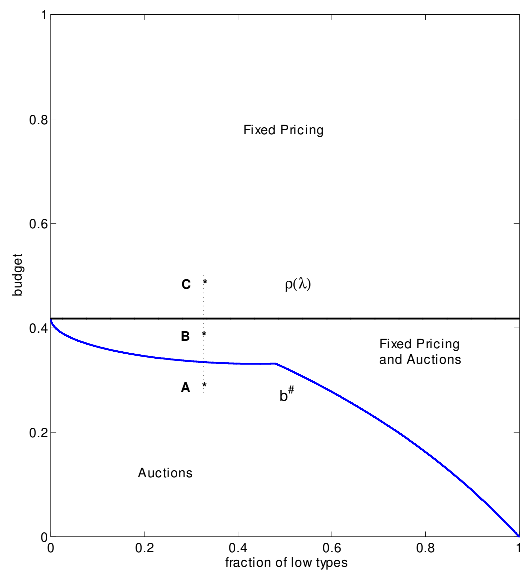

Figure 1a - Benchmark

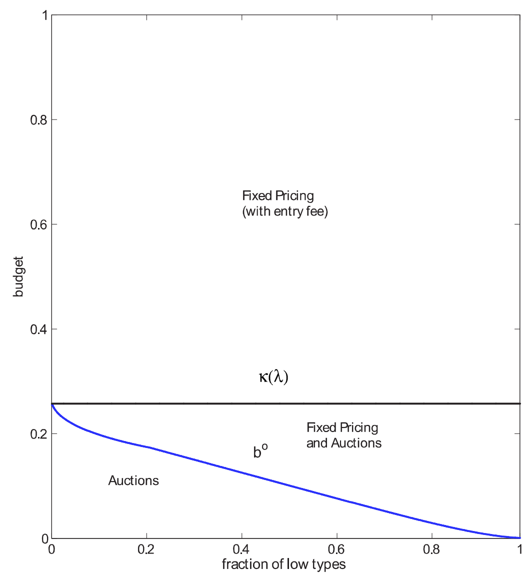

Figure 1b - Entry Fees

3.3 Constrained Efficiency

Consider a social planner whose objective is to maximize the total surplus while still being constrained with the same matching frictions in the decentralized economy (hence "constrained efficiency"). The planner can assign seller k with terms of trade $(m, r_m^k) \in \mathbb{M} \times [0,1]$ and queue lengths $(x_h^k, x_l^k) \in \mathbb{R}^2$ taking as given the demand distribution in (1). Recall that buyers are identical in their valuation of the good. If trade occurs at price $r \leq b$ the seller obtains payoff r while the buyer, no matter his type, obtains payoff $1 - r$; hence the total surplus at every meeting equals to 1. It follows that a sufficient condition to ensure that every meeting results in trade is not letting sellers to engage in ex-ante screening i.e. by keeping r_m^k below b.[9] Constrained efficiency is, then, equivalent to maximizing the total number of trades in the market. This is achieved by assigning each seller with the same total demand λ. To see why suppose, with some abuse of notation, that the measure of sellers is 1 and that the planner divides them into S equal groups assigning group k with queue lengths x_h^k and x_l^k. The planner solves

$$\max_{(x_h^k, x_l^k)\in\mathbb{R}^2} \sum_{k=1}^{S} \frac{1-z_0(x_h^k+x_l^k)}{S} \quad \text{s.t.} \quad \sum_{k=1}^{S} \frac{x_h^k}{S} = \lambda_h \quad \text{and} \quad \sum_{k=1}^{S} \frac{x_l^k}{S} = \lambda_l.$$

It is easy to show that the solution entails setting $x_h^k + x_l^k = \lambda$ for all k; i.e. an outcome is efficient if each store receives the same expected demand λ.

Remark 2 *The no-screening (fixed price) and the full-screening (auction) equilibria described in Proposition (5) are both constrained efficient since in either case each store receives the same expected demand λ. The partial screening equilibrium where both rules coexist, on the other hand, is inefficient as fixed price stores are too crowded whereas auction stores are too depleted (compared to the efficient level λ).*

Interestingly, if b is too large or too small then the equilibrium is efficient, but if b is moderate then the equilibrium is inefficient. The implication is that an improvement in the budget constraint—e.g. letting buyers pay in instalments—may result in efficiency loss. Consider, for instance, points A and B in Figure 1a. At point A the budget is severely low and the corresponding outcome is an auction equilibrium, which is efficient. Point B, with a slightly higher budget, lies in the partial screening territory, which is inefficient. Along the auction equilibrium at point A each store has the same number of customers; thus the number of matches forming and resulting in trade across the economy is as high as it can possibly be. Along the separating equilibrium at B some stores are too depleted while others are too crowded, which means that fewer number of matches are formed and fewer number of trades are created. Clearly, the improvement of the budget from A to B causes the number of trades to fall, and leads to a loss of efficiency.[10]

4 Entry Fees

In this section we extend the benchmark model by letting sellers charge an entry fee. Entry fees can be used in conjunction with fixed pricing as well as with auctions. The case with fixed pricing is considerably simpler to analyze and it delivers the basic intuition. So, given the space limit, we take the following approach: We analyze a version of the model where fixed price sellers, but not auction sellers, may ask for an entry fee. We characterize the outcome of this version and show that

[9] This is not a necessary condition. One can achieve efficiency with ex-ante screening, provided that high types are instructed to shop at expensive stores and low types are instructed to shop at cheap stores and each store receives the same *total* demand λ.

[10] It is easy to produce an example with the opposite conclusion: Point C lies in the fixed pricing zone, which is efficient. Moving from B to C improves efficiency.

the budget constraint becomes less binding. Then we discuss what would happen if one uses entry fees in conjunction with auctions.

To start, suppose that fixed price sellers, in addition to the list price r_f, may ask for an entry fee ϕ_f. For a moment ignore auctions, i.e. suppose $\mathbb{M} = \{\text{fixed pricing with entry fee}\}$. We will characterize the outcome when the budget constraint is slack and show that in order to avoid the budget constraint as much as possible, sellers should set the list price $r_f = 0$ and raise the revenue entirely from entry fees.

If the budget constraint is slack, then all buyers are served indiscriminately. Letting $x_f \equiv x_{h,f} + x_{l,f}$, the expected utility of a buyer (high type or low type) is given by

$$U_f = \frac{1 - z_0(x_f)}{x_f}(1 - r_f) - \phi_f.$$

The first part of U_f is the same as the expected utility in the benchmark model, given by (9), but now the buyer has to pay a fee, so we subtract ϕ_f. Note that the fee ϕ_f is paid whether or not the buyer is able to purchase the item. The list price r_f, however, is paid only if the buyer is selected to purchase the item. The expected profit of a seller is given by

$$\Pi_f = \sum_{n=1}^{\infty} z_n(x_f)\{n\phi_f + r_f\} = \{1 - z_0(x_f)\}r_f + x_f\phi_f.$$

The expression $\{1 - z_0(x_f)\}r_f$ is the expected profit in the benchmark, given by (12). With regard to fees, on average, the seller gets x_f customers and charges each one of them ϕ_f, so we add $x_f\phi_f$. Even though U_f and Π_f have different expressions than before, the relationship

$$\Pi_f = 1 - z_0(x_f) - x_f U_f$$

still holds. The queue length x_f, on and off the equilibrium path, is determined via the indifference condition: $x_f > 0$ if $U_f = \Omega$ else $x_f = 0$. The seller solves

$$\max_{x_f \in \mathbb{R}_+} 1 - z_0(x_f) - x_f\Omega.$$

The first order is given by $z_0(x_f) = \Omega$. Solving $U_f = z_0(x_f)$ for ϕ_f and r_f, we have

$$\phi_f + \frac{1 - z_0(x_f)}{x_f} r_f = \frac{1 - z_0(x_f) - z_1(x_f)}{x_f}. \tag{18}$$

There is a continuum of pairs (ϕ_f, r_f) satisfying this equation. One such pair is $\phi_f = 0$ and $r_f = \rho(x_f)$, which corresponds to the solution in the benchmark with no entry fee. Notice however, one can avoid the budget constraint as much as possible by setting $r_f = 0$ and $\phi_f = \kappa(x_f)$, where

$$\kappa(x_f) = \frac{1 - z_0(x_f) - z_1(x_f)}{x_f}. \tag{19}$$

To see why, note that the budget constraint is slack if

$$r_f + \phi_f \leq b. \tag{20}$$

In the budget constraint (20) the list price r_f and the entry fee ϕ_f have identical weights, 1; however in the FOC (18) the weight of r_f is smaller than the weight of ϕ_f; indeed $\frac{1-z_0(x_f)}{x_f} < 1$. Since the seller is indifferent between raising \$1 via either channel (list price or entry fee); the optimal way of avoiding the budget constraint is setting the list price $r_f = 0$ and raising the revenue entirely from ϕ_f. So, WLOG we will focus on this scenario.

Given that $r_f = 0$, it is straightforward to show that all sellers set the same entry fee $\phi_f = \kappa(x_f)$ and therefore get the same expected demand $x_f = \lambda$; so, the equilibrium entry fee is $\phi_f^* = \kappa(\lambda)$. Substituting this into the payoff functions above, we see that equilibrium payoffs for buyers and sellers are, respectively, $z_0(\lambda)$ and $\mu(\lambda)$, which are the same as in the equilibrium with homogenous buyers (Remark 1). One can sustain this outcome if the budget constraint is slack, i.e. if

$$\kappa(\lambda) \leq b.$$

Recall that in the benchmark model with no entry fee the budget constraint was slack if $\rho(\lambda) \leq b$ (Proposition 2). Since $\kappa(\lambda) < \rho(\lambda)$ it is clear that entry fees make the budget constraint less binding and enlarge the parameter space where the outcome with homogenous, financially unconstrained buyers remains intact. Now we can turn to the full-fledged model.

Proposition 6 *Suppose* $\mathbb{M} = \{$*fixed pricing with entry fee, auctions*$\}$. *If* $b \geq \kappa(\lambda)$ *then all sellers adopt fixed pricing and charge each customer* $\phi_f^* = \kappa(\lambda)$. *If* $b^\circ < b < \kappa(\lambda)$, *where* b° *is defined below, auctions and fixed pricing coexist. Auction stores set* $r_a^* = \min\{b, \hat{r}(x_{l,a}^*)\}$ *and serve both types of customers whereas fixed price stores set* $\phi_f^* = b$ *and serve low types only. If* $b \leq b^\circ$ *then all sellers adopt auctions and set the same reserve price* $r_a^* = \min\{b, \hat{r}(\lambda_l)\}$.

The proposition is practically the same as Proposition 5 only with new thresholds (see Fig 1b). The nature of the equilibria remains the same (i.e. if b is large then sellers compete with fixed pricing, if it is low then they choose auctions and if it is moderate then they mix); however the budget constraint is now less pronounced. Indeed a comparison between Figure 1a and 1b reveals that $\kappa < \rho$ and $b^\circ < b^\#$, i.e. entry fees shrink the parameter space where the budget constraint kicks in.

