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Collusive Communication Schemes in a First-Price Auction

Helmuts Āzacis* and Péter Vida**

May 2012

Abstract

We study optimal bidder collusion at first-price auctions when the collusive mechanism only relies on signals about bidders' valuations. We build on Fang and Morris (2006) when two bidders have low or high private valuation of a single object and additionally each receives a private noisy signal from an incentiveless center about the opponent's valuation. We derive the unique symmetric equilibrium of the first price auction for any symmetric, possibly correlated, distribution of signals, when these can only take two values. Next, we find the distribution of 2-valued signals, which maximizes the joint payoffs of bidders. We prove that allowing signals to take more than two values will not increase bidders' payoffs if the signals are restricted to be public. We also investigate the case when the signals are chosen conditionally independently and identically out of $n \geq 2$ possible values. We demonstrate that bidders are strictly better off as signals can take on more and more possible values. Finally, we look at another special case of the correlated signals, namely, when these are independent of the bidders' valuations. We show that in any symmetric 2-valued strategy correlated equilibrium, the bidders bid as if there were no signals at all and, hence, are not able to collude.

Keywords: Bidder-optimal signal structure; Collusion; (Bayes) correlated equilibrium; First price auction; Public and private signals.

JEL Classification Numbers: D44; D82.

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1 Introduction

In a standard independent private value (IPV) first price auction it is assumed that each bidder observes her own valuation but has no information about her opponents' valuations except for the distribution from which they are drawn. However, in real life auctions, bidders may hold or may have incentives to gather additional information about their opponents' valuations. For example, in procurement auctions while a firm's actual cost is private information, other firms may observe its investment in new technologies which serves as a noisy signal about the firm's actual cost. Fang and Morris (2006), Bergemann and Vällimäki (2006) and Kim and Che (2004) are examples of setups where bidders can benefit from such noisy signals and the seller expects less revenue in first price auctions compared to second price auctions.[1] Bidders can improve their position against the seller by acquiring information about each other's private information through various institutions as well. It has been recognized that trade associations, through gathering and sharing filtered (aggregated) industry specific information, can serve as collusion facilitating devices.[2] For example, Genesove and Mullin (1997) provide an interesting case study of the workings of the Sugar Institute, the trade association uniting the U.S. domestic sugar refiners from 1928 to 1936. They describe how the Sugar Institute collected, aggregated and disseminated the data about the industry among its members. One of their findings is that "the Sugar Institute revealed less information to its members than it knew" (Genesove and Mullin, 1997, p. 20). Indeed, theory also suggests that bidders are better off if the signals observed about the opponents' valuations contain some noise instead of being perfect signals in which case bidders would bid in a complete information environment. Hence, for collusive purposes, bidders may agree and commit to mechanisms which provide them with such noisy information about each other's valuations.

Our goal is to examine the extent to which bidders can improve their payoffs in a one-shot IPV first price auction when they have access to such mechanisms. In particular, we assume that the bidders have access to a center, an incentiveless third party, which facilitates collusion between them. For the collusion to be successful, the center must be able to learn the private valuations of bidders and then, conditional on this information, coordinate

[1] Information about the opponents' valuations has no impact on the equilibrium bidding strategies in the second price private value auctions, hence on the revenue of the seller.

[2] See, for example, Vives (1990) and the references in his Section 2.

the bids that the bidders submit in the auction. In this paper, we abstract away from the process how the center collects information about bidders' valuations.[3] Instead, we assume that the center already knows the realized valuation profile. Before bidding in the auction, the center, depending on the realized valuation profile, sends random (noisy) private signals to the bidders about each other's valuations. Of course, as in a standard IPV setup, each bidder also knows her own valuation. We emphasize that a signal does not alter bidder's valuation of the object, but it changes the bidder's beliefs about the opponent's valuation. Since the signals can be correlated, a signal can also convey information about the signal received by the opponent. After learning their valuations and the signals sent by the center, bidders are allowed to bid as they want. However, by altering bidders' beliefs about the opponents' valuations, the center affects equilibrium strategies of bidders in the first price auction. Therefore, we are interested in the following question: *what is the signal structure that the center should use to maximize bidders' payoffs?*[4]

As in Fang and Morris (2006), we consider the simplest IPV setup with only two bidders, whose valuations can either take low or high value. Fang and Morris (2006) show that the seller expects less revenue in the first price auction than in the second price auction when signals are drawn *independently* (that is, a signal only depends on the valuation of the opponent), the signal received by a bidder is her *private* information, and the signals can take one of *two* values. We generalize the signal structure in several directions and address the following questions. Can the center increase bidders' payoffs by using correlated signals? Should the signals be distributed publicly or privately? Are the bidders better off if the center uses richer language instead of 2-valued signals?

[3]In Section 6.1, we provide an example, which shows how the center can elicit this information from the bidders in a (weakly) incentive compatible manner. We assume that if a bidder reports some valuation to the center then she is not allowed to bid above it in the auction. Requiring full incentive compatibility leads to the analysis of communication equilibria of the first price auction; see, for example, Pavlov (2009) for partial results.

[4]Different correlated equilibrium concepts when applied to games with incomplete information are extensively studied in Forges (1993, 2006) and Bergemann and Morris (2011). According to the equilibrium classification of Forges (1993, 2006), we look for the best (from the bidders point of view) *Bayesian solution* of a *standard* IPV first-price auction, where bidders can make use of an omniscient mediator. According to Bergemann and Morris (2011), we look for the best *Bayes correlated equilibrium* of the standard IPV first-price auction. The two concepts coincide in the current setup; see the discussion in Section 4.1 of Bergemann and Morris (2011).

We analyze and compare four classes of signal structures. First, we keep the 2-valued signals as in Fang and Morris (2006), but allow them to be correlated. We characterize the unique symmetric equilibrium for any 2-valued symmetric signal structure and then find the signal structure that maximizes the bidders' joint payoff. Under the optimal symmetric signal structure, the signals received by the bidders are neither independent, nor perfectly correlated. Roughly speaking, a high valuation bidder, depending on the signal she receives, either learns her opponent's valuation or the signal that the opponent has received, but not both. Therefore, it is possible that both bidders have high valuation, and both know that the opponent has high valuation, but still each is unsure what the opponent knows. As a result, the bidders bid less aggressively.

An important subclass of the 2-valued symmetric signal structures contains those, in which the randomness does not depend on the bidders' realized valuation profile. In order to produce such signals, the center does not have to know the profile of bidders' valuations. Hence, the question: can the center induce collusive bidding without knowing the bidders' valuations, and just distribute correlated signals? We find that the answer is negative. To be more precise, the equilibrium when the bidders receive valuation-independent correlated signals and peg their bidding strategies to them, is known as a strategy-correlated equilibrium of the underlying first price auction.[5] We find the following as a by-product of the analysis of the optimal 2-valued (valuation-dependent) symmetric signal structures. No collusive equilibrium in symmetric correlated strategies exists if the center uses only 2-valued valuation-independent correlated signals. More precisely, in any symmetric strategy-correlated equilibrium, the bidders bid exactly as in the non-cooperative equilibrium, independently of the signals received. This result suggests that for the collusion to be profitable, the center must possess some information about the valuations.

We were unable to extend the general correlated signal structure to the case when the signals are allowed to take more than two values.[6] Instead, we next consider two extreme cases when the signals are either independent or

[5] We refer here to Aumann's (1974) strategic form correlated equilibrium, which has been applied to games with incomplete information by Cotter (1991) and Forges (1993). See also the references in Footnote 4.

[6] Though, we conjecture that introducing more signal values does not increase the bidders' payoffs, that is, the optimal 2-valued correlated signal structure gives the overall optimum.

perfectly correlated, and in each case we allow the signals to take more than two values. When the signals are perfectly correlated, the center, depending on the valuation profile of the bidders, effectively chooses a *single* signal out of n possible values and announces it publicly. We establish that within the class of public signals, the bidders can achieve their best payoffs with a 2-valued signal and allowing richer $n-$valued public signal does not improve bidders' payoffs. Even more, we show that there is no gain in having an asymmetric public signal structure since the optimum can be implemented with a symmetric one. This implies that, in terms of payoffs, any public signal structure is dominated by the optimal 2-valued correlated signal structure discussed previously. However, the optimal public signal structure has an attractive property that, unlike the case of the optimal 2-valued correlated signal structure, it is independent of the prior beliefs of bidders. Therefore, the center might prefer to use the optimal public signal if he is unsure what are the prior beliefs of bidders. In Section 6.2, we also provide an example of sequential auction, in which a public signal structure arises at the start of the second round as the seller publicly announces the winning bid of the first round. This example demonstrates how the seller might (unwillingly) enhance collusion between the bidders.

The structure of the optimal 2-valued public signal is as follows. When the signal takes one of the two values, bidder 1 knows that bidder 2 has high valuation, while bidder 2 is unsure whether bidder 1 has low or high valuation. When the signal takes the other value, the roles of the bidders are reversed. Thus, after observing the signal, the bidders are asymmetric. Let us call a bidder weak if she learns that the opponent has high valuation, and otherwise call her strong. To gain intuition for the optimal public signal structure, let us first consider what would be the expected payoff of high valuation bidder in the second price auction where bidding one's valuation is a dominant strategy. If the bidder was weak, she would expect zero payoff, that is, less than in the absence of signal. And opposite, if the bidder was strong, she would expect more than in the absence of signal. In the first price auction, on the contrary, both weak and strong bidder (with high valuation) expect as much as the strong bidder would expect in the second price auction, that is, both types expect more than in the absence of signal. This suggests that the public signal is optimal when the asymmetry between weak and strong types is maximal.

The result that a 2-valued public signal maximizes bidders' payoffs within the class of public signals is in sharp contrast with our findings for the class of signal structures where the signals are drawn independently and identically. We show that for any $n-$valued independent signal structure we can always construct an $(n+1)-$valued independent signal structure that results in strictly higher payoffs for the bidders. Compared to the bidders' payoffs under the optimal public signal structure, we find that the optimal 2-valued independent signal structure gives higher payoffs only when the prior probability of a bidder having low valuation is low. Otherwise, the optimal public signal structure leads to higher payoffs. Numerical results suggest that the same conclusions also hold for $n > 2$. Additionally, the numerical results indicate that the bidders' payoffs under the optimal 2-valued correlated signal structure always exceed those that result from any independent signal structure for any n and any prior beliefs.

We should emphasize that, except for the public signal case, we have restricted attention to symmetric signal structures. Apart from tractability issues that arise when dealing with asymmetric signal structures, this decision was motivated by a result in Fang and Morris (2006). They show in their Proposition 5 that even in 2-valued independent signal case, neither symmetric nor asymmetric equilibrium exists for generic values of parameters once we consider asymmetric signal structures. For all symmetric signal structures that we study we find a unique symmetric Nash equilibrium, therefore the existence of equilibrium is not an issue in our model. However, we do not show whether or not there also exist asymmetric equilibria.

In a seminal paper, McAfee and McMillan (1992) characterize the optimal collusive mechanisms when the center needs to elicit bidders' private valuations. However, they assume the center can enforce the bids. Marshall and Marx (2007) and Lopomo, Marx, and Sun (2011) extend the model of McAfee and McMillan (1992) by assuming the center cannot control the bids that the bidders submit at the auction but he can enforce side-payments between the bidders. In particular, Lopomo, Marx, and Sun (2011) show that in this case there is no collusive mechanism that improves bidders' payoffs relative to non-cooperative bidding even if the side-payments, which only depend on the reported valuations, are allowed. Despite this negative result, we believe it is interesting to study how the center should share his knowledge about bidders' valuations if he possessed such information and he could not control the bids directly or indirectly through monetary transfers.

There exists extensive literature that studies a similar question but from the seller's perspective. Namely, how should the seller disclose information about bidders' valuations in order to maximize his revenue. This question has been addressed both when the auction rules are fixed and in the mechanism design context. Among the former, closest to our setup is the one by Kaplan and Zamir (2000) who also consider the first-price private value auction. They find that the seller can increase his revenue through public announcements about bidders' valuations. We show that the opposite result holds in our model (see Remark 2). These differences can be reconciled with the help of Maskin and Riley (2000a) who compare seller's revenues in first-price and second-price auctions in the presence of asymmetries: the setup in Kaplan and Zamir (2000) is closer to the one considered in Proposition 4.3 (and Example 1) of Maskin and Riley (2000a), while our setup is closer to the one in Proposition 4.5 (and Example 3). In the context of mechanism design, when the seller additionally decides on the auction format, Skreta (2011) considers a model, in which similar to our model, bidder's type is multidimensional consisting of valuation and belief components. Skreta (2011) finds that in the IPV setup, the maximal revenue is obtained with full information disclosure but the optimal mechanism is different from the first-price auction.