The proof of the proposition is largely the same as before. The difference is that we need to work with the new expected payoffs when dealing with fixed pricing; specifically instead of (9) we have

$$U_{h,f} = U_{l,f} = -\phi_f + \frac{1 - z_0(x_{h,f} + x_{l,f})}{x_{h,f} + x_{l,f}}$$

and instead of (12) we have

$$\Pi_f = (x_{h,f} + x_{l,f})\,\phi_f.$$

Consequently we end up with different thresholds: $\kappa(\cdot)$ replaces $\rho(\cdot)$, b° replaces $b^\#$, and $x_{l,f}^\circ$

replaces $x_{l,f}^{\#}$ where $x_{l,f}^{\circ}$ solves $U_{l,f}(b,0,x_{l,f}^{\circ}) = U_{l,a}(r_a,\lambda_h,\lambda_l)$ and b° is the unique value of b satisfying $\Pi_f(b,0,x_{l,f}^{\circ}) = \Pi_a(r_a,\lambda_h,\lambda_l)$. With these modifications one can prove the proposition by repeating the previous proof almost step by step. (The proof, which is available upon request, is too repetitive hence it is omitted.)

The next question is what happens if sellers charge an entry fee with auctions. This case is somewhat tedious because entry fees affect not only how sellers compete but also how buyers bid, rendering expected utilities nontrivial. Given the space limit, we do not undertake this task, however based on the analysis so far we can make an educated guess on what would happen. The reason why auctions are competed away against fixed pricing is the fact that if b is sufficiently large then fixed pricing can avoid the budget constraint and provide all buyers the same market utility, but auctions cannot. With auctions high types are always prioritized over low types, that is $U_{h,a} > U_{l,a}$. With entry fees, expected payoffs $U_{h,a}$ and $U_{l,a}$ will have different closed form expressions but the inequality $U_{h,a} > U_{l,a}$ will remain.[11] Indeed, in a bidding contest high types inevitably will have an edge over low types and, as long as buyers pay the same entry fee, the inequality $U_{h,a} > U_{l,a}$ will persist. Consequently, we suspect, an unconstrained auction equilibrium will still fail to exist if b is sufficiently large.

5 Conclusion

In markets for most big ticket items (houses, automobiles, furniture, business equipment, etc.) a significant number of potential buyers are budget constrained. Despite its practical importance, the competing mechanism literature paid little attention on how the presence of financially constrained buyers affects trading mechanism selection. Indeed, as mentioned in the introduction, to the best of our knowledge, this paper is the first attempt investigating this problem in a fully competitive setup. Absent budget constraints, the existing literature capitulates that if buyers differ in their valuations then the unique equilibrium entails all sellers holding second price auctions (McAfee, 1993) whereas if buyers have identical valuations then a range of mechanisms are payoff equivalent and coexist in the same market (Eeckhout and Kircher, 2010). We show that these results are not robust to the presence of budget constrained buyers. Merely lowering the budgets of a few buyers renders the auction equilibrium as well as payoff equivalence results invalid.

Restriction attention to fixed pricing and auctions we fully characterize competitive search equilibria where sellers compete for scarce customers, some of which are budget constrained, and show that if buyers differ only slightly in terms of their ability to pay then sellers adopt fixed pricing

[11] Consider a second price auction, where an entry fee $\phi_a(n)$ is payable if n customers show up at the store. The fee ought to be indexed by, and in fact, falling in n because with a rising fee, or even a flat fee, if n is large then not all bidders would participate. One can show that in the unique symmetric pure strategy equilibrium of the second price auction low types bid $b - \phi_a(n)$ and high types bid $1 - \phi_a(n)$. Given these bidding strategies, the expected payoff for a buyer, conditional on being in a store with $n \geq 2$ buyers (including the buyer himself), are given by

$$u_{l,a}(n) = \frac{\theta^{n-1}}{n}(1-b) - \phi_a(n)\left[1 - \frac{\theta^{n-1}}{n}\right] \text{ and } u_{h,a}(n) = \theta^{n-1}(1-b) - \phi_a(n)\left[1 - \frac{1-\theta^n - n\theta^{n-1}(1-\theta)}{n(1-\theta)}\right].$$

Observe that for all $n \geq 2$ we have $u_{l,a}(n) < u_{h,a}(n)$. This inequality implies that $U_{l,a} < U_{h,a}$ because $U_{l,a}$ is a weighted sum of $u_{l,a}(n)$s and $U_{h,a}$ is a weighted sum of $u_{h,a}(n)$s.

whereas if they differ too much then they adopt auctions. In between these two extremes there is an intermediate region where they mix (i.e. both mechanisms coexist). A natural extension of the model is where sellers may ask for an entry fee. We show that with entry fees the nature of the equilibria remains the same—sellers choose fixed pricing if the budget is large, they choose auctions if the budget is low and they mix if it is moderate—but the thresholds are smaller than before; hence the budget constraint becomes less binding. Indeed, with entry fees sellers are able to raise the revenue from all buyers present at the store—not just from the one acquiring the item—which makes the item more affordable.

The result that entry fees make the budget constraint less binding is promising, but even then if b is low enough the unconstrained equilibrium still fails to exist. So, one deduces that no matter which mechanism sellers compete with there will be a lower bound on the budget b, below which the unconstrained equilibrium will fail to exist. An open question, then, is how to obtain that lower bound and how to design the optimal mechanism achieving that bound. Answering this question, of course, requires a more general setup than ours. Specifically the set of mechanisms should be large enough to encompass a wide range of selling rules, not just fixed pricing and auctions.

References

Baye, Michael R.; Dan Kovenock; and Casper G. de Vries (1996): The all-pay auction with complete information. *Economic Theory*, 8(2).

Burdett, Kenneth; Shouyong Shi; and Randall Wright (2001): Pricing and Matching with Frictions. *The Journal of Political Economy*, 109(5).

Burkett, Justin (2015): Endogenous budget constraints in auctions. *Journal of Economic Theory*, 158, Part A.

Camera, Gabriele and Cemil Selcuk (2009): Price Dispersion with Directed Search. *Journal of the European Economic Association*, 7(6).

Che, Yeon Koo and Ian Gale (1996): Expected revenue of all-pay auctions and first-price sealed-bid auctions with budget constraints. *Economics Letters*, 50(3).

Che, Yeon Koo and Ian Gale (1998): Standard Auctions with Financially Constrained Bidders. *Review of Economic Studies*, 65(1).

Che, Yeon Koo and Ian Gale (2000): The Optimal Mechanism for Selling to a Budget-Constrained Buyer. *Journal of Economic Theory*, 92(2).

Eeckhout, Jan and Philipp Kircher (2010): Sorting versus screening: Search frictions and competing mechanisms. *Journal of Economic Theory*, 145(4).

Galenianos, Manolis and Philipp Kircher (2012): On the Game-Theoretic Foundations of Competitive Search Equilibrium. *International Economic Review*, 53.

Geromichalos, Athanasios (Nov 2012): Directed search and optimal production. *Journal of Economic Theory*, 147(6).

Grimmett, Geoffrey and Dominic Welsh (1986): *Probability: An Introduction*. Oxford Science Publications.

Hafalir, Isa E.; R. Ravi; and Amin Sayedi (2012): A near Pareto optimal auction with budget constraints. *Games and Economic Behavior*, 74(2).

Hillman, Arye and Dov Samet (1987): Dissipation of contestable rents by small numbers of contenders. *Public Choice*, 54(1).

Julien, Benoit; John Kennes; and Ian King (2000): Bidding for Labor. *Review of Economic Dynamics*, 3(4).

Kotowski, Maciej (Jun 2015): First-Price Auctions with Budget Constraints. *Working Paper*.

Kultti, Klaus (1999): Equivalence of Auctions and Posted Prices,. *Games and Economic Behavior*, 27(1).

McAfee, R. Preston (1993): Mechanism Design by Competing Sellers. *Econometrica*, 61(6).

Peters, Michael (2000): Limits of Exact Equilibria for Capacity Constrained Sellers with Costly Search. *Journal of Economic Theory*, 95(2).

Selcuk, Cemil (May 2012): Trading mechanism selection with directed search when buyers are risk averse. *Economics Letters*, 115(2):207–210.

Shimer, Robert (2005): The Assignment of Workers to Jobs in an Economy with Coordination Frictions. *Journal of Political Economy*, 113(5).

Virag, Gabor (2011): High profit equilibria in directed search models. *Games and Economic Behavior*, 71(1).

Zheng, Charles Z. (2001): High Bids and Broke Winners. *Journal of Economic Theory*, 100(1).

Appendix

Proof of the Proposition 1. We focus on symmetric mixed strategies where type i bidders pick the same cdf $G_i(p) : [\underline{s}_i, \bar{s}_i] \to [0,1]$. A point p is an *increasing point* of the distribution function G_i if G_i is not constant in an ε neighborhood of p, i.e. if for each $\varepsilon > 0$ the probability of having a value in $(p-\varepsilon, p+\varepsilon)$ is positive. Conversely, p is a *constant point* if G_i is constant in an ε neighborhood of p. If there is an atom at p, then by definition, p is an increasing point. If the pair (G_h, G_l) corresponds to an equilibrium, then at each increasing point p of G_h high type buyers earn their equilibrium payoff u_h. Similarly at each increasing point p of G_l low types earn their equilibrium payoff u_l (Hillman and Samet, 1987; Baye et al., 1996). In words, in a mixed strategy equilibrium, players must make their equilibrium payoffs at all increasing points (including mass points) in the support of their equilibrium distribution functions. These claims are immediate from the definition of a mixed strategy equilibrium; Hillman and Samet (1987) provide a more formal discussion (see Proposition 2 therein).

We argue that $u_h \geq \theta^{n-1}(1-b)$ and $u_l \geq \frac{\theta^{n-1}}{n}(1-b)$, where θ is the probability that a customer is a low type. To see why, note that low types cannot bid more than b; so if a high type buyer bids slightly more than b, then even if he loses against all other high types, he can still win the item with probability θ^{n-1}. This would provide him a payoff $\theta^{n-1}(1-b)$. Similarly if a low type bids b, then in the worst case scenario he loses against all high types and ties with every other low type, so his payoff is at least $\frac{\theta^{n-1}}{n}(1-b)$. Since $u_h \geq \theta^{n-1}(1-b)$, a high type would not bid more than $1-\theta^{n-1}(1-b)$, i.e. $\bar{s}_h \leq 1-\theta^{n-1}(1-b)$.

The equilibrium cdf G_h cannot have a mass point anywhere on its support $[\underline{s}_h, \bar{s}_h]$. Indeed a mass point means tying with other bidders, in which case random rationing takes place and the surplus is divided among the tying bidders. A financially unconstrained bidder can always beat the tie and improve his payoff by placing a bid that is slightly above the mass point, which, of course is inconsistent with equilibrium. The argument applies to the entire support of G_h including the upper bound $\bar{s}_h$: since $\bar{s}_h$ is less than 1 there is room to beat a potential tie at $\bar{s}_h$. It follows that G_h is continuos on its support with no jumps (it does not have flat spots either, but more on this below).

Now turn to G_l. The equilibrium cdf G_l cannot have an atom anywhere below b for the same reason above, however it may have an atom at b as low types cannot bid more than b, so there are three possibilities: (i) the entire mass is at b or (ii) there is a partial mass at b, so G_l has a continuos tail in some region below b (iii) there is no mass at b, so G_l is continuos on its entire support. Below we rule out the second and the third scenarios, which means that the only possible scenario is the first one where low types bid b with probability one.

In scenarios (ii) and (iii) G_l is assumed to be continuos over some interval $[\underline{s}_l, \bar{s}_l]$ where $\underline{s}_l < \bar{s}_l \leq b$. Recall that G_h is also continuos over $[\underline{s}_h, \bar{s}_h]$, so there are three possible scenarios pertaining the lower bounds: either $\underline{s}_l < \underline{s}_h$ or $\underline{s}_l > \underline{s}_h$ or $\underline{s}_l = \underline{s}_h$. Suppose $\underline{s}_l < \underline{s}_h$. This implies $u_l = 0$. To see why, note that $\underline{s}_l$ is an increasing point of G_l and in a mixed strategy equilibrium any increasing point, including $\underline{s}_l$, must yield the equilibrium payoff u_l to the low type bidder. Bidding $p = \underline{s}_l$ yields a zero payoff because the bidder is sure to lose (everyone else is sure to bid more than $\underline{s}_l$ since $\underline{s}_l < \underline{s}_h$); a contradiction since $u_l > 0$. Now suppose $\underline{s}_l > \underline{s}_h$. Based on similar arguments, this implies $u_h = 0$; again a contradiction since $u_h > 0$. Finally if $\underline{s}_l = \underline{s}_h$ then $u_h = u_l = 0$, which again is a contradiction. In words, if both cdfs have continuos pieces, then the one with the lower bound on the far left is sure to yield a zero payoff. It follows that G_l cannot have a continuos bit anywhere below b; the entire mass must be at point b i.e. low types must bid b with probability 1.

Since low types bid b, the lower bound of G_h must lie above b, i.e. G_h must be spread over some interval $[\underline{s}_h, \bar{s}_h]$ where $\underline{s}_h \geq b$. Indeed if $\underline{s}_h < b$ then, by the arguments above, $u_h = 0$, which would contradict $u_h > 0$. Since low types bid b and high types are sure to bid more than b (recall that $\underline{s}_h \geq b$) the equilibrium payoff of a low type equals to $u_l = \frac{\theta^{n-1}}{n}(1-b)$.

Now turn to high types. As discussed earlier, G_h cannot have an atom, i.e. there are no jumps. We now argue that it cannot have intermittent flat spots either. By contradiction, suppose G_h is constant over some interval $(a_1, a_2) \subset [\underline{s}_h, \bar{s}_h]$. Both a_1 and a_2 are increasing points of G_h, hence both points must deliver the same payoff u_h. Since G_h is assumed to be flat over the interval (a_1, a_2), the probability of winning the auction is the same at both points (notice that $G_l = 1$ at both points because $\underline{s}_h \geq b$). This, however, means that the player gets a lower payoff by bidding a_2 than bidding a_1 since $a_2 > a_1$; a contradiction.

It follows that G_h is monotonically increasing on its support $[\underline{s}_h, \bar{s}_h]$; so, a high type buyer must earn the same payoff u_h at any point $p \in [\underline{s}_h, \bar{s}_h]$. The expected payoff associated with bidding p is given by

$$EU(p) = \sum_{j=0}^{n-1} \binom{n-1}{j} \theta^j (1-\theta)^{n-1-j} G_h^{n-1-j}(p)(1-p).$$

To understand why, note that in addition to the high type buyer in question, there are $n-1$ *other* buyers in the store. The buyer knows only his own type, however, given the Poisson arrival rates, he understands that with probability $\binom{n-1}{j}\theta^j (1-\theta)^{n-1-j}$ there are j low types and $n-1-j$ high types present in the auction (excluding himself). Consequently, his bid $p \in [\underline{s}_h, \bar{s}_h]$ wins the auction with probability $G_h^{n-1-j}(p)$ (observe that, since $\underline{s}_h \geq b$, low types are outbid with probability one). The binomial theorem implies that

$$EU(p) = [\theta + (1-\theta) G_h(p)]^{n-1}(1-p).$$

High types must earn their equilibrium payoff u_h at any point $p \in [\underline{s}_h, \bar{s}_h]$, i.e.

$$EU(p) = u_h \Rightarrow G_h(p) = \frac{\left(\frac{u_h}{1-p}\right)^{\frac{1}{n-1}} - \theta}{1-\theta}.$$

We know $G_h(\underline{s}_h) = 0$ and $G_h(\bar{s}_h) = 1$, hence $u_h = \theta^{n-1}(1-\underline{s}_h)$ and $u_h = 1 - \bar{s}_h$. Recall that $u_h \geq \theta^{n-1}(1-b)$ and that $\underline{s}_h \geq b$. This means $\underline{s}_h = b$ and $u_h = \theta^{n-1}(1-b)$ and therefore $\bar{s}_h = 1 - \theta^{n-1}(1-b)$. Substituting for u_h yields the expression of G_h in the body of the Proposition.

Proof of Lemma 2. Start with fixed pricing. Note that

$$x_{h,f}U_{h,f} + x_{l,f}U_{l,f} = 1 - z_0(x_{h,f} + x_{l,f}) - \{1 - z_0(x_{h,f} + x_{l,f})\}r_f,$$

where $U_{h,f}$ and $U_{l,f}$ are given by (9). Recall that $\Pi_f = \{1 - z_0(x_{h,f} + x_{l,f})\}r_f$; hence

$$\Pi_f = 1 - z_0\left(x_{h,f} + x_{l,f}\right) - x_{h,f}U_{h,f} - x_{l,f}U_{l,f}.$$

Note that if $r_f > b$ then $x_{l,f} = 0$ and the relationship still holds. Now consider auctions. Note that $z_1(x) = xz_0(x)$ and $z_0(x+y) = z_0(x)\,z_0(y)$. It follows that

$$\begin{aligned} x_{h,a}U_{h,a} + x_{l,a}U_{l,a} \quad &= \quad z_0\left(x_{h,a}\right) + z_1\left(x_{h,a}\right) - z_0\left(x_{h,a} + x_{l,a}\right) - z_1\left(x_{h,a} + x_{l,a}\right) r_a + \\ &\quad -b\left\{z_0\left(x_{h,a}\right) + z_1\left(x_{h,a}\right) - z_0\left(x_{h,a} + x_{l,a}\right) - z_1\left(x_{h,a} + x_{l,a}\right)\right\}, \end{aligned}$$

where $U_{h,a}$ and $U_{l,a}$ are given by (7) and (8). The expected profit Π_a is given by (11). A term by term comparison reveals that

$$\Pi_a = 1 - z_0\left(x_{h,a} + x_{l,a}\right) - x_{h,a}U_{h,a} - x_{l,a}U_{l,a},$$

confirming the validity of the relationship under auctions. Again if $r_a > b$ then $x_{l,a} = 0$ and the relationship is still valid.

Proof of Proposition 2. If $b \geq \rho(\lambda)$ then the fixed price equilibrium in the homogenous model (described in Remark 1) remains intact, because the equilibrium list price $\rho(\lambda)$ is less than b. The equilibrium is unique, because, as we show below, if $b \geq \rho(\lambda)$ then a separating equilibrium, where different sellers cater to different types of customers, fails to exist.

If $b \leq \mu(\lambda_h)$, then low types are screened out completely. To see why, ignore low types for a moment i.e. suppose that sellers target high types only. This reduces the buyer-seller ratio to λ_h. Since sellers face financially unconstrained buyers, per Remark 1, they post $\rho(\lambda_h)$ and consequently earn $\mu(\lambda_h)$. If b is too small then this outcome remains an equilibrium even if low types are present in the market. Indeed if $b \leq \mu(\lambda_h)$ then no seller would be willing to deviate and cater to low types, because doing so can at most bring in a revenue b which is less than $\mu(\lambda_h)$. Uniqueness, again, follows from the fact that (established below) if $b \leq \mu(\lambda_h)$ then a separating equilibrium fails to exits.

Now, suppose $\mu(\lambda_h) < b < \rho(\lambda)$. In this region b is neither large enough to slacken the budget constraint nor small enough to justify ignoring low types altogether. The only possible outcome is a separating equilibrium where a fraction of sellers are expensive, catering to high types, while remaining sellers are affordable catering to both types (as it turns out, high types avoid shopping at these stores). Below we characterize this equilibrium. To start, focus on an expensive store with $r_f > b$ and let $x_{h,f}$ be his expected demand consisting of high type buyers (clearly $x_{l,f} = 0$). The seller's expected profit equals to

$$\Pi_f = 1 - z_0(x_{h,f}) - x_{h,f} U_f\left(r_f, x_{h,f}, 0\right),$$

where

$$U_f\left(r_f, x_{h,f}, 0\right) = \frac{1 - z_0(x_{h,f})}{x_{h,f}}(1 - r_f).$$

The expected demand $x_{h,f}$ satisfies the indifference constraint $U_f(r_f, x_{h,f}, 0) = \Omega_h$. Substituting the constraint into the objective function, the seller solves

$$\max_{x_{h,f}} 1 - z_0(x_{h,f}) - x_{h,f}\Omega_h.$$

The FOC is given by $z_0(x_{h,f}) = \Omega_h$. The second order condition is negative; hence the solution to the FOC corresponds to the global maximum. Solving $U_f\left(r_f, x_{h,f}, 0\right) = z_0(x_{h,f})$ for r_f yields $r_f = \rho(x_{h,f})$. Substituting this into Π_f reveals that $\Pi_f = \mu\left(x_{h,f}\right)$. Note that all expensive sellers post the same list price. To see why, consider another expensive seller with price $r'_f > b$. His problem is similar: the FOC of his profit maximization problem is given by $z_0(x'_{h,f}) = \Omega_h$ and therefore $r'_f = \rho(x'_{h,f})$. Since both $z_0(x'_{h,f})$ and $z_0(x_{h,f})$ are equal to Ω_h, we have $x_{h,f} = x'_{h,f}$. Note that ρ is one-to-one, hence $r_f = r'_f$.