The rest of the paper is organized as follows. In Section 2, we set up the model. Next, we study the four classes of signal structures in the same order as discussed above. The general 2-valued signal structures, including the correlated signals that do not depend on the bidders' valuations, are studied in Section 3, the public signal structures in Section 4, and the independent signal structures in Section 5. In Section 6, we provide two examples that illustrate how bidders can collude even if they have no access to an omniscient center. We conclude in Section 7. All major proofs are relegated to the Appendix.

2 The Model

Two bidders, 1 and 2, compete for an object. When we refer to a generic bidder, we use she and we do not index the notation if it does not cause confusion. Bidders' valuations of the object are independently drawn from identical distributions. We assume that bidder's valuation of the object takes one of two possible values $\{V_L, V_H\}$, where we set $V_L = 0$ and $V_H = 1$.[7] The ex ante probability that a bidder's valuation v takes value V_L is denoted by $p \in (0, 1)$. Of course, the probability of V_H is $1 - p$.

[7] All results extend in the obvious way for any $0 \leq V_L < V_H$.

As in standard private value auction models, each bidder privately observes her own valuation v. Fang and Morris (2006) assume that each bidder also privately observes a noisy signal s about her opponent's valuation. For tractability, they assume that the noisy signal can only take two possible qualitative categories, $s \in \{L, H\}$.[8] Further, in Fang and Morris (2006), the signals are drawn, conditional on realized valuations, independently and identically from a given distribution function, implying that bidder's signal conveys no information about the signal received by the opponent. We generalize their setup by allowing the signals to be correlated and/or to take on values from a larger set, namely, from the set $N = \{1, 2, \ldots, n\}$.

In contrast to Fang and Morris (2006), we also assume that the bidders have access to an incentiveless center who knows the realized valuations of bidders and, depending on these, sends to the bidders private signals. We assume that prior to the auction, the bidders can choose and instruct the center, which distribution function to use for each realized profile of valuations. The goal is to identify the distribution that maximizes the joint payoff of bidders.

Since the bidders observe their signals privately, bidder l's type is a 2-tuple (v_l, s_l) where $s_l \in N$ denotes bidder l's $(l = 1, 2)$ signal about bidder m's $(m = 1, 2,\ m \neq l)$ type (v_m, s_m). The signals are generated as follows. For all $(i, j) \in N \times N$,

$$\begin{aligned}
\Pr\left((s_1, s_2) = (i, j) \,|\, (v_1, v_2) = (V_L, V_L)\right) &= r_{0.ij}, \\
\Pr\left((s_1, s_2) = (i, j) \,|\, (v_1, v_2) = (V_H, V_L)\right) &= r_{1.ij}, \\
\Pr\left((s_1, s_2) = (i, j) \,|\, (v_1, v_2) = (V_L, V_H)\right) &= r_{2.ij}, \\
\Pr\left((s_1, s_2) = (i, j) \,|\, (v_1, v_2) = (V_H, V_H)\right) &= r_{ij}.
\end{aligned}$$

We refer to $(r_{0.ij}, r_{1.ij}, r_{2.ij}, r_{ij})_{i \in N, j \in N}$ as a *signal structure*. Of course,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_{0.ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{1.ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{2.ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} = 1,$$

$0 \leq r_{0.ij} \leq 1$, $0 \leq r_{1.ij} \leq 1$, $0 \leq r_{2.ij} \leq 1$, and $0 \leq r_{ij} \leq 1$ for all $(i, j) \in N \times N$. The joint distribution of types is summarized in the following table.

[8] In their set up, bidders' valuations are sometimes allowed to take one of 3 possible values. In this case, dealing with 3-valued signals already becomes intractable.

	$(0,1)$	$\cdots$	$(0,n)$	$(1,1)$	$\cdots$	$(1,n)$
$(0,1)$	$p^2 r_{0.11}$	$\cdots$	$p^2 r_{0.1n}$	$p(1-p)\, r_{2.11}$	$\cdots$	$p(1-p)\, r_{2.1n}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$(0,n)$	$p^2 r_{0.n1}$	$\cdots$	$p^2 r_{0.nn}$	$p(1-p)\, r_{2.n1}$	$\cdots$	$p(1-p)\, r_{2.nn}$
$(1,1)$	$p(1-p)\, r_{1.11}$	$\cdots$	$p(1-p)\, r_{1.1n}$	$(1-p)^2\, r_{11}$	$\cdots$	$(1-p)^2\, r_{1n}$
$\vdots$	$\vdots$		$\vdots$	$\vdots$		$\vdots$
$(1,n)$	$p(1-p)\, r_{1.n1}$	$\cdots$	$p(1-p)\, r_{1.nn}$	$(1-p)^2\, r_{n1}$	$\cdots$	$(1-p)^2\, r_{nn}$

Remark 1 *When signals are generated independently of realized valuation profile, $r_{0.ij} = r_{1.ij} = r_{2.ij} = r_{ij}$ for all $(i,j) \in N \times N$. We will refer to such signal structures as correlating devices. The Bayesian Nash equilibria corresponding to a correlating device are known as strategy-correlated equilibria (Cotter, 1991; Forges, 1993).*

The bidders participate in a first price auction with zero reserve price,[9] where bidders simultaneously submit bids b depending on the realizations of v and s. The highest bidder gets the object and pays her bid to the seller. In the event of a tie, the bidder with higher valuation gets the object[10] and the tie-breaking can be arbitrary if bidders' valuations are the same. Next we present an immediate result about the equilibrium behavior of types with valuation V_L.

Lemma 1 *In any Bayesian Nash equilibrium of the first price auction, types $(0,i)$ for all $i \in N$ bid 0.*

The proof is analogous to that in Lemma A.1 of Fang and Morris (2006) and therefore is omitted.[11] Given the result of Lemma 1, in the continuation, we will only need to consider the strategies of high valuation types. Further, since in equilibrium types $(0,i)$ bid 0, we suppress i and refer to them collectively as type $v = 0$, and we refer to type $(1,i)$ for each $i \in N$ as type $s = i$. Consequently, the joint distribution of types becomes:

[9] The assumption of zero reserve price is made purely for simplicity, and all subsequent results extend in the natural way if a binding reserve price is introduced.

[10] On the use of this tie-breaking rule see Maskin and Riley (2000b), Kim and Che (2004), and Fang and Morris (2006). One way to justify it is to assume that in case of a tie, an auxiliary second price auction is held and the highest bid from the first price auction serves as the starting price.

[11] The proof on page 151 in Maskin and Riley (1985) can also be modified and adopted to prove Lemma 1.

	0	1	$\cdots$	n
0	p^2	$p(1-p)\,x_{2.1}$	$\cdots$	$p(1-p)\,x_{2.n}$
1	$p(1-p)\,x_{1.1}$	$(1-p)^2\,r_{11}$	$\cdots$	$(1-p)^2\,r_{1n}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
n	$p(1-p)\,x_{1.n}$	$(1-p)^2\,r_{n1}$	$\cdots$	$(1-p)^2\,r_{nn}$

where

$$x_{1.i} = \sum_{j=1}^{n} r_{1.ij} \quad \text{and} \quad x_{2.i} = \sum_{j=1}^{n} r_{2.ji}$$

for all $i \in N$. It follows that, without loss of generality, we could have assumed that only high valuation bidders receive a signal from the center in the model.

Remark 2 *If instead of maximizing the joint payoff of bidders, we were interested in maximizing seller's revenue, then in the optimum, the center must send either completely uninformative signals or, on the contrary, reveal fully the information about bidders' valuations. In both cases, the ex ante payoff of a bidder is $p(1-p)$. To see that the seller cannot extract more surplus from the bidders, note that a bidder can always guarantee an ex ante payoff of $p(1-p)$ by bidding 0, according to Lemma 1.*

3 The Optimal 2-Valued Signal Structure

We start by considering 2-valued symmetric signal structures: $n = 2$, $x_{1.i} = x_{2.i}$ for $i = 1, 2$, which we now write as x_i, and $r_{12} = r_{21}$. We also assume that $r_{12} > 0$. If $r_{12} = 0$, then given the equilibrium behavior of type 0 in Lemma 1, the equilibrium strategies of types 1 and 2 can be determined independently. In this case, the derivation of equilibrium follows Maskin and Riley (1985), and one can verify that the ex ante payoff of bidder is $p(1-p)$, that is, the same as in the absence of collusive communication. Similarly, we rule out the case when

$$\frac{x_1}{x_2} = \frac{r_{11}}{r_{21}} = \frac{r_{12}}{r_{22}}.$$

This case is equivalent to a single uninformative signal, which would again result in the payoff of $p(1-p)$ for each bidder.

Case 1: $\Pr(1\,\vert\,1) \geq \Pr(1\,\vert\,2)$, $\dfrac{\Pr(0\,\vert\,1)}{\Pr(0\,\vert\,2)} \geq \dfrac{\Pr(1\,\vert\,1)}{\Pr(1\,\vert\,2)}$	Case 3: $\Pr(1\,\vert\,1) < \Pr(1\,\vert\,2)$, $\Pr(2\,\vert\,1) \leq \Pr(2\,\vert\,2)$
Case 2: $\Pr(1\,\vert\,1) > \Pr(1\,\vert\,2)$, $\dfrac{\Pr(0\,\vert\,1)}{\Pr(0\,\vert\,2)} < \dfrac{\Pr(1\,\vert\,1)}{\Pr(1\,\vert\,2)}$	Case 4: $\Pr(1\,\vert\,1) < \Pr(1\,\vert\,2)$, $\Pr(2\,\vert\,1) > \Pr(2\,\vert\,2)$

Figure 1: Cases in Proposition 1

In order to find the optimal symmetric signal structure, we first characterize the equilibrium strategies. In particular, we restrict attention to symmetric equilibria.

Proposition 1 *For each 2-valued symmetric signal structure, there exists a unique symmetric equilibrium in the first price auction.*

The full version of proposition, containing the equilibrium strategies, is stated in the Appendix. Here we highlight the main properties of the equilibrium.

Let $\Pr(j|i)$ denote the conditional probability that a bidder of type i, $i = 1, 2$, assigns to the opponent being of type j, $j = 0, 1, 2$. Since we can always rename the signals, without loss of generality, we assume that $\Pr(0|1) \geq \Pr(0|2)$ holds. Obviously, the equilibrium strategies depend on the signal structure and, consequently, on the conditional probabilities. In particular, when deriving the symmetric equilibrium in Proposition 1, we distinguish four cases which are summarized in Figure 1.[12] These cases are mutually exclusive and cover all possibilities.

In the proof to Proposition 1, we show that high valuation types use mixed strategies in the equilibrium, and the supports of these strategies are intervals. Let $\underline{b}_i$ and $\overline{b}_i$ denote the lower and upper endpoints of the interval

[12] If $\Pr(0|2) = 0$ in Case 1, then we set $\Pr(0|1) / \Pr(0|2) = \infty$.

for the mixed strategy of type i, $i = 1, 2$. We show that types 1 and 2 bid on adjacent intervals in Cases 1 and 3: $0 = \underline{b}_1 \leq \underline{b}_2 = \overline{b}_1 < \overline{b}_2$. In Case 2, the support of type 1 is a subset of the support of type 2 and both supports have a common lower endpoint: $0 = \underline{b}_1 = \underline{b}_2 < \overline{b}_1 < \overline{b}_2$. In Case 4, the support of type 2 is a subset of the support of type 1 and both supports have a common upper endpoint: $0 = \underline{b}_1 \leq \underline{b}_2 < \overline{b}_1 = \overline{b}_2$.

Before we identify the optimal 2-valued signal structure, we first discuss a special case when the signal distribution is constant over the realized valuation profile. That is, we want to know if there exists a symmetric strategy-correlated equilibrium that improves on the non-cooperative payoffs. When the center uses a correlating device, the probabilities must satisfy the following additional restrictions: $x_1 = r_{11} + r_{12}$ and $x_2 = r_{12} + r_{22}$. They imply that $\Pr(0|1) = \Pr(0|2) = p$. Figure 1 then tells us that any strategy-correlated equilibrium falls under Case 2 or 4. One can verify from Proposition 1 that the equilibrium strategies of high valuation types corresponding to these cases reduce to

$$F_i(b) = \frac{p}{1-p} \frac{b}{1-b}$$

for $b \in [0, 1-p]$, which are the equilibrium strategies of the standard first price auction (Maskin and Riley, 1985).