Now consider an affordable store with the list price $\tilde{r}_f \leq b$ and let $\tilde{x}_{h,f}$ and $\tilde{x}_{l,f}$ be the queue lengths. We conjecture that (to be verified) affordable stores attract low types only, i.e. $\tilde{x}_{h,f} = 0$ and $\tilde{x}_{l,f} > 0$ which means $U_f\left(\tilde{r}_f, 0, \tilde{x}_{l,f}\right) = \Omega_l < \Omega_h$. Recall that $U_f\left(r_f, x_{h,f}, 0\right) = \Omega_h$. It follows that $U_f\left(\tilde{r}_f, 0, \tilde{x}_{l,f}\right) < U_f\left(r_f, x_{h,f}, 0\right)$ i.e.

$$\frac{1 - z_0(\tilde{x}_{l,f})}{\tilde{x}_{l,f}}(1 - \tilde{r}_f) < \frac{1 - z_0(x_{h,f})}{x_{h,f}}(1 - r_f).$$

Since $r_f > \tilde{r}_f$ we have $\tilde{x}_{l,f} > x_{h,f}$, i.e. affordable stores are more crowded than expensive stores. We now show that $\tilde{r}_f = b$. The affordable store's problem is

$$\max_{\tilde{x}_{l,f}} 1 - z_0(\tilde{x}_{l,f}) - \tilde{x}_{l,f}\Omega_l.$$

Differentiating wrt $\tilde{x}_{l,f}$ yields the first order condition $\Omega_l = z_0\left(\tilde{x}_{l,f}\right)$. Notice, however, the FOC cannot hold with equality, i.e. $\Omega_l \neq z_0(\tilde{x}_{l,f})$. Indeed, if $\Omega_l = z_0(\tilde{x}_{l,f})$ then $U\left(\tilde{r}_f, 0, \tilde{x}_{l,f}\right) = z_0(\tilde{x}_{l,f})$ which, in turn, implies that $\tilde{r}_f = \rho(\tilde{x}_{l,f})$. Recall that expensive stores post $r_f = \rho(x_{h,f})$. Since $\tilde{x}_{l,f} > x_{h,f}$ we have $\rho(\tilde{x}_{l,f}) > \rho(x_{h,f})$ i.e. $\tilde{r}_f > r_f$; a contradiction because r_f must exceed $\tilde{r}_f$. From the affordable seller's point of view, setting $\tilde{r}_f = \rho(\tilde{x}_{l,f}) > b$ maximizes the expected profit, however this interior solution is outside of the feasible region $[0, b]$. Global concavity of the objective function implies that the seller ought to post the corner price $\tilde{r}_f = b$. This argument applies to all affordable stores; hence all such stores post $\tilde{r}_f = b$ and earn $\tilde{\Pi}_f = \{1 - z_0\left(\tilde{x}_{l,f}\right)\}b$.

Since all expensive stores post the same list price r_f and all affordable stores post the same list price b the feasibility equations in (15) become

$$\varphi\tilde{x}_{h,f} + (1-\varphi)\, x_{h,f} = \lambda_h \quad \text{and} \quad \varphi\tilde{x}_{l,f} + (1-\varphi)\, x_{l,f} = \lambda_l,$$

where φ denotes the fraction of affordable stores. Substituting $\tilde{x}_{h,f} = 0$ and $x_{l,f} = 0$ we have

$$x^*_{h,f} = \lambda_h / (1-\varphi) \quad \text{and} \quad \tilde{x}^*_{l,f} = \lambda_l/\varphi. \tag{21}$$

Recall that $\tilde{x}_{l,f} > x_{h,f}$; so we must have $\varphi < \lambda_l/\lambda = \sigma$. The value of φ is pinned down by the equal profit condition $\Pi_f = \tilde{\Pi}_f$, where

$$\Pi_f = \mu\left(\lambda_h / (1-\varphi)\right) \quad \text{and} \quad \tilde{\Pi}_f = \{1 - z_0\left(\tilde{x}_{l,f}\right)\}b.$$

Note that Π_f rises whereas $\tilde{\Pi}_f$ falls in φ. Letting $\Delta \equiv \Pi_f - \tilde{\Pi}_f$, note that $d\Delta/d\varphi > 0$ and

$$\Delta(0) = \mu(\lambda_h) - b \quad \text{and} \quad \Delta(\sigma) = \mu(\lambda) - \{1 - z_0(\lambda)\}b.$$

It follows that if $\mu(\lambda_h) < b < \rho(\lambda)$ then, by the Intermediate Value Theorem, there exists a unique $\varphi^* \in (0, \sigma)$ satisfying $\Pi_f = \tilde{\Pi}_f$, i.e. there exists a unique separating equilibrium. Note that outside this parameter region a separating equilibrium fails to exist. To complete the proof, we need to show that $\Omega_l < \Omega_h$, which verifies our earlier conjecture that high types indeed stay away from affordable stores. Note that

$$\Omega_h = z_0(x^*_{h,f}) \ \text{ and } \ \Omega_l = \frac{1 - z_0(x^*_{h,f})}{\tilde{x}^*_{l,f}}(1-b).$$

The fact that $\Pi_f = \tilde{\Pi}_f$ implies $b = \mu(x^*_{h,f})/(1 - z_0(\tilde{x}^*_{l,f}))$; thus

$$\Omega_h - \Omega_l = \frac{z_0(x^*_{h,f})}{\tilde{x}^*_{l,f}}\{q - 1 + e^{-q}\}, \ \text{ where } q \equiv \tilde{x}^*_{l,f} - x^*_{h,f}.$$

Note that q is positive because $\tilde{x}^*_{l,f} > x^*_{h,f}$. The expression in curly brackets is positive for all $q > 0$; hence $\Omega_h > \Omega_l$. This verifies the earlier conjecture and completes the proof of Proposition 2.

Proof of Lemma 3. The proof consists of two steps:
Step 1. We show that no auction seller sets a reserve price above b. By contradiction, suppose a store has $\tilde{r}_a > b$. Since $\tilde{r}_a$ is exceeds b we have $\tilde{x}_{h,a} > 0$ and $\tilde{x}_{l,a} = 0$. The expected utility of a high type buyer visiting this store equals to

$$\tilde{U}_{h,a} \equiv U_{h,a}(\tilde{r}_a, \tilde{x}_{h,a}, 0) = z_0(\tilde{x}_{h,a})(1 - \tilde{r}_a),$$

whereas the expected profit of the store equals to

$$\tilde{\Pi}_a \equiv \Pi_a(\tilde{r}_a, \tilde{x}_{h,a}, 0) = 1 - z_0(\tilde{x}_{h,a}) - \tilde{x}_{h,a}\tilde{U}_{h,a}.$$

These expressions are obtained by substituting $\tilde{x}_{l,a} = 0$ into (7) and (11). Below we demonstrate that if this store posts b instead of $\tilde{r}_a$ then he can do better than $\tilde{\Pi}_a$ while providing high types with the same utility $\tilde{U}_{h,a}$. If the seller posts b then he gets some low types as well as some high types, so let $x'_{l,a}$ and $x'_{h,a}$ denote his new expected demands. Substituting $r_a = b$ into (7) and (8) yields buyers' expected utilities visiting this store:

$$U'_{h,a} \equiv U_{h,a}(b, x'_{h,a}, x'_{l,a}) = z_0(x'_{h,a})\,(1-b) \text{ and } U'_{l,a} \equiv U_{l,a}(b, x'_{h,a}, x'_{l,a}) = z_0(x'_h)\frac{1 - z_0\,(x'_l)}{x'_l}\,(1-b)\,.$$

Combining these expressions with (13) we obtain the expected profit of the seller who posts b:

$$\Pi'_a \equiv \Pi_a(b, x'_{h,a}, x'_{l,a}) = 1 - z_0(x'_{h,a}) - z_1(x'_{h,a}) + bz_1(x'_{h,a}) + bz_0(x'_{h,a})[1 - z_0(x'_{l,a})].$$

We will show that $\Pi'_a > \tilde{\Pi}_a$ when $U'_{h,a} = \tilde{U}_{h,a}$ i.e. when the seller provides high types with the same level of utility. Note that

$$U'_{h,a} = \tilde{U}_{h,a} \Leftrightarrow z_0(x'_{h,a})\,(1-b) = z_0(\tilde{x}_{h,a})\,(1-\tilde{r}_a)\,,$$

which implies that $x'_{h,a} > \tilde{x}_{h,a}$ since $r_a > b$. The last term in Π'_a is positive; hence, to show $\Pi'_a > \tilde{\Pi}_a$ it suffices to show that Δ is positive, where

$$\Delta \equiv z_0(\tilde{x}_{h,a}) - z_0(x'_{h,a}) + z_1(x'_{h,a})(1-\tilde{r}_a) - z_1(\tilde{x}_{h,a})\,(1-b)\,.$$

Substitute $z_0(x'_{h,a})\,(1-b) = z_0(\tilde{x}_{h,a})\,(1-\tilde{r}_a)$ into Δ and rearrange to obtain

$$\Delta = z_0(x'_{h,a})\,\{e^q - 1 - (1-b)\,q\}\,,$$

where $q \equiv x'_{h,a} - \tilde{x}_{h,a} > 0$. The expression inside the curly brackets is positive for all $q > 0$, hence $\Delta > 0$, and therefore $\Pi'_a > \tilde{\Pi}_a$, i.e. the deviation is profitable. So, no auction seller sets a reserve price above b.

Step 2. We show that every auction store must attract both types of customers i.e. there cannot be an outcome where a store attracts low types only or high types only. There are three possibilities for an auction store: (a) It attracts high types only i.e. $x_{h,a} > 0$ and $x_{l,a} = 0$. (b) It attracts low types only, i.e. $x_{h,a} = 0$ and $x_{l,a} > 0$. (c) It attracts both types i.e. $x_{h,a} > 0$ and $x_{l,a} > 0$. Scenario (a) is not possible. To see why suppose indeed $x_{h,a} > 0$ and $x_{l,a} = 0$. This means that $U_{h,a}\,(r_a, x_{h,a}, 0) = \Omega_h$ and $U_{l,a}\,(r_a, x_{h,a}, 0) < \Omega_l$. Notice, however, $U_{l,a}\,(r_a, x_{h,a}, 0) = U_{h,a}\,(r_a, x_{h,a}, 0)$. This, in turn, means that $\Omega_h < \Omega_l$; a contradiction, since $\Omega_h \geq \Omega_l$.

Scenario (b) is not possible, either. By way of contradiction, suppose that there is a seller who attracts low types only, i.e. his queue lengths are $x_{h,a} = 0$ and $x_{l,a} > 0$. This means that $U_{h,a}\,(r_a, 0, x_{l,a}) < \Omega_h$ and $U_{l,a}\,(r_a, 0, x_{l,a}) = \Omega_l$. Furthermore, since $x_l > 0$ we have $U_{l,a}\,(r_a, 0, x_{l,a}) <$

$U_{h,a}(r_a, 0, x_{l,a})$ and therefore $\Omega_l < \Omega_h$. The expected profit of this seller equals to

$$\Pi_a = 1 - z_0(x_{l,a}) - x_{l,a}U_{l,a}.$$

Since $U_{l,a} = \Omega_l$, the sellers problem is

$$\max_{x_{l,a}} 1 - z_0(x_{l,a}) - x_{l,a}\Omega_l.$$

The first order condition yields $\Omega_l = z_0(x_{l,a})$. The second order condition is always negative; hence the solution of FOC corresponds to a global maximum. Solving $U_{l,a} = z_0(x_{l,a})$ for r_a yields $r_a = \eta$ where

$$\eta \equiv \frac{1 - z_0(x_{l,a}) - z_1(x_{l,a})}{z_1(x_{l,a})}(1 - b).$$

Observe that the reserve price r_a needs be less than or equal to b (Step 1); hence there are two cases: Either $\eta \leq b$ and therefore $\Omega_l = z_0(x_{l,a})$. Or $\eta > b$, in which case $r_a = b$ and therefore $\Omega_l > z_0(x_{l,a})$. Considering both possibilities, we have $\Omega_l \geq z_0(x_{l,a})$.

Under our conjecture this store is attracting low types only; so high types must be shopping elsewhere. Since $\mathbb{M} = \{\text{auctions}\}$, the stores where high types shop must be auction stores. We already know that no auction store caters to high types only (scenario (a) above); so, some auction stores must be catering to both types of customers. Consider such a store and let r'_a be its reserve price and $x'_{h,a} > 0$ and $x'_{l,a} > 0$ be its queue lengths. Since $x'_{h,a}$ and $x'_{l,a}$ are both positive we have $U'_{h,a} = \Omega_h$ and $U'_{l,a} = \Omega_l$. The store's expected profit equals to

$$\Pi'_a = 1 - z_0(x'_{h,a} + x'_{l,a}) - x'_{h,a}\Omega_h - x'_{l,a}\Omega_l.$$

Both stores must be earning equal profits, i.e. (i) $\Pi_a = \Pi'_a$. We have already established that (ii) $U_{l,a} = \Omega_l = U'_{l,a}$ and (iii) $U_{h,a} < \Omega_h = U'_{h,a}$. Below we show that these relationships cannot hold together, rendering scenario (b) infeasible. To start, note that the equal profit condition requires $\Delta = 0$, where

$$\Delta \equiv \Pi'_a - \Pi_a = z_0(x_{l,a}) + x_{l,a}\Omega_l - z_0(x'_{h,a} + x'_{l,a}) - x'_{h,a}\Omega_h - x'_{l,a}\Omega_l.$$

Fix $x'_{h,a} + x'_{l,a}$ and note that if $x_{l,a} = x'_{h,a} + x'_{l,a}$ then $\Delta < 0$ because $\Omega_h > \Omega_l$. In addition $\partial\Delta/\partial x_{l,a} = \Omega_l - z_0(x_{l,a})$, which is positive because $\Omega_l \geq z_0(x_{l,a})$. It follows that $\Delta = 0$ is possible only if $x_{l,a} > x'_{h,a} + x'_{l,a}$. Since $U_{l,a} = U'_{l,a} = \Omega_l$ and $U_{h,a} < U'_{h,a} = \Omega_h$ we have $U'_{h,a} - U'_{l,a} > U_{h,a} - U_{l,a}$, which is equivalent to

$$\frac{x_{l,a} - 1 + z_0(x_{l,a})}{x_{l,a}} < z_0(x'_{h,a})\frac{x'_{l,a} - 1 + z_0(x'_{l,a})}{x'_{l,a}}. \tag{22}$$

Note that the expression $[x - 1 + z_0(x)]/x$ rises in x. Since $x_{l,a} > x'_{h,a} + x'_{l,a}$ we have

$$\frac{x_{l,a} - 1 + z_0(x_{l,a})}{x_{l,a}} > \frac{x'_{l,a} - 1 + z_0(x'_{l,a})}{x'_{l,a}},$$

which contradicts (22) because $z_0(x'_{h,a}) < 1$. To sum up, the equal profit condition in (i) cannot hold together with restrictions (ii) and (iii); thus, scenario (b), where an auction store attracts low types only fails to exist. Hence the only possible scenario is (c). This outcome is characterized in Proposition 3.

Proof of Proposition 3. Lemma 3 established that every auction seller must be attracting both types of customers. So consider a seller with reserve price r_a and queue lengths $x_{h,a} > 0$ and $x_{l,a} > 0$. His expected profit equals to

$$\Pi_a = 1 - z_0(x_{h,a} + x_{l,a}) - x_{h,a}U_{h,a} - x_{l,a}U_{l,a},$$

where $U_{h,a}$ and $U_{l,a}$ are given by (7) and (8). The seller's problem is

$$\max_{r_a,\, x_{h,a},\, x_{l,a}} \Pi_a \text{ subject to } U_{h,a} = \Omega_h \text{ and } U_{l,a} = \Omega_l.$$

Both constraints must bind as the store must be attracting both types of customers. The FOC is given by

$$\frac{d\Pi_a}{dr_a} = [z_0(x_{l,a} + x_{h,a}) - \Omega_h]\frac{dx_{h,a}}{dr_a} + [z_0(x_{l,a} + x_{h,a}) - \Omega_l]\frac{dx_{l,a}}{dr_a} = 0.$$

The General Implicit Function Theorem implies that

$$\frac{dx_{h,a}}{dr_a} = \frac{\det \overline{B}}{\det A} \text{ and } \frac{dx_{l,a}}{dr_a} = \frac{\det \underline{B}}{\det A},$$

where

$$A = \begin{bmatrix} \frac{\partial U_{h,a}}{\partial x_{h,a}} & \frac{\partial U_{h,a}}{\partial x_{l,a}} \\ \frac{\partial U_{l,a}}{\partial x_{h,a}} & \frac{\partial U_{l,a}}{\partial x_{l,a}} \end{bmatrix}, \ \overline{B} = \begin{bmatrix} -\frac{\partial U_{h,a}}{\partial r_a} & \frac{\partial U_{h,a}}{\partial x_{l,a}} \\ -\frac{\partial U_{l,a}}{\partial r_a} & \frac{\partial U_{l,a}}{\partial x_{l,a}} \end{bmatrix}, \ \underline{B} = \begin{bmatrix} \frac{\partial U_{h,a}}{\partial x_{h,a}} & -\frac{\partial U_{h,a}}{\partial r_a} \\ \frac{\partial U_{l,a}}{\partial x_{h,a}} & -\frac{\partial U_{l,a}}{\partial r_a} \end{bmatrix}.$$

Inspecting (7) and (8) one can verify that for $i = h, l$ we have

$$\frac{\partial U_{i,a}}{\partial r_a} = -z_0(x_{l,a} + x_{h,a}),\ \frac{\partial U_{i,a}}{\partial x_{h,a}} = -U_{i,a},\ \frac{\partial U_{h,a}}{\partial x_{l,a}} = -z_0(x_{l,a} + x_{h,a})\,(b - r_a)$$

$$\frac{\partial U_{l,a}}{\partial x_{l,a}} = -z_0(x_{l,a} + x_{h,a})(1 - r_a) - z_0(x_{h,a})\,(1 - b)\,\frac{\mu(x_{l,a}) - x_{l,a}z_1(x_{l,a})}{x_{l,a}^2}.$$

Observe that (i) $0 < U_{l,a} < U_{h,a}$ and (ii) $\partial U_{l,a}/\partial x_{l,a} < \partial U_{h,a}/\partial x_{l,a} < 0$. It follows that

$$\begin{aligned}
\det A &= U_{l,a}\frac{\partial U_{h,a}}{\partial x_{l,a}} - U_{h,a}\frac{\partial U_{l,a}}{\partial x_{l,a}} > 0,\\
\det \overline{B} &= -\frac{z_0^2(x_{h,a})z_0(x_{l,a})}{x_{l,a}^2}\,(1-b)\,\mu(x_{l,a}) < 0,\\
\det \underline{B} &= \frac{z_0^2(x_{h,a})z_0(x_{l,a})}{x_{l,a}^2}\,(1-b)\,(1-x_{l,a}-z_0(x_{l,a})) < 0.
\end{aligned}$$

Substitute $U_{h,a} = \Omega_h$ and $U_{l,a} = \Omega_l$ into $d\Pi_a/dr_a$ to obtain

$$\frac{d\Pi_a}{dr_a} = \frac{z_0(x_{h,a})}{\det A}[c_1 r_a - c_2],$$

where

$$\begin{aligned}
c_1 &= z_0(x_{l,a})\left(\det \overline{B} + \det \underline{B}\right) < 0,\\
c_2 &= (1-b)\left[(1-z_0(x_{l,a}))\det \overline{B} + \mu(x_{l,a})\det \underline{B}/x_{l,a}\right] < 0.
\end{aligned}$$

Solving the FOC for r_a we obtain

$$\frac{d\Pi_a}{dr_a} = 0 \Leftrightarrow r_a = \frac{c_2}{c_1} \equiv \hat{r}\left(x_{l,a}\right).$$

Straightforward algebra reveals that

$$\hat{r}(x) = \frac{x - z_1(x) - xz_1(x)}{z_0(x) - z_0^2(x) + xz_1(x) - z_1(x)}\,(1-b)\,. \tag{23}$$

To verify the SOC note that $z_0(x_{h,a})/\det A > 0$ since $\det A > 0$. It follows that

$$sign\left(d\Pi_a/dr_a\right) = sign\left(c_1 r_a - c_2\right).$$

Observe that c_1 and c_2 are negative constants since $\det \overline{B}$ and $\det \underline{B}$ are both negative and independent of r_a. Therefore $d\Pi_a/dr_a > 0$ for all $r_a < \hat{r}$ and $d\Pi_a/dr_a < 0$ for all $r_a > \hat{r}$, which means that the objective function is globally concave and $r_a = \hat{r}$ is the unique maximum. Recall, however the reserve price must be less than or equal to b (Lemma 3), so if $\hat{r} \leq b$ then the seller posts $r_a = \hat{r}$ but if $\hat{r} > b$ then he posts $r_a = b$.

To prove uniqueness we show that all auction stores post the same reserve price. Consider two auction sellers, k and j, one with price $r_a^k = \min\{b, \hat{r}(x_{l,a}^k)\}$ and queue lengths $x_{h,a}^k$ and $x_{l,a}^k$ and the other with price $r_a^j = \min\{\hat{r}(x_{l,a}^j), b\}$ and queue lengths $x_{h,a}^j$ and $x_{l,a}^j$. Both stores must be providing their customers with the same market utilities, i.e. $U_{h,a}(r_a^k, x_{h,a}^k, x_{l,a}^k) = U_{h,a}^j(r_a^j, x_{h,a}^j, x_{l,a}^j) = \Omega_h$ and $U_{l,a}(r_a^k, x_{h,a}^k, x_{l,a}^k) = U_{l,a}^j(r_a^j, x_{h,a}^j, x_{l,a}^j) = \Omega_l$. Basic algebra reveals that these relationships hold iff

$\xi(x^k_{l,a}) = \xi(x^j_{l,a})$ where

$$\xi(x^k_{l,a}) \equiv \frac{z_0(x^k_{l,a})(1-r^k_a) + \left[1 - z_0(x^k_{l,a})\right](1-b)}{z_0(x^k_{l,a})(1-r^k_a) + \left[1 - z_0(x^k_{l,a}) - z_1(x^k_{l,a})\right](1-b)/x^k_{l,a}}.$$

There are three sub-cases here:

- Both prices are interior, i.e. $r^k_a = \hat{r}(x^k_{l,a}) < b$ and $r^j_a = \hat{r}(x^j_{l,a}) < b$. Substituting $r^k_a = \hat{r}(x^k_{l,a})$ reveals that ξ is a 1-1 function; thus the fact that $\xi(x^k_{l,a}) = \xi(x^j_{l,a})$ implies that $x^k_{l,a} = x^j_{l,a}$ and therefore $r^k_a = r^j_a$.