Corollary 1 *For any two-valued symmetric correlating device there exists a unique symmetric strategy-correlated equilibrium, which coincides with the unique Bayesian Nash equilibrium of the first price auction in the absence of any signals.*

This result motivates us in continuation to study signal structures that depend on bidders' valuations.

Given the equilibrium strategies in Proposition 1, we are in a position to find the symmetric signal structure that maximizes bidders' joint payoff.

Theorem 1 *Among symmetric 2-valued signal structures, $(x_1, x_2, r_{11}, r_{12}, r_{22}) = \left(1, 0, 0, \frac{\sqrt{p}}{1+\sqrt{p}}, \frac{1-\sqrt{p}}{1+\sqrt{p}}\right)$ maximizes bidders' joint payoffs. In the optimum, bidder's ex ante payoff is $(1-p)\sqrt{p}$.*

The distribution of types corresponding to the optimal signal structure is shown in the following table.

	0	1	2
0	p^2	$p\left(1-p\right)$	0
1	$p\left(1-p\right)$	0	$\left(1-p\right)^2\frac{\sqrt{p}}{1+\sqrt{p}}$
2	0	$\left(1-p\right)^2\frac{\sqrt{p}}{1+\sqrt{p}}$	$\left(1-p\right)^2\frac{1-\sqrt{p}}{1+\sqrt{p}}$

The optimal signal structure falls under Case 3 in Proposition 1, where the support of type 1's strategy has collapsed in a single point.

Corollary 2 *Given the optimal symmetric 2-valued signal structure, type 1 bids* 0 *with probability 1, while type 2 randomizes according to*

$$F_2\left(b\right)=\frac{\sqrt{p}}{1-\sqrt{p}}\frac{b}{1-b}$$

on the interval $\left[0,1-\sqrt{p}\right]$ *in the equilibrium.*

One can easily verify that type 1 is indeed indifferent between bidding 0 and anything in $\left(0,1-\sqrt{p}\right]$. Further, in the optimum, both types expect the same payoff of $\sqrt{p}$. Hence, the effect of collusive signals is almost the same as increasing the prior of the opponent having low valuation from p to $\sqrt{p}$.

Under the optimal signal structure, type 1 learns the signal that the opponent observes but is unsure about the opponent's valuation. On the other hand, type 2 learns that the opponent has high valuation but is unsure about the signal that the opponent observes. Consequently, type 1 knows that if the opponent has high valuation, she will bid relatively aggressively. Therefore, type 1 finds it optimal only to compete against the low valuation opponent. Although type 2 knows that she faces the high valuation bidder and therefore needs to bid relatively aggressively, her bid is suppressed by the possibility that the opponent bids very low.

Clearly, the payoff found in Theorem 1 is strictly higher than the payoff of $p\left(1-p\right)$ that a bidder would expect in the absence of collusive signals. For example, when $p=0.25$, the former payoff is double the latter one. If the bidders were able to capture the entire surplus, then the ex ante payoff of bidder would be $\left(1-p\right)\left(p+\frac{1}{2}\left(1-p\right)\right)$. Hence, there still exists a scope for improvement upon the optimal 2-valued signal payoffs, but we have not been able to prove if allowing signals to take more values will strictly increase the bidders' payoffs. However, as the next section demonstrates, when the signals are restricted to be public or, equivalently, perfectly correlated, adding more values does not improve on the optimal 2-valued public signals.

4 The Optimal Public Signal Structure

We now consider a situation when the signals are public. We allow signals to take an arbitrary number of values, $n \geq 2$, and the signal structure to be asymmetric. Consider the second table in Section 2. When the signals are public, in each row i, the probabilities r_{ij} are strictly positive for at most one j, and in each column j, the probabilities r_{ij} are strictly positive for at most one i. Since we can reorder and rename rows and columns, we assume that $r_{ij} \geq 0$ if $i = j$ and $r_{ij} = 0$ otherwise. Given this assumption, we say that both bidders observe one common signal, instead of a pair of signals.[13]

Proposition 2 *Given a public signal $i \in N$, if $r_{ii} = 0$, then both bidders submit bids equal to 0. If $x_{1.i} = x_{2.i} = 0$ and $r_{ii} > 0$, then both bidders submit bids equal to 1. If $x_{l.i} \geq x_{m.i}$, $x_{l.i} > 0$, and $r_{ii} > 0$, then the equilibrium of the first price auction is as follows:*

1. *Bidder l with $v_l = 1$ randomizes according to*

$$F_{l.i}(b) = \frac{px_{l.i}}{px_{l.i} + (1-p) r_{ii}} \frac{px_{m.i} + (1-p) r_{ii}}{(1-p) r_{ii}} \frac{1}{1-b} - \frac{px_{m.i}}{(1-p) r_{ii}} \quad (1)$$

   on the interval $\left[0, \bar{b}_i\right]$, and puts a mass $F_{l.i}(0) > 0$ on bid 0 if $x_{l.i} > x_{m.i}$;

2. *Bidder m with $v_m = 1$ randomizes according to*

$$F_{m.i}(b) = \frac{px_{l.i}}{(1-p) r_{ii}} \frac{b}{1-b} \quad (2)$$

   on the interval $\left[0, \bar{b}_i\right]$, where

$$\bar{b}_i = \frac{(1-p) r_{ii}}{px_{l.i} + (1-p) r_{ii}}.$$

The equilibrium payoff of a bidder with valuation $v = 1$ is

$$\frac{px_{l.i}}{px_{l.i} + (1-p) r_{ii}}. \quad (3)$$

[13] More precisely, we should define types $(0, i)$ and $(1, i)$ as in the first table in Section 2. Then, when a public signal i has occurred, it is common knowledge that each bidder is of either type $(0, i)$ or type $(1, i)$. However, given that all types $(0, i)$ will bid 0, they are suppressed in type 0.

Given the equilibrium strategies, we now provide a signal structure that maximizes bidders' joint payoff. In particular, we prove that there exists a 2-valued public signal structure that achieves the maximal payoff. In the proof to the following theorem, we show that we can increase the joint payoff if we aggregate all signals for which $x_{1.i} \geq x_{2.i}$ into one single signal and all the remaining signals into the second signal.

Theorem 2 *To achieve the maximal joint payoff with public signals, it is enough that the signal takes two values,* $n = 2$*. The optimal 2-valued public signal structure is* $(x_{1.1}, x_{1.2}, x_{2.1}, x_{2.2}, r_{11}, r_{22}) = (1, 0, 0, 1, 0.5, 0.5)$*. In the optimum, each bidder's ex ante payoff is* $p\,(1-p)\,\frac{2}{1+p}$*.*

The distribution of types corresponding to the optimal public signal structure is summarized in the following table:

	0	1	2
0	p^2	0	$p\,(1-p)$
1	$p\,(1-p)$	$\frac{(1-p)^2}{2}$	0
2	0	0	$\frac{(1-p)^2}{2}$

The optimum can also be implemented with a symmetric public signal structure by exchanging columns corresponding to types 1 and 2 of bidder 2 in the above table. Therefore, there is no gain in allowing asymmetric public signal structures. Since the optimal public signal structure is a special case of signal structures considered in the previous section, the ex ante payoff of a bidder now is obviously lower than the one found in Theorem 1.

The structure of optimal public signals is similar to the one found in the previous Section. One of the high valuation types, call her a weak bidder, learns that the opponent has high valuation, while the other high valuation type, call her a strong bidder, is unsure about the valuation of the opponent. Furthermore, a weak bidder only bids against a strong bidder and vice versa. As was suggested in the Introduction, it is optimal to create as high asymmetry between both types as possible. As a result, both types benefit. A strong bidder believes that the opponent has low valuation with high probability, and therefore does not bid aggressively, which in turn allows a weak bidder to bid less aggressively even so she knows that the opponent has high valuation. For the signal structure in Theorem 1, the strong bidder, that is, type 1 is bidding zero in the equilibrium. It is sustained by the fact that the weak bidder, that is, type 2 is unsure about the signal that the opponent receives. However, with public signals such uncertainty is absent and, as a result, the strong bidder must now bid above zero with a positive probability. Therefore, the optimal public signal structure leads to lower payoffs for the bidders compared to the one in Theorem 1. Nevertheless, this signal structure has an attractive feature that it does not depend on the initial prior. Therefore, if the center does not know p, he might prefer using the optimal public signal structure to facilitate collusion.

5 Independent Private Signal Structures

For the rest of the paper we assume that the signals are independently and identically distributed: $x_{1.i} = x_{2.i}$ for all $i \in N$, which we write as x_i, and $r_{ij} = y_i y_j$ for all $i, j \in N$. Of course, $\sum_{i \in N} x_i = \sum_{i \in N} y_i = 1$, $0 \leq x_i \leq 1$, and $0 \leq y_i \leq 1$ for all $i \in N$. We assume that there is no $i \in N$ such that $x_i = y_i = 0$. That is, each signal $i \in N$ has ex ante positive probability to appear, otherwise we are back to the case with less signals. Without loss of generality we assume that

$$\frac{x_1}{y_1} > \frac{x_2}{y_2} > ... > \frac{x_n}{y_n}.$$

If this relationship is not satisfied, we can always rename the signals. That is, we can always name the signal with the highest ratio as 1, and so on.[14] We prove later that if $\frac{x_i}{y_i} = \frac{x_{i+1}}{y_{i+1}}$ for some i then signals i and $i+1$ can be considered as one single signal with probabilities $x_{i'} = x_i + x_{i+1}$, $y_{i'} = y_i + y_{i+1}$ and the corresponding symmetric equilibria are the same in terms of the payoffs. Therefore, we will maintain these assumptions for the rest of the paper. We will now denote a signal structure by $(x, y)_n = (x_j, y_j)_{j \in N}$.

The main result of this section is that once the signals are restricted to be independent, there is no finite valued signal structure that would maximize the bidders' payoffs, meaning that it is always possible to increase the payoffs by allowing signals to take more values. First, however, we present the symmetric equilibrium strategies. In the equilibrium, the bidder with the lowest signal randomizes on an interval starting at V_L and bidders with higher and higher signals randomize on higher and higher adjacent intervals.

[14] If $y_i = 0$ then we set $\frac{x_i}{y_i} = \infty$.

Proposition 3 *The unique symmetric equilibrium of the first price auction with independent private signal structure* $(x,y)_n$ *is as follows:*

1. *Bidder of type 1 bids 0 if* $y_1 = 0$*, otherwise she randomizes over* $\left[\bar{b}_0, \bar{b}_1\right]$ *according to the cumulative distribution function*
$$F_1(b) = \frac{px_1}{(1-p)\,y_1^2}\frac{b}{1-b},$$
where $\bar{b}_0 \equiv 0$ *and*
$$\bar{b}_1 = 1 - \frac{px_1}{px_1 + (1-p)\,y_1^2}.$$

2. *Bidder of type* i*, for* $i = 2, ..., n$*, randomizes over* $\left[\bar{b}_{i-1}, \bar{b}_i\right]$ *according to the cumulative distribution function*
$$F_i(b) = \frac{px_i + (1-p)\,y_i \sum_{k=1}^{i-1} y_k}{(1-p)\,y_i^2}\frac{b - \bar{b}_{i-1}}{1-b}, \tag{4}$$
where
$$\bar{b}_i = 1 - \frac{px_i + (1-p)\,y_i \sum_{t=1}^{i-1} y_k}{px_i + (1-p)\,y_i \sum_{k=1}^{i} y_k}\left(1 - \bar{b}_{i-1}\right). \tag{5}$$

Remark 3 *If for some* $i \in N$*,* $\frac{x_i}{y_i} = \frac{x_{i+1}}{y_{i+1}}$ *then* i *and* $i+1$ *have the same expected payoff for any bid in* $[\bar{b}_{i-1}, \bar{b}_{i+1}]$*. Moreover, it is easy to see that if we replace signals* i *and* $i+1$ *with* i' *having probabilities* $x_{i'} = x_i + x_{i+1}$*,* $y_{i'} = y_i + y_{i+1}$ *then in the corresponding equilibrium with* $n-1$ *signals it is true that* $\bar{b}_{i'} = \bar{b}_{i+1}$ *and the strategies of types different from* i' *do not change. This shows that our original assumption with strict inequalities is indeed without loss of generality.*

The expected payoff of type $i \in N$ is
$$K(i,(x,y)) = \frac{px_i + (1-p)\,y_i \sum_{k=1}^{i-1} y_k}{px_i + (1-p)\,y_i}\left(1 - \bar{b}_{i-1}\right), \tag{6}$$
and each bidder's ex ante expected payoff is
$$P(x,y) = (1-p)\sum_{i \in N}(px_i + (1-p)y_i)K(i,(x,y)). \tag{7}$$

Example 1 *Let* $n = 2$ *and* $x_1 = y_2 = q$ *and* $x_2 = y_1 = 1 - q$*. That is, when signal 1 is equally indicative of value* V_L *as signal 2 is of value* V_H*. Then the equilibrium described in Proposition 3 is the same as in Proposition 1 of Fang and Morris (2006). They show that among these special signal structures there is an optimal* q *which minimizes the seller's revenue. However, this signal structure is not optimal for the bidders among all independent private signals. Indeed, the expected payoff of bidder when there are 2 signals is*

$$\begin{aligned} P(x,y) &= (1-p)px_1\left(1+\frac{px_2+(1-p)\,y_2y_1}{px_1+(1-p)\,y_1^2}\right) \\ &= (1-p)px_1\frac{p+(1-p)\,y_1}{px_1+(1-p)\,y_1^2}. \end{aligned}$$

The first order condition with respect to (w.r.t.) x_1 *is*

$$\frac{p\,(1-p)^2\,y_1^2\,(p+(1-p)\,y_1)}{(px_1+(1-p)\,y_1^2)^2},$$

which is strictly positive if $y_1 > 0$.[15] *Therefore, it is optimal to set* $x_1 = 1$*. The first order condition w.r.t.* y_1 *is*

$$(1-p)\,px_1\frac{-\left((1-p)\,y_1+p\right)^2+p\,(p+(1-p)\,x_1)}{(px_1+(1-p)\,y_1^2)^2},$$

which when set equal to 0 *and imposing* $x_1 = 1$ *implies that*

$$y_1 = \frac{\sqrt{p}}{1+\sqrt{p}}.$$

The second derivative w.r.t. y_1 *is negative when evaluated at the optimal value of* y_1*. This indicates that, if signals can take on 2 different values,* $x_1 = y_2 = q$ *and* $x_2 = y_1 = 1-q$ *does not hold under the optimal independent signal structure.*

Bidder's payoff, given the optimal 2-valued independent signal structure, is

$$(1-p)\,\frac{\sqrt{p}\,(1+\sqrt{p})}{2}.$$

For example, when $p = 0.25$*, bidder's ex ante payoff is* 0.2813*. For comparison, Fang and Morris (2006) show that* q *that minimizes the seller's revenue is* 0.7639*, which results in the bidder's payoff of* 0.2628*.*

[15] If $y_1 = 0$, then $P\,(x,y) = p\,(1-p)$, that is, the same payoff as in the case when there are no signals at all.

The following theorem shows what happens with the ex ante payoff as we increase the number of values that the signals can take.

Theorem 3 *For any n-valued signal structure, $(x,y)_n$ there exists an $(n+1)$-valued signal structure, $(x',y')_{n+1}$ such that the following is true:*

$$P((x,y)_n) < P((x',y')_{n+1}).$$

To prove the theorem we take an arbitrary $(x,y)_n$ and we show that if $x_n > 0$ then it cannot be optimal. Hence, we assume that $x_n = 0$ and show that we can always introduce an additional signal value that strictly improves the bidder's payoff.

Example 2 *To illustrate the approach adopted in the proof of the theorem, consider the optimal 2-valued independent signal structure found in Example 1. We construct the 3-valued signal structure by reassigning the probabilities as follows:*

$$\begin{array}{ll} x_1' = (1-\delta)\,x_1, & y_1' = (1-\delta)\,y_1, \\ x_2' = \delta x_1, & y_2' = \delta y_1 + \epsilon, \\ x_3' = 0, & y_3' = 1 - y_1 - \epsilon, \end{array}$$

Choosing $\delta = 0.5$ and $\epsilon = 0.1$, we find that bidder's ex ante payoff is 0.2841 *when $p = 0.25$. Hence, the bidder's payoff is strictly higher than the one obtained under the optimal 2-valued independent signal structure.*

Further, the optimal 3-valued independent signal structure is[16]

$$\begin{array}{ll} x_1 = 0.7295, & y_1 = 0.1941, \\ x_2 = 0.2705, & y_2 = 0.1941, \\ x_3 = 0, & y_3 = 0.6118, \end{array}$$

which leads to the bidder's payoff of 0.2884.

To find the optimal independent signal structures for $n > 2$ and the corresponding payoffs, we needed to resort to numerical computations. Figure 2 illustrates that although the bidder's payoff strictly increases in n, the gain is small when comparing the optimal payoffs for $n = 2$ and $n = 500$.[17] (Note

[16] The optimum was calculated numerically.

[17] Our numerical calculations suggest that $y_1 = y_2 = ... = y_{n-1}$ holds in the optimum. We imposed this constraint in our calculations when n was large.

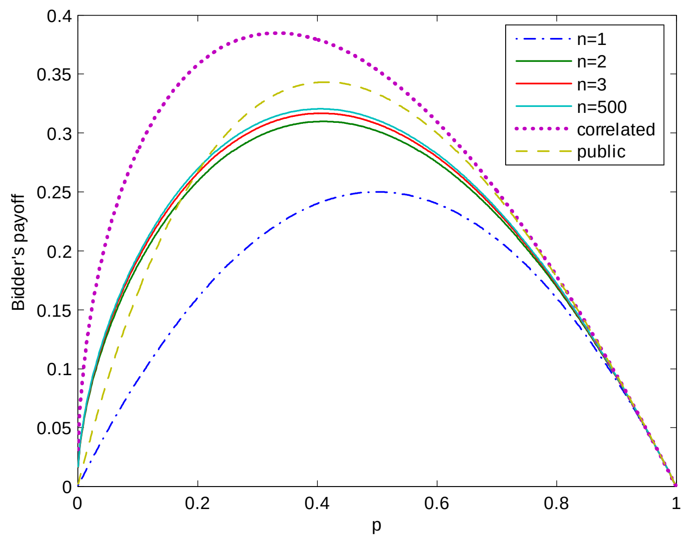

Figure 2: Bidder's payoff for various signal structures

that $n = 1$ is equivalent to the case when there are no signals at all.) Further, the independent signal structures dominate the optimal public signal structure for low values of p, while the opposite result holds for high values of p. However, it can be verified that on average (that is, assuming that the value of p is drawn from a uniform distribution) the optimal public signal structure outperforms the optimal 2-valued independent signal structure. Furthermore, as can be seen from Example 1, the optimal 2-valued independent signal structure depends on the knowledge of prior. Numerical results suggest that the same conclusions hold for $n > 2$. These arguments speak in favour of the use of optimal public signal structure to facilitate collusion. Finally, we can see that the optimal 2-valued correlated signal structure outperforms all other considered signal structures.

6 Collusion without an Omniscient Center

Our previous analysis hinges on the assumption that the bidders have access to a center, which possesses knowledge about their valuations. In the first example that follows, we show how the center can elicit this information from the bidders in a weakly incentive compatible manner by naturally restricting the bids that the bidders can submit after reporting their valuation types to the center. In the second example, the bidders can extract information about each others' valuation from the first round of a sequential auction and bid collusively in the second round. In both examples we stress that the "role" of the omniscient center might be unwillingly played by a party, whose interests differ from those of the bidders.

6.1 Eliciting Bidders' Valuations

To justify the assumption that the center possesses knowledge about the bidders' valuations, here we provide a simple mechanism that allows the center to elicit bidders' private valuations. For that, it is enough to assume that no bidder is allowed to bid above the valuation that she has reported to the center. To be more precise, the mechanism is as follows. First, the bidders privately report their valuations to the center; then, conditional on the reported valuations, the center sends private signals to the bidders; finally, the bidders bid weakly below their reported valuations in the first price auction.

We now argue that a bidder does not have incentives to misreport her valuation if the other bidder reports her valuation truthfully. From Lemma 1 it follows that a type 0 bidder always expects a payoff of zero and has no incentives to exaggerate her valuation. It is also easy to argue that in equilibrium, a high valuation type irrespective of the signal that she receives will never bid strictly below 0 (more generally, below V_L). Suppose a high valuation bidder reports to the center that her valuation is 0 and afterwards she bids 0. She can expect a strictly positive payoff only if there is a tie. We assume that the tie is broken according to the reported valuations and not the true ones. Note that this is consistent with the interpretation in Footnote 10 that ties are broken with the help of second price auction, and our assumption that no bidder can bid above her reported valuation. It follows that a high valuation bidder who has misreported her valuation can only win against the low valuation opponent. But she can achieve the same or even higher payoff by reporting her true valuation and then bidding 0. Hence, there is an equilibrium, in which both bidders reveal their valuations truthfully.

To justify the restriction on bids, we could think of a situation when a regulator, before firms set their prices in a Bertrand duopoly, collects information about their marginal costs. Either during their dealings with the regulator or through the regulator's annual reports about the industry, the firms can obtain noisy information about rival's cost. Hence, here the regulator (unwillingly) plays the role of the center. Clearly, the firms have incentives to exaggerate about their costs and then set low prices. In this case, collusion can not be maintained in equilibrium. Suppose, however, that the regulator does not allow a firm to set price below its reported cost. Such a restriction can be reasonable as the regulator might have concerns about the quality of the service that the firms provide etc.. Therefore, after learning that a firm has set a lower price than its reported cost, the regulator can intervene for the protection of consumers. Though, as our analysis implies, by doing this, the regulator facilitates collusion between the firms and harms consumers.

We conjecture that the result of this subsection also holds for more general setups besides the one with binary valuations.

6.2 A Sequential Auction

Suppose two identical items are sold in a sequential first price auction. There are only two bidders but each bidder demands both items. Furthermore, we assume that each bidder has the *same* valuation for both items, which can be either 0 or 1. The auction proceeds as follows. The bidders submit sealed bids for the first item, and the auctioneer only announces the winning bid. Next, bidders submit sealed bids for the second item, the winner of this item is determined and the payments are made.

Now we present the equilibrium of this sequential auction. Let b_1 and b_2 denote the bids submitted by bidders 1 and 2 in the first round. A stylized distribution of types before the second round is:

	0	$(1, b_1)$	$(1, b_2)$
0	p^2	0	$p(1-p)f(b_2)$
$(1, b_1)$	$p(1-p)f(b_1)$	$(1-p)^2 F(b_1) f(b_1)$	0
$(1, b_2)$	0	0	$(1-p)^2 F(b_2) f(b_2)$

A bidder of type $(1, b_i)$ means that the bidder has high valuation and bidder i has won the first round auction with a bid of b_i. $f(b_i)$ denotes the probability that bidder i submits the bid b_i in the first round. The seller's announcement of the winning bid works as a public signal, which allows the bidders to update their beliefs about the opponent's valuation. Thus, bidder i of type $(1, b_i)$ now believes that the opponent has low valuation with probability

$\frac{p}{p+(1-p)F(b_i)}$, while bidder i of type $(1, b_j)$, $j \neq i$ believes that the opponent has high valuation for sure. From Proposition 2 we know that both bidders of type $\left(1, \hat{b}\right)$, where $\hat{b} = \max\{b_1, b_2\}$, expect the same payoff from the second round, equal to

$$\frac{p}{p + (1-p) F\left(\hat{b}\right)},$$

irrespective of who submitted the winning bid in the first round.

Expected payoff of bidder i with high valuation from both rounds is

$$\begin{aligned} & (1-b_i)(p + (1-p) F(b_i)) \\ + \quad & (p + (1-p) F(b_i)) \times \frac{p}{p + (1-p) F(b_i)} \\ + \quad & (1-p)(1-F(b_i)) \int_{b_i}^{\bar{b}} \frac{p}{p + (1-p) F(b_j)} \frac{f(b_j)}{1 - F(b_j)} db_j, \end{aligned}$$

which can be written as

$$(1-b_i)(p + (1-p) F(b_i)) + p - p \ln(p + (1-p) F(b_i)).$$

At $b_i = 0$, the expected payoff is

$$2p - p \ln p.$$

Thus, the first stage equilibrium strategy is implicitly defined by

$$(1-b_i)(p + (1-p) F(b_i)) - p \ln(p + (1-p) F(b_i)) = p(1 - \ln p)$$

over $\left[0, \bar{b}\right]$ where

$$\bar{b} = 1 - p + p \ln p.$$

We derive two conclusions from this example. First, the expected payoff of high valuation bidder, $p(2 - \ln p)$ is higher than the one she would obtain if the seller did not disclose any information after the first round of auction. In the latter case, the expected payoff of high valuation bidder would be $2p$. Thus, one can say that the seller is effectively facilitating the collusion between the bidders. Second, if the bidders behaved myopically and updated their beliefs based only on the identity of the winner, but not on the winning bid, then the distribution of types before the second round would exactly correspond to the one derived from the optimal public signal structure in

Theorem 2.[18] In this case, the expected payoff of bidder i with high valuation is

$$(1 - b_i)(p + (1 - p) F(b_i)) + \frac{p}{p + (1 - p)\frac{1}{2}}.$$

It follows that in the first round, the bidder would bid exactly as in a standard first price auction in the absence of any signals. The bidder's equilibrium payoff is equal to $p + \frac{2p}{1+p}$, which is less than $p(2 - \ln p)$ but still higher than $2p$. Thus, even if the distribution of types before the second round is different from the one that would be obtained from the optimal public signal structure, the bidders benefit from conditioning their second round beliefs on the winning bid of the first round, since it allows them already to suppress their bids in the first round.