- Both prices are corner, i.e. $r^k_a = b$ and $r^j_a = b$, which, of course means that $r^k_a = r^j_a$.

- One price is interior and the other is corner, i.e. $r^k_a = b$ while $r^j_a = \hat{r}(x^j_{l,a}) < b$. Straightforward algebra reveals that one cannot have $U^k_{h,a} = U^j_{h,a}$ and $U^k_{l,a} = U^j_{l,a}$ together with the equal profit condition $\Pi^k_a = \Pi^j_a$. Hence such an outcome fails to exist.

Since all sellers post the same reserve price, symmetry in buyers visiting strategies implies that the queue length at each store is identical—that is $x^k_{h,a} = \lambda_h$ and $x^k_{l,a} = \lambda_l$ for all k. This completes the proof.

Proof of Lemma 4. The proof consists of two steps.
Step 1. We will prove that auction stores advertise $r_a = \min\{b, \hat{r}(x_{l,a})\}$ and cater to both types of customers. This claim appears to be a repetition of Lemma 3 above; however, that lemma was based on a setting with $\mathbb{M} = \{\text{auctions}\}$ whereas in here $\mathbb{M} = \{\text{fixed pricing, auctions}\}$, which poses new alternatives where buyers may shop. First note that no auction seller sets $r_a > b$ as he can do better by setting $r_a = b$. This claim is established in the proof of Lemma 3; the proof remains valid here. As for customer demographics, note that an auction store with price $r_a \leq b$ faces three scenarios: (a) it attracts high types only and low types stay away (b) it attracts low types only and high types stay away (c) it attracts both types. Scenario (a) can be ruled out using the same arguments in the proof of Lemma 3, but there is a subtlety here. One can construct a scenario where all low types shop at fixed price stores and high types at auction stores and all players earn the same expected utility; however, such an outcome requires coordination among buyers on where to shop when indifferent. Specifically, low types ought to coordinate among themselves not to show up at an auction store (and high types ought to coordinate among themselves not to show up at fixed prices stores) even though they are indifferent. Given the large number of buyers in the market, such coordination is not plausible, so we rule out this possibility. Scenario (b) is also impossible, and its proof is largely the same as before; however there is an additional possibility here: if the store attracts low types only then high types might be shopping at fixed price stores. Below we rule out this possibility, which will leave scenario (c) as the only possible outcome.

If the auction seller indeed attracts low types only then $x_{h,a} = 0$ and $x_{l,a} > 0$. This means that

$U_{l,a}(r_a, 0, x_{l,a}) = \Omega_l$ and $U_{h,a}(r_a, 0, x_{l,a}) < \Omega_h$, where

$$U_{h,a}(r_a, 0, x_{l,a}) = z_0\left(x_{l,a}\right)(1 - r_a) + \left[1 - z_0\left(x_{l,a}\right)\right](1 - b)\,.$$

Since $x_{l,a} > 0$ we have $U_{l,a} < U_{h,a}$ and therefore $\Omega_l < \Omega_h$. Now consider a fixed price store where high types might be shopping. There are two cases: (i) $r_f > b$ or (ii) $r_f \leq b$.

Case 1. If $r_f > b$ then $x_{l,f} = 0$ and $x_{h,f} > 0$. The expected utility of a high type buyer at this store is given by $U_{h,f}(r_f, x_{h,f}, 0)$. The fact that $x_{h,f} > 0$ implies $U_{h,f} = \Omega_h$. Recall that $U_{h,a} < \Omega_h$; hence $U_{h,f} > U_{h,a}$. Since $r_f > b$ we have $U_{h,f}(r_f, x_{h,f}, 0) < 1 - b$. On the other hand, note that $U_{h,a}(r_a, 0, x_{l,a}) \geq 1 - b$, since $r_a \leq b$. It follows that $U_{h,a} > U_{h,f}$; a contradiction.

Case 2. Now suppose $r_f \leq b$. Since the price is affordable, high types and low types obtain the same expected utility there, i.e. $U_{h,f} = U_{l,f}$. The fact that high types shop at the fixed price store implies $U_{h,f} = \Omega_h$. The market utility of a low type Ω_l, by definition, must be greater than or equal to his expected payoff at the fixed price store i.e. $\Omega_l \geq U_{l,f}$. Furthermore recall that $\Omega_l \leq \Omega_h$. It follows that $\Omega_l = \Omega_h$. The fact that the auction store attracts low types only implies $U_{h,a} < \Omega_h$ and $U_{l,a} = \Omega_l$. Since $U_{h,a} > U_{l,a}$ we have $\Omega_h > \Omega_l$. This, of course, contradicts $\Omega_h = \Omega_l$. Hence this possibility, too, is ruled out.

It follows that an auction store must attract both types of customers i.e. queue lengths $x_{h,a}$ and $x_{l,a}$ are both positive satisfying $U_{h,a} = \Omega_h$ and $U_{l,a} = \Omega_l$. Hence an auction seller solves

$$\max_{r_a,\, x_{h,a}, x_{l,a}} \Pi_a \text{ subject to } U_{h,a} = \Omega_h \text{ and } U_{l,a} = \Omega_l.$$

This problem is analyzed in the proof of Proposition 3. Following the same steps, one can show that all auction sellers post the same reserve price $r_a = \min\{b, \hat{r}\left(x_{l,a}\right)\}$, where $\hat{r}$ is given by (23). This completes the proof of Step 1.

Step 2. We will show that all fixed price sellers post $r_f = \min\{b, \rho(x_{h,f} + x_{l,f})\}$. We start by demonstrating that no seller posts a price above b. By way of contradiction, suppose a fixed price seller advertises $r_f > b$. Letting $x_{h,f}$ denote his queue length consisting of high types (clearly $x_{l,f} = 0$), the seller solves

$$\max_{r_f, x_{h,f}} \{1 - z_0(x_{h,f})\} r_f \quad \text{s.t.} \quad \frac{1 - z_0(x_{h,f})}{x_{h,f}}(1 - r_f) = \Omega_h.$$

This problem is analyzed earlier. The solution entails the seller posting $r_f = \rho(x_{h,f})$ and earning $\mu(x_{h,f})$, where ρ and μ are given by (17) and (16). High type buyers visiting this store earn $U_{h,f} = z_0(x_{h,f})$. Below we show that if this seller switches to auctions then he can earn more than $\mu(x_{h,f})$ while still providing high types with utility Ω_h. Thus, the above outcome cannot be an equilibrium.

Per Lemma 4, if the seller switches to auctions, then he must attract both types of customers and post $r_a = \min\{\hat{r}(x_{l,a}), b\}$. Since he attracts both types of customers we have $U_{h,a}(r_a, x_{h,a}, x_{l,a}) = \Omega_h$ and $U_{l,a}(r_a, x_{h,a}, x_{l,a}) = \Omega_l$. Note that $d\hat{r}/db < 0$; so fix $x_{l,a}$ and let $\bar{b}$ be the unique value of b

satisfying $\hat{r}(\bar{b}, x_{l,a}) = \bar{b}$. There are two cases:

Case 1. Suppose that $b \leq \bar{b}$. Since $\hat{r} \geq b$ the seller posts $r_a = b$. Basic algebra reveals that

$$\hat{r} \leq \frac{1 - z_0(x_{l,a})}{z_0(x_{l,a})} (1 - b) .$$

Since $\hat{r} \geq b$ it follows that $z_0(x_{l,a}) < 1 - b$. Substitute $r_a = b$ into (7) and (11) to obtain

$$U_{h,a} = z_0(x_{h,a}) (1 - b) \quad \text{and} \quad \Pi_a = 1 - (1 - b) \{z_0(x_{h,a}) + z_1(x_{h,a})\} - bz_0(x_{h,a} + x_{l,a}).$$

Recall that (i) $U_{h,f} = z_0(x_{h,f})$, (ii) $U_{h,f} = \Omega_h$ and (iii) $U_{h,a} = \Omega_h$. This means that

$$z_0(x_{h,f}) = z_0(x_{h,a}) (1 - b) \Leftrightarrow x_{h,f} - x_{h,a} = - \ln (1 - b) .$$

Now we can check profits:

$$\begin{aligned} \Pi_a - \mu(x_{h,f}) &= (1 + x_{h,f}) z_0(x_{h,f}) - (1 - b) \{z_0(x_{h,a}) + z_1(x_{h,a})\} - bz_0(x_{h,a} + x_{l,a}) \\ &= (x_{h,f} - x_{h,a}) (1 - b) z_0(x_{h,a}) - bz_0(x_{h,a} + x_{l,a}) \\ &= - (1 - b) z_0(x_{h,a}) \ln (1 - b) - bz_0(x_{h,a} + x_{l,a}). \end{aligned}$$

It follows that $\Pi_a > \mu \Leftrightarrow -\frac{1-b}{b} \ln (1 - b) > z_0(x_{l,a})$. Recall that $z_0(x_{l,a}) < 1 - b$. Therefore, $\Pi_a > \mu$ if

$$-\frac{1 - b}{b} \ln (1 - b) \geq 1 - b.$$

It is straightforward to verify that this inequality is satisfied for all $b \in (0, 1)$. Hence the deviation is profitable.

Case 2. Now suppose $b > \bar{b}$. The previous step establishes that switching to auctions is profitable even if the seller sets the corner reserve price $r_a = b$. Now that the budget is sufficiently high, the deviating seller can set the interior reserve price $r_a = \hat{r}(b, x_{l,a})$ and earn even more. More specifically the previous step has $\Pi_a(b', x_{h,a}, x_{l,a}) > \mu$ for all $b' \leq \bar{b}$. The concavity of the objective function implies that $\Pi_a (b', \cdot) < \Pi_a(\hat{r}(b', x_{l,a}), \cdot)$. It is straightforward to show that $d\Pi_a/db > 0$ thus $\Pi_a(\hat{r}(b', x_{l,a}), \cdot) < \Pi_a(\hat{r}(b, x_{l,a}), \cdot)$ for all $b > \bar{b} \geq b'$. Combining these inequalities, we have $\mu < \Pi_a(\hat{r}(b, x_{l,a}), \cdot)$ for all $b > \bar{b}$ i.e. the deviation is profitable.