7 Conclusions

We have considered the simplest IPV setup with only two bidders, whose valuations can either take low or high value. After bidders have learnt their own valuations but before submitting their bids in the first price auction, bidders receive private random signals from a center where the randomness depends on the bidders' realized valuation profile. Given the signals, bidders update their beliefs about their opponent's valuation, form beliefs about the signal received by the opponent and submit their bids. We have characterized the equilibrium bidding functions and compared bidders' equilibrium payoffs across different signal structures, and we have found the following. When the signals can take on one of two possible values, we have calculated the optimal symmetric (correlated) signal structure. We have shown that the center must use the information about the valuations of bidders, or otherwise the bidders cannot collude at all. That is, there exists no collusive symmetric strategy-correlated equilibrium for 2-valued valuation-independent symmetric signal structures. If we restrict our attention to perfectly correlated signals (that is, public signals) the center can attain the optimal payoff with only 2-valued public signals. If two signals are chosen independently and identically, the more values the signal can take on the better payoffs bidders can achieve

[18] One can assume that each bidder submits a mixed bidding strategy to a computer, and the computer randomizes accordingly and submits a bid. Hence, the bidder does not know the bid submitted by her own computer. In this case it is enough if the seller only announces the identity of the winner of the first item.

in the symmetric equilibrium. Numerical results suggest that the optimum achieved with 2-valued correlated signals is the overall optimum among the signal structures that have been studied. However, the optimal signal structure depends on the bidders' prior beliefs. It is shown that the optimum achieved with public signals is instead independent of the prior. Hence, a center which does not know the bidders' prior might prefer to use public signals to enhance bidders' collusion. We have also provided an example how a public signal structure emerges endogenously without the help of center after the first round of a sequential auction, hence resulting in collusive bidding in the second round of that auction.

References

AUMANN, R. J. (1974): "Subjectivity and correlation in randomized strategies," *Journal of Mathematical Economics*, 1(1), 67–96.

BERGEMANN, D., AND S. MORRIS (2011): "Correlated Equilibrium in Games with Incomplete Information," DP 1822, Cowles Foundation.

BERGEMANN, D., AND J. VÄLLIMÄKI (2006): "Information in mechanism design," in *Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress*, ed. by R. Blundell, W. K. Newey, and T. Persson, vol. 1, chap. 5, pp. 186–221. Cambridge University Press.

COTTER, K. D. (1991): "Correlated equilibrium in games with type-dependent strategies," *Journal of Economic Theory*, 54, 48–68.

FANG, H., AND S. MORRIS (2006): "Multidimensional private value auctions," *Journal of Economic Theory*, 126, 1–30.

FORGES, F. (1993): "Five legitimate definitions of correlated equilibrium in games with incomplete information," *Theory and Decision*, 35(3), 277–310.

——— (2006): "Correlated Equilibrium in Games with Incomplete Information Revisited," *Theory and Decision*, 61(4), 329–344.

GENESOVE, D., AND W. P. MULLIN (1997): "The Sugar Institute Learns to Organize Information Exchange," WP 5981, NBER.

KAPLAN, T. R., AND S. ZAMIR (2000): "The Strategic Use of Seller Information in Private-Value Auctions," Working Paper.

KIM, J., AND Y.-K. CHE (2004): "Asymmetric information about rivals' types in standard auctions," *Games and Economic Behavior*, 46, 383–397.

LOPOMO, G., L. M. MARX, AND P. SUN (2011): "Bidder collusion at first-price auctions," *Review of Economic Design*, 15, 177–211.

MARSHALL, R. C., AND L. M. MARX (2007): "Bidder collusion," *Journal of Economic Theory*, 133, 374–402.

MASKIN, E. S., AND J. G. RILEY (1985): "Auction Theory with Private Values," *The American Economic Review*, 75(2), 150–155.

——— (2000a): "Asymmetric Auctions," *Review of Economic Studies*, 67(3), 413–438.

——— (2000b): "Equilibrium in Sealed High Bid Auctions," *Review of Economic Studies*, 67(3), 439–454.

MCAFEE, R. P., AND J. MCMILLAN (1992): "Bidding Rings," *The American Economic Review*, 82(3), 579–599.

PAVLOV, G. (2009): "Communication Equilibria in All Pay Auctions," Working Paper.

SKRETA, V. (2011): "On the informed seller problem: optimal information disclosure," *Review of Economic Design*, 15(1), 1–36.

VIVES, X. (1990): "Trade Association Disclosure Rules, Incentives to Share Information, and Welfare," *The RAND Journal of Economics*, 21(3), 409–430.

8 Appendix

Proposition 1 *The unique symmetric equilibrium of the first price auction is as follows:*

Case 1 *If* $\Pr(1|1) \geq \Pr(1|2) > 0$ *and*

$$\frac{\Pr(0|1)}{\Pr(0|2)} \geq \frac{\Pr(1|1)}{\Pr(1|2)},$$

then

1. *type 1 randomizes on* $\left[0, \bar{b}_1\right]$ *according to*

$$F_1(b) = \frac{\Pr(0|1)}{\Pr(1|1)} \frac{b}{1-b} \tag{8}$$

where

$$\bar{b}_1 = \frac{\Pr(1|1)}{\Pr(0|1) + \Pr(1|1)}; \tag{9}$$

2. *type 2 randomizes on* $\left[\bar{b}_1, \bar{b}_2\right]$ *according to*

$$F_2(b) = \frac{\Pr(0|2) + \Pr(1|2)}{\Pr(2|2)} \frac{b - \bar{b}_1}{1-b} \tag{10}$$

where

$$\bar{b}_2 = 1 - (\Pr(0|2) + \Pr(1|2)) \left(1 - \bar{b}_1\right). \tag{11}$$

Case 2 *If* $\Pr(1|1) > \Pr(1|2) > 0$ *and*

$$\frac{\Pr(1|1)}{\Pr(1|2)} > \frac{\Pr(0|1)}{\Pr(0|2)},$$

then

1. *type 1 randomizes on* $\left[0, \bar{b}_1\right]$ *according to*

$$F_1(b) = \frac{\Pr(2|2)\Pr(0|1) - \Pr(2|1)\Pr(0|2)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)} \frac{b}{1-b}$$

where

$$\bar{b}_1 = \frac{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}{\Pr(2|2) - \Pr(2|1)};$$

2. *type 2 randomizes on* $\left[0, \overline{b}_1\right]$ *according to*

$$F_2(b) = \frac{\Pr(1|1)\Pr(0|2) - \Pr(1|2)\Pr(0|1)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)} \frac{b}{1-b}$$

and on $\left[\overline{b}_1, \overline{b}_2\right]$ *according to*

$$F_2(b) = \frac{\Pr(0|2)}{\Pr(2|2)} \frac{b}{1-b} - \frac{\Pr(1|2)}{\Pr(2|2)}$$

where $\overline{b}_2 = 1 - \Pr(0|2)$.

Case 3 *If* $\Pr(1|1) < \Pr(1|2)$ *and* $0 < \Pr(2|1) \leq \Pr(2|2)$, *then the types bid as in Case 1.*

Case 4 *If* $\Pr(1|1) < \Pr(1|2)$ *and* $\Pr(2|1) > \Pr(2|2)$, *then*

1. *type 1 randomizes on* $[0, \underline{b}_2]$ *according to (8), where*

$$\underline{b}_2 = \frac{\Pr(1|1)(\Pr(0|1) - \Pr(0|2))}{\Pr(1|2)\Pr(0|1) - \Pr(1|1)\Pr(0|2)},$$

and on $\left[\underline{b}_2, \overline{b}_2\right]$ *according to*

$$\begin{aligned} F_1(b) &= \frac{\Pr(0|1)(\Pr(2|2) - \Pr(2|1))}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)} \frac{1}{1-b} \\ &+ \frac{\Pr(2|1)\Pr(0|2) - \Pr(2|2)\Pr(0|1)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)} \end{aligned}$$

where $\overline{b}_2 = 1 - \Pr(0|1)$;

2. *type 2 randomizes on* $\left[\underline{b}_2, \overline{b}_2\right]$ *according to*

$$\begin{aligned} F_2(b) &= \frac{\Pr(0|1)(\Pr(1|1) - \Pr(1|2))}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)} \frac{1}{1-b} \\ &+ \frac{\Pr(1|2)\Pr(0|1) - \Pr(1|1)\Pr(0|2)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}. \end{aligned}$$

Proof of Proposition 1. Without loss of generality, let $\Pr(0|1) \geq \Pr(0|2)$. We also assume that $\Pr(2|1) > 0$, $\Pr(1|2) > 0$ and $\Pr(k|1) \neq \Pr(k|2)$ for some $k \in \{0,1,2\}$ hold. As discussed in the main text, if these assumptions are not satisfied, the expected payoff of bidder is $p(1-p)$.

It is easy to argue that the supremum of all submitted bids does not exceed 1 in any equilibrium. Also, by bidding 1, a type i bidder, $i = 1, 2$, expects a payoff of 0. To sustain it as a part of equilibrium, she must assign zero probability to the opponent bidding anything less than 1. However, given that $x_1 + x_2 = 1$ and $r_{12} > 0$ holds, there is a type of opponent who will bid for sure strictly less than 1.[19] It follows that the supremum of all submitted bids is strictly less than 1, and each type i bidder, $i = 1, 2$, expects a strictly positive payoff and wins with a strictly positive probability in any equilibrium.

Suppose now that bidder 2 of a high valuation type bids $0 < \tilde{b} < 1$ with a strictly positive probability in an equilibrium. Then bidder 1 of either high valuation type, instead of bidding in the interval $\left[\tilde{b} - \delta, \tilde{b}\right]$ for some $\delta > 0$, is better off by bidding $\tilde{b} + \epsilon$ where $\epsilon > 0$ is sufficiently small. But then bidder 2 is better off to bid $\tilde{b} - \delta$ instead of $\tilde{b}$, which contradicts our assumption that she bids $0 < \tilde{b} < 1$ with a strictly positive probability. Thus, only the bid of 0 can possibly occur with a strictly positive probability. Further, the high valuation bidders cannot tie with a positive probability at 0 in the equilibrium, as either bidder will instead prefer to bid just above 0. In particular, it implies that if type i bids 0 with a positive probability then $r_{ii} = 0$.

Let $F_i(b)$ be the distribution function according to which type i, $i = 1, 2$, randomizes in an equilibrium. Consider the union of supports of $F_1(b)$ and $F_2(b)$. We claim that this union of supports is a connected set. Suppose to the contrary that bidder 2 with high valuation bids $\tilde{b}$ but does not bid in the interval $\left[\tilde{b} - \delta, \tilde{b}\right)$. But then bidder 1 with high valuation prefers bidding $\tilde{b} - \delta$ instead of $\tilde{b}$. In the same way we can argue that the lower limit of the union of supports is 0.

Since there are no ties between high valuation bidders in equilibrium, the expected payoff of type i bidder from submitting a bid b is given by

$$\{\Pr(0|i) + \Pr(i|i) F_i(b) + \Pr(j|i) F_j(b)\}(1-b). \tag{12}$$

[19] If, for example, $x_2 = r_{12} = 0$, then type 2 would bid 1 with probability 1 in the equilibrium.

Let π_i denote the expected payoff of type i, $i = 1, 2$. Equation (12) implies that $\pi_i \geq \Pr(0|i)$. We have already argued that there is a type i such that $\underline{b}_i = 0$. Therefore, either $\pi_1 = \Pr(0|1)$ or $\pi_2 = \Pr(0|2)$ or both must hold. If $\pi_i = \Pr(0|i)$ for $i = 1, 2$, then the ex ante payoffs are the same as in the absence of collusive communication. If $\overline{b}_1 = \overline{b}_2$, then $\pi_1 = \pi_2 = \max\{\Pr(0|1), \Pr(0|2)\} = \Pr(0|1)$. If $\overline{b}_i > \overline{b}_j$, then $\pi_j \geq \pi_i = 1 - \overline{b}_i$.

Now we derive what are the restrictions on the conditional probabilities for one or another equilibrium configuration to exist, and next, using the identified restrictions, we determine what is the equilibrium configuration for each signal structure.

Suppose there is an equilibrium in which $0 = \underline{b}_i < \underline{b}_j$ holds. Then $\Pr(i|i) > 0$, and the expected payoff of type i from bidding on the interval $\left[0, \underline{b}_j\right]$ is

$$\{\Pr(0|i) + \Pr(i|i) F_i(b)\}(1 - b) = \Pr(0|i),$$

which implies that

$$F_i(b) = \frac{\Pr(0|i)}{\Pr(i|i)} \frac{b}{1-b} \tag{13}$$

for $0 \leq b \leq \underline{b}_j$. Consider now type j, who deviates and bids below $\underline{b}_j$. The expected payoff is

$$\begin{aligned} & \left\{\Pr(0|j) + \Pr(i|j) \frac{\Pr(0|i)}{\Pr(i|i)} \frac{b}{1-b}\right\}(1-b) \\ = \;& \Pr(0|j) + \left\{\Pr(i|j) \frac{\Pr(0|i)}{\Pr(i|i)} - \Pr(0|j)\right\} b. \end{aligned}$$

It must be that the term in curly brackets is positive or otherwise type j will find it profitable to deviate. Hence,

$$\frac{\Pr(0|i)}{\Pr(0|j)} \geq \frac{\Pr(i|i)}{\Pr(i|j)} \tag{14}$$

must hold.

Suppose there is an equilibrium in which $\overline{b}_i > \overline{b}_j$ holds. Then $\Pr(i|i) > 0$, and the expected payoff of type i from bidding on the interval $\left[\overline{b}_j, \overline{b}_i\right]$ is

$$\{\Pr(0|i) + \Pr(j|i) + \Pr(i|i) F_i(b)\}(1 - b) = 1 - \overline{b}_i,$$

which implies that

$$F_i(b) = \frac{1}{\Pr(i|i)} \left[\frac{1-\bar{b}_i}{1-b} - (\Pr(0|i) + \Pr(j|i)) \right] \tag{15}$$

for $b \in [\bar{b}_j, \bar{b}_i]$. Consider now type j, who deviates and bids above $\bar{b}_j$. The expected payoff is

$$\begin{aligned} & \left\{ \Pr(0|j) + \Pr(j|j) + \frac{\Pr(i|j)}{\Pr(i|i)} \left[\frac{1-\bar{b}_i}{1-b} - (\Pr(0|i) + \Pr(j|i)) \right] \right\} (1-b) \\ = & \frac{\Pr(i|j)}{\Pr(i|i)} (1-\bar{b}_i) + \left\{ \Pr(0|j) + \Pr(j|j) - \frac{\Pr(i|j)}{\Pr(i|i)} (\Pr(0|i) + \Pr(j|i)) \right\} (1-b). \end{aligned}$$

It must be that the term in curly brackets is positive or otherwise type j will find it profitable to deviate. Hence,

$$\frac{\Pr(0|j) + \Pr(j|j)}{\Pr(i|j)} \geq \frac{\Pr(0|i) + \Pr(j|i)}{\Pr(i|i)}$$

or

$$\Pr(i|i) \geq \Pr(i|j) \tag{16}$$

must hold.

Suppose that $0 = \underline{b}_i \leq \underline{b}_j < \bar{b}_i \leq \bar{b}_j$, that is, there exists an interval, in which both types bid in an equilibrium. For the bids in this interval we can set the expression in (12) equal to π_i and multiply both sides with $\Pr(j|j)$. Similarly, we multiply the analogous expression for type j with $\Pr(j|i)$:

$$\begin{aligned} \Pr(j|j)\,\pi_i &= \Pr(j|j)\{\Pr(0|i) + \Pr(i|i)\,F_i(b) + \Pr(j|i)\,F_j(b)\}(1-b), \\ \Pr(j|i)\,\pi_j &= \Pr(j|i)\{\Pr(0|j) + \Pr(i|j)\,F_i(b) + \Pr(j|j)\,F_j(b)\}(1-b). \end{aligned}$$

Subtracting the second line from the first and re-arranging, we obtain the expression for $F_i(b)$ in the region where the supports intersect:

$$F_i(b) = \frac{\Pr(j|j)\,\pi_i - \Pr(j|i)\,\pi_j}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)} \frac{1}{1-b} + \frac{\Pr(j|i)\Pr(0|j) - \Pr(j|j)\Pr(0|i)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}. \tag{17}$$

The conditional probabilities and payoffs must be such that $F_i(b)$ is a strictly increasing function in this region. Also note that if $\Pr(2|2)\Pr(1|1) = \Pr(2|1)\Pr(1|2)$, then the supports of both types cannot overlap.

We are now in the position to determine the equilibria for all possible values of conditional probabilities. To simplify the analysis, we initially find a symmetric equilibrium, in which the support of bids for each type is connected. Afterwards, we will argue that it is the unique symmetric equilibrium by showing that there are no symmetric equilibria, in which the support for either type is disconnected.

Case 1. $\Pr(1|1) \geq \Pr(1|2) > 0$ and

$$\frac{\Pr(0|1)}{\Pr(0|2)} \geq \frac{\Pr(1|1)}{\Pr(1|2)} \tag{18}$$

imply that $0 < \Pr(2|1) < \Pr(2|2)$ and

$$\frac{\Pr(0|2)}{\Pr(0|1)} < \frac{\Pr(2|2)}{\Pr(2|1)} \tag{19}$$

also hold. According to (14), conditions (18) and (19), in turn, imply that $0 = \underline{b}_1 \leq \underline{b}_2$ must hold. We argue by contradiction that $\underline{b}_2 < \bar{b}_1$ cannot happen in the equilibrium. If $\pi_1 \geq \Pr(0|1)$ and $\pi_2 = \Pr(0|2)$, then $F_2(b)$ is a non-increasing function in (17) for $(i,j) = (2,1)$. Therefore, if $\underline{b}_2 < \bar{b}_1$, then $\pi_1 = \Pr(0|1)$ and $\pi_2 > \Pr(0|2)$, which also implies that $F_2(\underline{b}_2) = 0$. However, evaluating (17) for $(i,j) = (2,1)$ at $\underline{b}_2$, we find that

$$1 - \underline{b}_2 = \frac{\Pr(1|1)\,\pi_2 - \Pr(1|2)\Pr(0|1)}{\Pr(1|1)\Pr(0|2) - \Pr(1|2)\Pr(0|1)} > 1,$$

or $\underline{b}_2 < 0$. Thus, it follows that $\bar{b}_1 = \underline{b}_2$ must hold, that is, the types bid on adjacent intervals. Thus, type 1 bids on $\left[0, \bar{b}_1\right]$ according to (13) or (8), where $\bar{b}_1$ is defined by $F_1\left(\bar{b}_1\right) = 1$ and is given in (9). Type 2 bids on $\left[\bar{b}_1, \bar{b}_2\right]$ according to (15). Since type 2 must be indifferent between bidding $\bar{b}_1$ and $\bar{b}_2$, it follows that $\bar{b}_2$ is given by (11). This allows to rewrite (15) into the form given in (10).

Case 2. $\Pr(1|1) > \Pr(1|2) > 0$ and

$$\frac{\Pr(1|1)}{\Pr(1|2)} > \frac{\Pr(0|1)}{\Pr(0|2)} \tag{20}$$

imply that $0 < \Pr(2|1) < \Pr(2|2)$ and

$$\frac{\Pr(0|2)}{\Pr(0|1)} < \frac{\Pr(2|2)}{\Pr(2|1)} \tag{21}$$

also hold. According to (14), conditions (20) and (21) imply that it must be the case that $\underline{b}_1 = \underline{b}_2 = 0$. If $0 = \underline{b}_i < \underline{b}_j$, then type j would have incentives to bid below $\underline{b}_j$. $\Pr(1|1) > 0$ and $\Pr(2|2) > 0$ imply that there cannot be a mass point at 0, and the payoffs are $\pi_1 = \Pr(0|1)$ and $\pi_2 = \Pr(0|2)$, which results in the same ex ante payoffs as in the case without collusive communication. Further, given the assumed conditions on probabilities, we can verify that the distribution function in (17) is well-defined for $i = 1, 2$, that is, it is positively sloped. Also, according to (17), the values of b at which $F_1(b) = 1$ and $F_2(b) = 1$ hold, respectively, are

$$\begin{aligned} b &= \frac{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}{\Pr(2|2) - \Pr(2|1)}, \\ b &= \frac{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}{\Pr(1|1) - \Pr(1|2)}. \end{aligned}$$

Since the former value of b is smaller than the latter, it means that $F_1(b)$ reaches 1 earlier. Therefore,

$$\overline{b}_1 = \frac{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}{\Pr(2|2) - \Pr(2|1)} > 0$$

and $\overline{b}_2 = 1 - \pi_2 = 1 - \Pr(0|2) \geq \overline{b}_1$. Further, one can verify that distribution functions (15) and (17) for $(i, j) = (2, 1)$ evaluated at $\overline{b}_1$, give the same answer:

$$F_2\left(\overline{b}_1\right) = \frac{\Pr(0|2)\Pr(1|1) - \Pr(0|1)\Pr(1|2)}{\Pr(2|2)\Pr(0|1) - \Pr(2|1)\Pr(0|2)} > 0.$$

To summarize, in the equilibrium, the types 1 and 2 bid according to (17), where $\pi_1 = \Pr(0|1)$ and $\pi_2 = \Pr(0|2)$, on the interval $\left[0, \overline{b}_1\right]$, and type 2 also bids on $\left[\overline{b}_1, \overline{b}_2\right]$ according to (15).

Case 3. According to (16), conditions $\Pr(1|1) < \Pr(1|2)$ and $\Pr(2|1) \leq \Pr(2|2)$ imply that $\overline{b}_1 \leq \overline{b}_2$ holds. But then $\pi_2 = 1 - \overline{b}_2$, and it must be the case that $\pi_1 \geq \pi_2$ or otherwise type 1 would deviate and bid $\overline{b}_2$. We argue that $\underline{b}_2 < \overline{b}_1$ cannot happen in the equilibrium. Conditions $\Pr(1|1) < \Pr(1|2)$ and $\Pr(2|1) \leq \Pr(2|2)$ together with $\pi_1 \geq \pi_2$ imply that $F_i(b)$ in (17) is a decreasing function for one of the types, depending on the sign of $\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)$. Therefore, $\overline{b}_1 = \underline{b}_2$ must hold, and the types bid as in Case 1.

Case 4. According to (16), conditions $\Pr(1|1) < \Pr(1|2)$ and $\Pr(2|1) > \Pr(2|2)$ imply that there cannot be an equilibrium in which $\overline{b}_i > \overline{b}_j$, as type

j will want to deviate and bid above $\bar{b}_j$. Hence, if there is an equilibrium, $\bar{b}_1 = \bar{b}_2$ must hold. This in turn implies that $\pi_1 = \pi_2$. As we know that $\pi_i \geq \Pr(0|i)$ for $i = 1, 2$ and there exists a type j for whom $\pi_j = \Pr(0|j)$, it must be the case that $\pi_1 = \pi_2 = \Pr(0|1)$. This in turn implies that $F_2(\underline{b}_2) = 0$, and $0 = \underline{b}_1 \leq \underline{b}_2$. Also, since $\pi_i = 1 - \bar{b}_i$ for $i = 1, 2$, then $\bar{b}_1 = \bar{b}_2 = 1 - \Pr(0|1)$.