So, the store must post $r_f \leq b$. Since r_f is below b we have $U_{l,f} = U_{h,f}$. The seller's problem is

$$\max_{r_f, x_{l,f}, x_{h,f}} \; \{1 - z_0(x_{l,f} + x_{h,f})\} r_f \quad \text{s.t.} \quad \frac{1 - z_0(x_{l,f} + x_{h,f})}{x_{l,f} + x_{h,f}} (1 - r_f) = \Omega.$$

This is analyzed earlier. The objective function is globally concave; the FOC is given by $z_0(x_{l,f} + x_{h,f}) = \Omega$; hence $r_f = \rho(x_{l,f} + x_{h,f})$. The constraint $r_f \in [0, b]$ along with the concavity of the objective function implies that if $\rho(x_{l,f} + x_{h,f}) > b$ then $r_f = b$. It follows that $r_f = \min\{b, \rho(x_{l,f} + x_{h,f})\}$. Following the steps in the proof of Proposition 2 it is easy to show that all fixed price sellers post the same list price: either they all post the same interior price $\rho(x_{h,f} + x_{l,f})$ or they all post the corner price b. In either case the queue lengths at each store must be identical. This completes the proof.

Proof of Proposition 4. Conjecture an outcome where all sellers compete with auctions. Per Lemma 4, all sellers set the same reserve price $r_a = \min\{b, \hat{r}(x_{l,a})\}$ and cater to both types of customers. Symmetry in buyers' visiting strategies implies that $x_{l,a} = \lambda_l$ and $x_{h,a} = \lambda_h$; so, the total demand at each store is $\lambda_l + \lambda_h = \lambda$.

Claim 1. In the auction equilibrium market utilities satisfy $\Omega_h > z_0(\lambda) > \Omega_l$.

Since $b < 1$ we have $U_{h,a} > U_{l,a}$ and therefore $\Omega_h > \Omega_l$. Recall from the proof of Proposition 3 that the FOC of an auction seller is given by

$$\frac{d\Pi_a}{dr_a} = [z_0(x_{h,a} + x_{l,a}) - \Omega_h]\frac{dx_{h,a}}{dr_a} + [z_0(x_{l,a} + x_{h,a}) - \Omega_l]\frac{dx_{l,a}}{dr_a} = 0.$$

Both $\frac{dx_{h,a}}{dr_a}$ and $\frac{dx_{l,a}}{dr_a}$ are negative; thus for $\frac{d\Pi_a}{dr_a} = 0$ to hold, the expressions in the first and the second square brackets must have opposite signs. Since $\Omega_h > \Omega_l$ the first expression is negative and the second one is positive; that is $\Omega_h > z_0(x_{l,a} + x_{h,a}) > \Omega_l$. In equilibrium $x_{l,a} = \lambda_l$ and $x_{h,a} = \lambda_h$; hence the inequality $\Omega_h > z_0(\lambda) > \Omega_l$ follows.

Claim 2. If b is sufficiently large then the auction equilibrium fails to exist.

The fact that $\Omega_h > \Omega_l$ presents a deviation opportunity. Below we show that if a seller switches to fixed pricing and targets low type customers (by providing them with the same utility Ω_l) then he will earn more. Note that if the fixed price seller provides his customers with payoff Ω_l then he will attract low types only. Indeed the facts (i) $U_{h,f} = U_{l,f}$ and (ii) $\Omega_h > \Omega_l$ imply that $U_{h,f} < \Omega_h$; hence $x_{h,f} = 0$. The seller's problem is

$$\max_{r_f, x_{l,f}} 1 - z_0(x_{l,f}) - x_{l,f}U_{l,f} \quad \text{s.t. } U_{l,f} = \Omega_l.$$

The problem is analyzed earlier. The FOC is given by $z_0(x_{l,f}) = \Omega_l$ and therefore seller posts $r_f = \rho(x_{l,f})$ and earns $\mu(x_{l,f})$. For this solution to be feasible we need $\rho(x_{l,f}) < b$, which is the case if b is sufficiently large. The fact that $\Omega_h > z_0(\lambda) > \Omega_l$ and $z_0(x_f) = \Omega_l$ implies

$$z_0(\lambda) > z_0(x_{l,f}) \Rightarrow x_{l,f} > \lambda$$

i.e. the fixed price store attracts more buyers than an auction store. We now compare profits. An auction seller earns

$$\Pi_a = 1 - z_0(\lambda) - \lambda_h\Omega_h - \lambda_l\Omega_l$$

whereas the fixed price seller earns

$$\mu(x_{l,f}) = 1 - z_0(x_{l,f}) - x_{l,f}\Omega_l.$$

We want to show that $\mu > \Pi_a$. Since $\Omega_h > \Omega_l$, it suffices to show $\Delta > 0$, where

$$\Delta \equiv z_0(\lambda) - z_0(x_{l,f}) + (\lambda - x_{l,f})\,\Omega_l.$$

Substituting $\Omega_l = z_0(x_{l,f})$ into Δ and re-arranging we have $\Delta = e^q - 1 - q$, where $q \equiv x_{l,f} - \lambda > 0$. Note that $\Delta > 0$ because q is positive; hence the fixed price seller earns more. This completes the proof.

Proof of Proposition 5. Since $\mathbb{M} = \{\text{fixed pricing, auctions}\}$, there are three equilibria to consider:

- E1. All sellers adopt fixed pricing.
- E2. Some sellers adopt fixed pricing while others adopt auctions.
- E3. All sellers adopt auctions.

Start with E2, i.e. consider an outcome where both rules coexist in the same market. Per Lemma 4, sellers competing with auctions advertise the same reserve price $r_a = \min\{b, \hat{r}(x_{l,a})\}$ and cater to both types of customers, which means that $x_{h,a}$ and $x_{l,a}$ are both positive satisfying $U_{h,a} = \Omega_h$ and $U_{l,a} = \Omega_l$. Since $U_{h,a} > U_{l,a}$ we have $\Omega_h > \Omega_l$. In addition, sellers competing with fixed pricing advertise the same price $r_f = \min\{\rho(x_{h,f} + x_{l,f}), b\}$; however it is not clear what type of customers they attract. There are three possible scenarios:
(i) They attract high types only i.e. $x_{l,f} = 0$, $x_{h,f} > 0$. This implies $U_{h,f} = \Omega_h$ and $U_{l,f} < \Omega_l$. Since $U_{h,f} = U_{l,f}$ we have $\Omega_h < \Omega_l$; a contradiction because $\Omega_h > \Omega_l$.
(ii) They attract both types i.e. $x_{h,f} > 0$, $x_{l,f} > 0$. This implies $U_{l,f} = \Omega_l$ and $U_{h,f} = \Omega_h$, which in turn indicates that $\Omega_l = \Omega_l$; again, a contradiction because $\Omega_h > \Omega_l$.
(iii) They attract low types only i.e. $x_{l,f} > 0$, $x_{h,f} = 0$. This scenario is possible as it implies $U_{l,f} = \Omega_l$ and $U_{h,f} < \Omega_h$, which is consistent with $\Omega_h > \Omega_l$. Below we will explore this scenario.

Claim 1. Along E2 fixed price stores are more crowded than auction stores, that is $x_{l,f} > x_{l,a} + x_{h,a}$.
Since $x_{h,f} = 0$, we have $r_f = \min\{\rho(x_{l,f}), b\}$. If $r_f = \rho(x_{l,f})$ then buyers visiting fixed price stores earn $U_{l,f}(\rho, 0, x_{l,f}) = z_0(x_{l,f})$. If $r_f = b$ then they earn $U_{l,f}(b, 0, x_{l,f}) > z_0(x_{l,f})$. Combining both possibilities we have $U_{l,f} \geq z_0(x_{l,f})$. Now, turn to auction stores. Claim 1 in the proof of Proposition 4 reveals that $\Omega_h > z_0(x_{l,a} + x_{h,a}) > \Omega_l$ and therefore $U_{h,a} > z_0(x_{l,a} + x_{h,a}) > U_{l,a}$. Since $U_{l,f} = U_{l,a} = \Omega_l$ and $U_{l,a} < z_0(x_{l,a} + x_{h,a})$ we have $z_0(x_{l,f}) < z_0(x_{l,a} + x_{h,a})$, which in turn indicates that $x_{l,a} + x_{h,a} < x_{l,f}$.

Claim 2. If $b < \rho(\lambda)$ then along E2 fixed price stores post $r_f = b$.

We have already established that sellers competing with the same rule set the same reserve price; hence the feasibility equations in (15) become

$$\alpha x_{l,a} + (1-\alpha)\,x_{l,f} = \lambda_l \quad \text{and} \quad \alpha x_{h,a} + (1-\alpha)\,x_{h,f} = \lambda_h,$$

where α represents the fraction of sellers adopting auctions. Furthermore recall that $x_{l,f} > 0$, $x_{h,f} = 0$, $x_{h,a} > 0$ and $x_{l,a} > 0$; hence

$$x_{h,a} = \lambda_h/\alpha,\ x_{h,f} = 0 \quad \text{and} \quad \alpha x_{l,a} + (1-\alpha)\, x_{l,f} = \lambda_l.$$

Since (i) $\alpha\,(x_{l,a} + x_{h,a}) + (1-\alpha)\, x_{l,f} = \lambda_h + \lambda_l = \lambda$ and (ii) $x_{l,f} > x_{h,a} + x_{l,a}$, we have

$$x_{l,f} > \lambda > x_{l,a} + x_{h,a}.$$

This, in turn, means that $\alpha \in (\lambda_h/\lambda, 1)$. Since $x_{l,f} > \lambda$ we have $\rho(x_{l,f}) > \rho\,(\lambda)$. So, if $b < \rho\,(\lambda)$ then all fixed price stores post $r_f = \min\{b, \rho(x_{l,f})\} = b$.