Since $F_2(\underline{b}_2) = 0$, from (17) for $(i, j) = (2, 1)$, we can solve out for $\underline{b}_2$:

$$\underline{b}_2 = \frac{\Pr(1|1)(\Pr(0|1) - \Pr(0|2))}{\Pr(1|2)\Pr(0|1) - \Pr(1|1)\Pr(0|2)} \geq 0.$$

Further, from (17) for $(i, j) = (1, 2)$, we find that

$$F_1(\underline{b}_2) = \frac{\Pr(0|1) - \Pr(0|2)}{\Pr(1|2) - \Pr(1|1)} \geq 0. \tag{22}$$

Also note that (13) for $i = 1$ gives the same expression for $F_1(\underline{b}_2)$. We can also verify that $F_1(\underline{b}_2) < 1$. It holds if $\Pr(0|1) + \Pr(1|1) < \Pr(0|2) + \Pr(1|2)$, which is indeed true as it is equivalent to $\Pr(2|1) > \Pr(2|2)$. Finally, one can also verify that $\underline{b}_2 < \bar{b}_1 = \bar{b}_2$. Hence, we have characterized the equilibrium, in which type 1 bids according to (13) in $[0, \underline{b}_2]$, and both types bid according (17), where $\pi_1 = \pi_2 = \Pr(0|1)$, on the interval $\left[\underline{b}_2, \bar{b}_2\right]$.

Finally, we prove that in any symmetric equilibrium, the support of equilibrium bids for each type must be connected. Suppose to the contrary that there exists an interval $\left(\underline{b}, \bar{b}\right)$ such that only type i bids in this interval, while type j (and possibly type i) bids on two disconnected intervals, and $\underline{b}$ is the upper limit of the first of these intervals, while $\bar{b}$ is the lower limit of the second of these intervals. Note that $F_j(\underline{b}) = F_j\left(\bar{b}\right)$ and $\Pr(i|i) > 0$ hold. The expected payoff of type i from bidding in the interval $\left(\underline{b}, \bar{b}\right)$ is

$$\pi_i = \{\Pr(0|i) + \Pr(j|i) F_j(\underline{b}) + \Pr(i|i) F_i(b)\}(1 - b),$$

which implies that

$$F_i(b) = \frac{1}{\Pr(i|i)}\left[\frac{\pi_i}{1 - b} - (\Pr(0|i) + \Pr(j|i) F_j(\underline{b}))\right].$$

Consider now type j, who deviates and bids in $\left(\underline{b}, \bar{b}\right)$. The expected payoff is

$$\begin{aligned}
&\left\{\Pr(0|j) + \Pr(j|j) F_j(\underline{b}) + \frac{\Pr(i|j)}{\Pr(i|i)}\left[\frac{\pi_i}{1 - b} - (\Pr(0|i) + \Pr(j|i) F_j(\underline{b}))\right]\right\}(1 - b) \\
= \; & \frac{\Pr(i|j)}{\Pr(i|i)}\pi_i + \left\{\Pr(0|j) + \Pr(j|j) F_j(\underline{b}) - \frac{\Pr(i|j)}{\Pr(i|i)}(\Pr(0|i) + \Pr(j|i) F_j(\underline{b}))\right\}(1 - b).
\end{aligned}$$

The above expression is linear in b. Further, note that in equilibrium, type j is indifferent between bidding $\underline{b}$ and $\overline{b}$. Therefore, he must be indifferent among all bids in the interval $\left(\underline{b}, \overline{b}\right)$, which implies that the expression in the curly brackets is zero. Then,[20]

$$F_j\left(\underline{b}\right) = F_j\left(\overline{b}\right) = \frac{\Pr\left(0|i\right)\Pr\left(i|j\right) - \Pr\left(0|j\right)\Pr\left(i|i\right)}{\Pr\left(2|2\right)\Pr\left(1|1\right) - \Pr\left(2|1\right)\Pr\left(1|2\right)}, \tag{23}$$

and

$$\pi_j = \frac{\Pr\left(i|j\right)}{\Pr\left(i|i\right)}\pi_i. \tag{24}$$

It immediately follows from (23) that there is at most one discontinuity in the support for each type.

We again consider all four cases.

Case 1. According to $\Pr\left(1|1\right) \geq \Pr\left(1|2\right)$, $\Pr\left(2|1\right) < \Pr\left(2|2\right)$, and (19), it is true that $F_1\left(\underline{b}\right) < 0$ for $(i, j) = (2, 1)$ in (23). Therefore, the support of bids for type 1 is connected. Further, from the previous analysis of Case 1, we already know that $\underline{b}_1 = 0$. It means that type $i = 1$ must bid not only in the interval $\left(\underline{b}, \overline{b}\right)$, but also in $[0, \underline{b}]$. But then there is an interval, in which both types bid, which has already been ruled out by the previous analysis. Hence, it must be that the support of bids for type $j = 2$ is also connected.

Case 2. According to $\Pr\left(1|1\right) > \Pr\left(1|2\right)$, $\Pr\left(2|1\right) < \Pr\left(2|2\right)$, (20), and (21), we obtain that $F_1\left(\underline{b}\right) < 0$ and $F_2\left(\underline{b}\right) < 0$ in (23), which is a contradiction. Therefore, the support of bids for each type must be connected.

Case 3. We already know that the supports of both types cannot overlap and $\overline{b}_1 < \overline{b}_2$ holds. Therefore, if the support of bids for type 1 is disconnected, then so is the support of bids for type 2. Thus, it is sufficient to argue that the support of bids for type 2 must be connected. We already know that $\pi_1 \geq \pi_2$. However, equation (24) for $(i, j) = (1, 2)$ together with $\Pr\left(1|1\right) < \Pr\left(1|2\right)$, implies that $\pi_1 < \pi_2$. Hence, we have obtained a contradiction.

Case 4. We know that in any equilibrium $\overline{b}_1 = \overline{b}_2$ and $\pi_1 = \pi_2$ must hold. Then, (24) implies that $\Pr\left(i|j\right) = \Pr\left(i|i\right)$, which contradicts the assumptions about conditional probabilities that $\Pr\left(1|1\right) < \Pr\left(1|2\right)$ and

[20] Alternatively, the expression in the curly brackets is zero if $\Pr\left(k|1\right) = \Pr\left(k|2\right)$ for all k, but this case has been ruled out.

$\Pr(2|1) > \Pr(2|2)$.

This completes the proof that the support of bids must always be connected for each type in a symmetric equilibrium. This also implies that the equilibrium that we have found before is the unique symmetric equilibrium for each distribution of signals. ■

Proof of Theorem 1. We structure the proof along the cases identified in Proposition 1.

Case 1 and 3. These cases result in the same equilibrium structure whereby the types bid on adjacent intervals. Combining them, the restrictions on probabilities such that the types bid on adjacent intervals are

$$\begin{aligned}\Pr(0|1) &\geq \Pr(0|2),\\ \Pr(2|2) &\geq \Pr(2|1),\\ \frac{\Pr(0|1)}{\Pr(1|1)} &\geq \frac{\Pr(0|2)}{\Pr(1|2)},\end{aligned}$$

or equivalently,

$$\begin{aligned}\frac{px_1}{px_1+(1-p)(r_{11}+r_{12})} &\geq \frac{px_2}{px_2+(1-p)(r_{12}+r_{22})},\\ \frac{(1-p)r_{22}}{px_2+(1-p)(r_{12}+r_{22})} &\geq \frac{(1-p)r_{12}}{px_1+(1-p)(r_{11}+r_{12})},\\ \frac{x_1}{r_{11}} &\geq \frac{x_2}{r_{12}}.\end{aligned}$$

The payoff of high valuation bidder before she receives a signal is

$$\begin{aligned}&\Pr(1)\pi_1+\Pr(2)\pi_2\\ =\;&\Pr(1)\Pr(0|1)+\Pr(2)\left(1-\bar{b}_2\right)\\ =\;&\Pr(1)\Pr(0|1)+\Pr(2)\left(\Pr(0|2)+\Pr(1|2)\right)\frac{\Pr(0|1)}{\Pr(0|1)+\Pr(1|1)}\\ =\;&px_1\left(1+\frac{px_2+(1-p)r_{12}}{px_1+(1-p)r_{11}}\right)\\ =\;&px_1\frac{p+(1-p)(r_{11}+r_{12})}{px_1+(1-p)r_{11}},\end{aligned}\tag{25}$$

where we have used (9) and (11), and $\Pr(i)$ for $i=1,2$ denotes the probability that the bidder will be of type i.

Hence, the problem that we are solving is[21]

$$\max_{x_1,x_2,r_{11},r_{12},r_{22}} px_1 \frac{p+(1-p)(r_{11}+r_{12})}{px_1+(1-p)r_{11}} \tag{P1}$$

subject to

$$\begin{aligned}
x_1+x_2 &= 1,\\
r_{11}+2r_{12}+r_{22} &= 1,\\
\frac{x_1}{r_{11}} &\geq \frac{x_2}{r_{12}},\\
\frac{px_1}{px_1+(1-p)(r_{11}+r_{12})} &\geq \frac{px_2}{px_2+(1-p)(r_{12}+r_{22})},\\
\frac{(1-p)r_{22}}{px_2+(1-p)(r_{12}+r_{22})} &\geq \frac{(1-p)r_{12}}{px_1+(1-p)(r_{11}+r_{12})},
\end{aligned}$$

and all probabilities $(x_1, x_2, r_{11}, r_{12}, r_{22})$ must be non-negative. The non-negativity constraints together with the above equality constraints ensure that each of the probabilities is also less than 1. The inspection of this program tells that the objective function is increasing in x_1. Therefore, setting $x_1 = 1$ and $x_2 = 0$ is optimal, since it does not violate any of the constraints. Using these results, we simplify our problem to

$$\max_{r_{11},r_{12},r_{22}} \frac{r_{12}}{p+(1-p)r_{11}}$$

subject to

$$\begin{aligned}
r_{11}+2r_{12}+r_{22} &= 1,\\
r_{22}(p+(1-p)r_{11}) &\geq (1-p)r_{12}^2,
\end{aligned}$$

and (r_{11}, r_{12}, r_{22}) must be non-negative. Note that we have taken a monotone transformation of the objective function.

Suppose the above inequality does not bind. Then it is always possible to increase the payoff by raising r_{12} by a small amount, and correspondingly decreasing r_{22}. (If $r_{22} = 0$, then $r_{12} = 0$ must also hold. But then the objective also takes zero value, which is clearly not a maximum.) Therefore,

[21] Since we restrict attention to symmetric signal structures and symmetric equilibria, it is enough to maximize the payoff of single bidder.

$r_{22}(p+(1-p)r_{11}) = (1-p)r_{12}^2$ holds. Using both equalities we can solve for r_{11} and r_{12} as functions of r_{22}

$$r_{11} = 1-2\frac{\sqrt{r_{22}}}{\sqrt{1-p}}+r_{22}, \tag{26}$$

$$r_{12} = \frac{\sqrt{r_{22}}}{\sqrt{1-p}}-r_{22}, \tag{27}$$

and write the objective function as

$$\max_{r_{22}} \frac{\sqrt{r_{22}}}{\frac{1}{\sqrt{1-p}}-\sqrt{r_{22}}},$$

which is always increasing in r_{22}.

To now, we have ignored the non-negativity constraints that $r_{11} \geq 0$ and $r_{12} > 0$ must also be satisfied. One can verify from (27) that $r_{12} > 0$ is satisfied for all $0 < r_{22} \leq 1$, while from (26), r_{11} is always decreasing in r_{22}. Therefore, r_{22} can be raised up to the point where $r_{11} = 0$, implying that

$$r_{12} = \frac{\sqrt{p}}{1+\sqrt{p}},$$

$$r_{22} = \frac{1-\sqrt{p}}{1+\sqrt{p}}.$$

Evaluating (25) at the optimum gives that the payoff of high valuation bidder before she receives a signal is $\sqrt{p}$. Therefore, bidder's ex ante payoff is $(1-p)\sqrt{p}$ in the optimum.

Case 2. The payoffs of types 1 and 2 are, respectively, $\pi_1 = \Pr(0|1)$ and $\pi_2 = \Pr(0|2)$, therefore the ex ante payoff of bidder is the same as in the case without collusive communication, $p(1-p)$.