Claim 3. If $b^{\#} < b < \rho\,(\lambda)$ then there exists a unique separating *equilibrium where a fraction $\alpha^* \in (\lambda_h/\lambda, 1)$ of sellers compete with auctions while remaining sellers compete with fixed pricing.*

We start by exploring how the expected profits Π_a and Π_f respond to a change in α. Recall that the expected profit of an auction seller is given by

$$\begin{aligned}\Pi_a\,(r_a, x_{h,a}, x_{l,a}) = \quad & z_1\,(x_{h,a} + x_{l,a})\, r_a + 1 - z_0\,(x_{h,a}) - z_1\,(x_{h,a}) \\ & + b\{z_0\,(x_{h,a}) + z_1\,(x_{h,a}) - z_0\,(x_{h,a} + x_{l,a}) - z_1\,(x_{h,a} + x_{l,a})\},\end{aligned}$$

where $r_a = \min\{\hat{r}(x_{l,a}), b\}$. Furthermore, per Claim 2, fixed price sellers earn

$$\Pi_f(b, 0, x_{l,f}) = \{1 - z_0(x_{l,f})\}b.$$

Note that

$$\frac{d\Pi_f}{d\alpha} = bz_0(x_{l,f})\frac{dx_{l,f}}{d\alpha},$$

where $dx_{l,f}/d\alpha = (\lambda_l - x_{l,a})\,/\,(1-\alpha)^2$, which is positive since $\lambda > x_{l,a}$. It follows that $d\Pi_f/d\alpha > 0$. Now turn to Π_a. We have

$$\begin{aligned}\frac{d\Pi_a}{d\alpha} \quad = \quad & z_1\,(x_{h,a})\, b\frac{dx_{h,a}}{d\alpha} + z_0\,(x_{h,a} + x_{l,a})\, r_a \frac{d\,(x_{h,a} + x_{l,a})}{d\alpha} \\ & + z_1\,(x_{h,a} + x_{l,a})\,(b - r_a)\,\frac{d\,(x_{h,a} + x_{l,a})}{d\alpha} + z_1\,(x_{h,a} + x_{l,a})\,\frac{\partial r_a}{\partial x_{l,a}}\frac{dx_{l,a}}{d\alpha},\end{aligned}$$

where $dx_{h,a}/d\alpha = -\lambda_h/\alpha^2 < 0$ and $dx_{l,a}/d\alpha = -(x_{l,f} - \lambda)/\alpha^2$, which is negative since $x_{l,f} > \lambda$. Note that $b - r_a$ is non-negative since $r_a = \min\,(\hat{r}, b)$. In addition, $\partial r_a/\partial x_{l,a}$ is either positive (if $\hat{r} \le b$) or zero (if $\hat{r} > b$). It follows that $d\Pi_a/d\alpha < 0$. Thus $\Delta \equiv \Pi_f - \Pi_a$ rises in α.

Recall that $\alpha \in (\lambda_h/\lambda, 1)$. At the lower bound where $\alpha = \lambda_h/\lambda$ we have $x_{h,a} = \lambda$, $x_{l,a} = 0$, $x_{l,f} = \lambda$, and therefore, $\hat{r}\,(x_{l,a}) = 0$. Auction sellers earn $\Pi_a\,(0, \lambda, 0) = \mu\,(\lambda)$ and fixed price sellers earn $\Pi_f(b, 0, \lambda) = \{1 - z_0\,(\lambda)\}b$. If $b < \rho\,(\lambda)$ then $\Pi_f < \Pi_a$ i.e. $\Delta < 0$. At the upper bound where $\alpha = 1$ we have $x_{h,a} = \lambda_h$, $x_{l,a} = \lambda_l$, and $x_{l,f} = x^{\#}_{l,f}$, where $x^{\#}_{l,f}$ solves $U_{l,f}(b, 0, x^{\#}_{l,f}) = U_{l,a}(r_a, \lambda_h, \lambda_l)$, i.e.

$$\frac{1-z_0(x_{l,f}^{\#})}{x_{l,f}^{\#}}(1-b)=U_{l,a}(r_a,\lambda_h,\lambda_l). \tag{24}$$

Fixed price sellers earn $\Pi_f(b,0,x_{l,f}^{\#})=\{1-z_0(x_{l,f}^{\#})\}b$ whereas auction sellers earn $\Pi_a(r_a,\lambda_h,\lambda_l)$. If $b^{\#}<b$, where $b^{\#}$ is the unique value of b satisfying

$$\Pi_a(r_a,\lambda_h,\lambda_l)=\{1-z_0(x_{l,f}^{\#})\}b, \tag{25}$$

then $\Pi_a<\Pi_f$, i.e. $\Delta>0$. Recall that Δ rises in α. The Intermediate Value Theorem implies that if $b^{\#}<b<\rho(\lambda)$ then there exists a unique $\alpha^*\in(\lambda_h/\lambda,1)$ satisfying the equal profit condition $\Delta=0$; i.e. the separating equilibrium exists and it is unique.

Claim 4. If $b^{\#}<b$ then E3, where all seller adopt auctions, fails to exist. If $b<\rho(\lambda)$ then E1, where all sellers adopt fixed pricing, fails to exist.

Consider E3, i.e. conjecture an equilibrium where all sellers adopt auctions. In such an equilibrium sellers earn $\Pi_a(r_a,\lambda_h,\lambda_l)$. Recall that if $b^{\#}<b$ then $\Pi_f(b,0,x_{l,f}^{\#})>\Pi_a(r_a,\lambda_h,\lambda_l)$, which implies that a seller can earn more than Π_a by deviating to fixed pricing and posting b. Such a seller would attract low types only and he can provide them with the same expected utility they were getting at auction stores; indeed, recall that $x_{l,f}^{\#}$ satisfies $U_{l,f}(b,0,x_{l,f}^{\#})=U_{l,a}(r_a,\lambda_h,\lambda_l)$. Since there is a profitable deviation, E3 fails to exist in the region $b^{\#}<b$.

Now consider E1 i.e. conjecture an outcome where all sellers compete via fixed pricing. We have established that if $\mathbb{M}=\{\text{auctions, fixed pricing}\}$ then all fixed price stores advertise the same price $r_f=\min\{\rho(x_{l,f}+x_{h,f}),b\}$. Since buyers follow symmetric visiting strategies, all such stores have the same queue lengths $x_{h,f}=\lambda_h$ and $x_{l,f}=\lambda_l$; hence $x_{l,f}+x_{h,f}=\lambda$. It follows that when $b<\rho(\lambda)$ all sellers post $r_f=\min\{\rho(\lambda),b\}=b$ earning $\Pi_f=\{1-z_0(\lambda)\}b$, while providing buyers

$$U_{h,f}=U_{l,f}=\frac{1-z_0(\lambda)}{\lambda}(1-b).$$

Note that since $\rho(\lambda)>b$ we have $U_{h,f}>z_0(\lambda)$. We will show that a seller can earn more by switching to auctions while providing high types with the same utility, i.e. satisfying $U_{h,a}=U_{h,f}$. Note that low types would stay away from the deviating store, i.e. $x_{l,a}=0$. Indeed if $x_{l,a}>0$ then $U_{l,a}<U_{h,a}$. Since $U_{h,a}=U_{h,f}$ and $U_{h,f}=U_{l,f}$, this would mean $U_{l,a}<U_{l,f}$, which in turn would imply $x_{l,a}=0$; a contradiction.

Since $x_{h,a}>0$ and $x_{l,a}=0$ we have

$$U_{h,a}=z_0(x_{h,a})(1-r_a) \text{ and } \Pi_a=1-z_0(x_{h,a})-x_{h,a}U_{h,a}.$$

The seller's problem is $\max_{r_a,\,x_{h,a}}\Pi_a$ s.t $U_{h,a}=U_{h,f}$, taking $U_{h,f}$ as given. Substituting the constraint, the seller solves

$$\max_{x_{h,a}} 1-z_0(x_{h,a})-x_{h,a}U_{h,f}.$$

The first order condition is given by $z_0(x_{h,a}) = U_{h,f}$, which in turn implies that $U_{h,a} = z_0(x_{h,a})$. Solving this equation for the reserve price yields $r_a = 0$, which means that the seller earns $\Pi_a = \mu(x_{h,a})$, where μ is given by (16). We can now compare profits. Observe that

$$\begin{aligned}
\mu(x_{h,a}) - \Pi_f &= 1 - z_0(x_{h,a}) - z_1(x_{h,a}) - \{1 - z_0(\lambda)\}b \\
&= z_0(\lambda) - z_0(x_{h,a}) + (\lambda - x_{h,a})z_0(x_{h,a}) \\
&= z_0(x_{h,a})\{e^{-q} - 1 + q\}.
\end{aligned}$$

where $q \equiv x_{h,a} - \lambda$. The second line follows from the fact that $z_0(x_{h,a}) = U_{h,f} = \{1 - z_0(\lambda)\}(1-b)/\lambda$. Note that $x_{h,a} < \lambda$ since $z_0(x_{h,a}) > z_0(\lambda)$; thus the expression in the third line is positive. It follows that $\mu(x_{h,a}) > \Pi_f$, i.e. the deviating seller earns more; hence a E1 fails to exist if $b < \rho(\lambda)$.

Per claims 3 and 4, if $b^{\#} < b < \rho(\lambda)$ then E1 and E3 fail to exist. E2—the separating equilibrium characterized above—is the unique competitive search equilibrium in this parameter region. Similarly, if $b^{\#} \geq b$ then E1 and E2 fail to exist. The only possible equilibrium is E3, where all sellers adopt auctions. Lemma 4 characterizes this outcome: all sellers set a reserve price $r_a = \min\{b, \hat{r}(x_{l,a})\}$ and cater to both types of customers. Since all sellers adopt the same rule and set the same reserve price we have $x_{h,a} = \lambda_h$ and $x_{l,a} = \lambda_l$. What remains to be done is to verify that if $b^{\#} \geq b$ then no seller deviates from E3 by offering fixed pricing. If such a deviation takes place then the deviating seller attracts low types only, and since b is too low, he posts $r_f = b$. Let $x_{l,f}$ denote his expected demand consisting of low types (note that $x_{h,f} = 0$). Low types visiting this store ought to be rewarded with the same expected utility they obtain along E3, that is, $U_{l,f}(b, 0, x_{l,f}) = U_{l,a}(r_a, \lambda_h, \lambda_l)$. Recall that the unique value of $x_{l,f}$ satisfying this relationship is $x_{l,f} = x_{l,f}^{\#}$. We know that if $b^{\#} \geq b$ then $\Pi_f(b, 0, x_{l,f}^{\#}) \leq \Pi_a(r_a, \lambda_h, \lambda_l)$ i.e. the deviating seller cannot earn more.

If $b \geq \rho(\lambda)$ then E2 and E3 fail to exist, but E1 is feasible. Since $b \geq \rho(\lambda)$ the budget constraint is slack, so E1 corresponds to the fixed price equilibrium in the homogenous model (described in Remark 1). Recall that along such an outcome all buyers earn the same market utility $z_0(\lambda)$ whereas sellers earn the same expected profit $\mu(\lambda)$. We need to verify that if $b \geq \rho(\lambda)$ then no seller deviates from E1 by switching to auctions. If such a deviation takes place then the deviating seller attracts high types only (see the proof of Claim 4). Since potential customers are financially unconstrained, the solution of his problem is the same as the solution in the homogenous model: Letting $x_{h,a}$ denote his expected demand, he provides his customers with $z_0(x_{h,a})$ while he earns $\mu(x_{h,a})$. High type buyers visiting this store ought to be rewarded with the same expected utility they obtain along E1, that is $z_0(x_{h,a}) = z_0(\lambda)$, which implies $x_{h,a} = \lambda$. It follows that the deviating seller's expected profit is $\mu(\lambda)$, which is the same as his expected profit along E1; hence there is no profitable deviation. This completes the proof. As an aside, note that in this parameter region one can provide all buyers the same utility $z_0(\lambda)$ by instructing low types to shop at fixed price stores and high types at auction stores. However, such an outcome requires coordination among buyers on where to shop when indifferent e.g. low types coordinate among themselves not to show up at an auction store, even though they are indifferent. The fixed price equilibrium in E1 requires no such coordination.