Case 4. The payoff of high valuation bidder in this equilibrium is $\Pr(0|1)$, irrespective of her type. Hence, we are solving the following problem:

$$\max \Pr(0|1)$$

subject to $\Pr(0|1) \geq \Pr(0|2)$, $\Pr(1|1) < \Pr(1|2)$, and $\Pr(2|1) > \Pr(2|2)$. Constraint $\Pr(1|1) < \Pr(1|2)$ can be ignored as it is implied by the other two inequalities. Thus,

$$\max_{x_1,x_2,r_{11},r_{12},r_{22}} \frac{px_1}{px_1+(1-p)(r_{11}+r_{12})} \tag{P2}$$

subject to

$$\begin{aligned}
x_1 + x_2 &= 1, \\
r_{11} + 2r_{12} + r_{22} &= 1, \\
\frac{px_1}{px_1 + (1-p)(r_{11}+r_{12})} &\geq \frac{px_2}{px_2 + (1-p)(r_{12}+r_{22})}, \\
\frac{(1-p)\, r_{12}}{px_1 + (1-p)(r_{11}+r_{12})} &> \frac{(1-p)\, r_{22}}{px_2 + (1-p)(r_{12}+r_{22})},
\end{aligned}$$

and all probabilities $(x_1, x_2, r_{11}, r_{12}, r_{22})$ must be non-negative. The objective is increasing in x_1 and decreasing in r_{11} and r_{12}. Consider decreasing r_{11} by 2δ, while increasing r_{12} by δ. Then, none of the constraints is violated and the objective has increased. Therefore, it is optimal to set $r_{11} = 0$. We can re-write the inequality constraints as

$$\frac{px_2}{px_2 + (1-p)(r_{12}+r_{22})} \leq \frac{px_1}{px_1 + (1-p)\, r_{12}} < \frac{px_2 + (1-p)\, r_{12}}{px_2 + (1-p)(r_{12}+r_{22})}.$$

We want

$$\frac{px_1}{px_1 + (1-p)\, r_{12}}$$

to be as high as possible but strictly less than the right-most expression in the above constraint. It follows that this program does not have a maximum. If we sat the second inequality as equality in the above constraint, we would obtain that $\Pr(0|1) = 1 - \Pr(2|2)$. This, together with $\Pr(1|1) = 0$, implies that $F_1(\underline{b}_2) = 1$ in (22), that is, the types bid on adjacent intervals, which contradicts the equilibrium structure of Case 4.

Although program (P2) does not have a maximum, we still need to verify that its supremum does not exceed the maximum that we have found for program (P1). To find the supremum, we can rewrite (P2) in the following form:

$$\max_{x_1, x_2, r_{12}, r_{22}} p + (1-p)\, r_{12} \tag{P3}$$

subject to

$$\begin{aligned}
x_1 + x_2 &= 1, \\
2r_{12} + r_{22} &= 1, \\
\frac{px_1}{px_1 + (1-p)\, r_{12}} &\geq \frac{px_2}{px_2 + (1-p)(r_{12}+r_{22})}, \\
\frac{px_1}{px_1 + (1-p)\, r_{12}} &= \frac{px_2 + (1-p)\, r_{12}}{px_2 + (1-p)(r_{12}+r_{22})},
\end{aligned}$$

and all probabilities $(x_1, x_2, r_{12}, r_{22})$ must be non-negative, where the objective of (P3) is obtained by combining the three equality constraints to express x_1 as a function of r_{12}. At the same time, we know that the solution to program (P1) satisfies $r_{11} = 0$ and the third inequality holds as equality. If we impose these constraints on (P1) from the outset, we obtain program (P3). (The first inequality in (P1) is automatically satisfied when $r_{11} = 0$ and therefore can be ignored.) Hence, we conclude that (P1) and (P3) have the same solution. This completes the proof that the optimal signal structure is given as the solution to program (P1). ■

Proof of Proposition 2. Given a public signal $i \in N$, if $r_{ii} = 0$, then the bidder with a high valuation knows that the opponent has a low valuation and therefore both will bid 0. If $x_{1.i} = x_{2.i} = 0$ and $r_{ii} > 0$, then it is common knowledge that the valuations of both bidders are equal to 1. It is a standard argument to show that the unique equilibrium involves both bidders submitting bids equal to 1.

Suppose that $x_{l.i} \geq x_{m.i}$, $x_{l.i} > 0$,and $r_{ii} > 0$. First, note that $F_{l.i}\left(\bar{b}_i\right) = F_{m.i}\left(\bar{b}_i\right) = 1$ is satisfied and so the distribution functions (1) and (2) are well defined. The expected payoff of bidder l with valuation $v_l = 1$ is given by

$$\left\{\frac{px_{l.i}}{px_{l.i} + (1-p)\, r_{ii}} + \frac{(1-p)\, r_{ii}}{px_{l.i} + (1-p)\, r_{ii}} F_{m.i}(b)\right\}(1-b)\,. \tag{28}$$

Substituting (2) into (28), we can verify that bidder l is indifferent among all bids in the interval $\left[0, \bar{b}_i\right]$, earning the expected payoff given in (3). Similarly, the expected payoff of bidder m with valuation $v_m = 1$ is

$$\left\{\frac{px_{m.i}}{px_{m.i} + (1-p)\, r_{ii}} + \frac{(1-p)\, r_{ii}}{px_{m.i} + (1-p)\, r_{ii}} F_{l.i}(b)\right\}(1-b)\,. \tag{29}$$

Substituting (1) into (29), we can verify that bidder m is also indifferent among all bids in the interval $\left[0, \bar{b}_i\right]$, earning the same expected payoff given in (3). Obviously, no bidder has incentives to bid above $\bar{b}_i$, while any bid below 0 would give a payoff of 0. Thus, we can conclude that (1) and (2) represent the equilibrium strategies of high valuation bidders when they observe the public signal i.

To prove that this equilibrium is unique, we can argue as in the proof of Proposition 1 that each bidder of type i will submit a bid according to an atomless distribution function, except possibly at 0; ties occur with zero probability; the supports of both distribution functions coincide; the common support is connected with the lower limit being equal to 0. Then the payoff of bidders 1 and 2 are given by (28) and (29), respectively. The common support also implies that the equilibrium payoffs of both bidders are the same and equal to the expression in (3). Equating (28) and (29) with (3) gives (1) and (2). Hence, the equilibrium is unique. ■

Proof of Theorem 2. We partition all signals into two sets, S and $N \backslash S$. The former contains all signals i such that $x_{1.i} > x_{2.i}$, while the latter contains all signals such that $x_{1.i} < x_{2.i}$. Signals i for whom $x_{1.i} = x_{2.i}$ holds are assigned arbitrarily as long as each set is non-empty.

Using the results of Proposition 2, the joint ex ante payoff of bidders is $p(1-p)$ times the following expression

$$\sum_{i \in S} x_{1.i} + \sum_{i \in N \backslash S} x_{2.i} \frac{p x_{1.i} + (1-p) r_{ii}}{p x_{2.i} + (1-p) r_{ii}} + \sum_{i \in N \backslash S} x_{2.i} + \sum_{i \in S} x_{1.i} \frac{p x_{2.i} + (1-p) r_{ii}}{p x_{1.i} + (1-p) r_{ii}}.$$

The first two terms represent the payoff of bidder 1, and the other two terms - the payoff of bidder 2. Using

$$\begin{aligned} \sum_{i \in S} x_{1.i} + \sum_{i \in N \backslash S} x_{1.i} &= 1, \\ \sum_{i \in S} x_{2.i} + \sum_{i \in N \backslash S} x_{2.i} &= 1, \end{aligned}$$

we can rewrite the joint payoff as

$$2 - \sum_{i \in N \backslash S} x_{1.i} + \sum_{i \in N \backslash S} x_{2.i} \frac{p x_{1.i} + (1-p) r_{ii}}{p x_{2.i} + (1-p) r_{ii}} - \sum_{i \in S} x_{2.i} + \sum_{i \in S} x_{1.i} \frac{p x_{2.i} + (1-p) r_{ii}}{p x_{1.i} + (1-p) r_{ii}}. \tag{30}$$

The expression in (30) is increasing in $x_{1.i}$ for all $i \in S$ but decreasing for all $i \in N \backslash S$. Similarly, (30) is increasing in $x_{2.i}$ for all $i \in N \backslash S$ but decreasing for all $i \in S$. Therefore, $x_{1.i} = 0$ for all $i \in N \backslash S$ and $x_{2.i} = 0$ for all $i \in S$, and we simplify (30) to

$$2 + \sum_{i \in N \backslash S} x_{2.i} \frac{(1-p) r_{ii}}{p x_{2.i} + (1-p) r_{ii}} + \sum_{i \in S} x_{1.i} \frac{(1-p) r_{ii}}{p x_{1.i} + (1-p) r_{ii}}.$$

One can verify that

$$\frac{ab}{a+b} + \frac{cd}{c+d} \leq \frac{(a+c)(b+d)}{a+b+c+d}.$$

(This inequality can be rewritten as $(ad-bc)^2 \geq 0$.) Therefore,

$$\begin{aligned}\sum_{i\in N\backslash S} x_{2.i}\frac{(1-p)\,r_{ii}}{px_{2.i}+(1-p)\,r_{ii}} &\leq \frac{(1-p)\sum_{i\in N\backslash S} r_{ii}}{p+(1-p)\sum_{i\in N\backslash S} r_{ii}},\\ \sum_{i\in S} x_{1.i}\frac{(1-p)\,r_{ii}}{px_{1.i}+(1-p)\,r_{ii}} &\leq \frac{(1-p)\sum_{i\in S} r_{ii}}{p+(1-p)\sum_{i\in S} r_{ii}},\end{aligned}$$

where we have additionally used the fact that $\sum_{i\in N\backslash S} x_{2.i} = \sum_{i\in S} x_{1.i} = 1$. Therefore, we can increase the joint payoff if we aggregate all signals $i \in S$ into a (new) signal 1 and all signals $i \in N\backslash S$ into a (new) signal 2 such that $\tilde{x}_{1.1} = 1$, $\tilde{x}_{1.2} = 0$, $\tilde{x}_{2.1} = 0$, $\tilde{x}_{2.2} = 1$, $\tilde{r}_{11} = \sum_{i\in S} r_{ii}$, and $\tilde{r}_{22} = \sum_{i\in N\backslash S} r_{ii} = 1-\tilde{r}_{11}$. Given these signals 1 and 2, we can apply Proposition 2 to verify that the joint equilibrium payoff is indeed given by

$$2+\frac{(1-p)\,\tilde{r}_{11}}{p+(1-p)\,\tilde{r}_{11}}+\frac{(1-p)\,\tilde{r}_{22}}{p+(1-p)\,\tilde{r}_{22}}. \tag{31}$$

This completes the proof that it is sufficient that the public signal takes one of two values in order to achieve the maximal joint payoff.

To find the optimal distribution of signals, it remains to maximize (31) subject to $\tilde{r}_{11}+\tilde{r}_{22} = 1$. It follows that $\tilde{r}_{11} = \tilde{r}_{22} = 0.5$. According to (3), a high valuation bidder expects a payoff of $\frac{2p}{1+p}$ irrespective of the public signal that she observes. The ex ante payoff of bidder is $p\,(1-p)\,\frac{2}{1+p}$. ■

Proof of Proposition 3. First, note that $F_i\left(\bar{b}_i\right) = 1$ for all $i \in N$ is satisfied and so the mixed strategies are well defined. Suppose that bidder 2 follows the strategy given in Proposition 3 and $y_1 > 0$. Consider bidder 1 of type $i \in N$. Her expected payoff when bidding $b \in \left[\bar{b}_{i-1}, \bar{b}_i\right]$ is

$$\left\{\frac{px_i+(1-p)\,y_i\sum_{k=1}^{i-1} y_k}{px_i+(1-p)\,y_i}+\frac{(1-p)\,y_i^2}{px_i+(1-p)\,y_i}F_i\,(b)\right\}(1-b)\,.$$

Substituting $F_i\,(b)$ from (4) yields a positive constant

$$\frac{px_i+(1-p)\,y_i\sum_{k=1}^{i-1} y_k}{px_i+(1-p)\,y_i}\left(1-\bar{b}_{i-1}\right).$$

Therefore, bidder 1 is indeed indifferent between any bid in the interval $\left[\bar{b}_{i-1}, \bar{b}_i\right]$.

Suppose now that bidder 1 of type i bids in an interval $\left[\bar{b}_{j-1}, \bar{b}_j\right]$ for $j \neq i$. Her expected payoff is

$$\left\{\frac{px_i + (1-p)\, y_i \sum_{k=1}^{j-1} y_k}{px_i + (1-p)\, y_i} + \frac{(1-p)\, y_i y_j}{px_i + (1-p)\, y_i} F_j(b)\right\}(1-b)$$

$$= \frac{px_i + (1-p)\, y_i \sum_{k=1}^{j-1} y_k}{px_i + (1-p)\, y_i}(1-b)$$

$$+ \frac{(1-p)\, y_i y_j}{px_i + (1-p)\, y_i} \frac{px_j + (1-p)\, y_j \sum_{k=1}^{j-1} y_k}{(1-p)\, y_j^2}\left(b - \bar{b}_{j-1}\right)$$

$$= \frac{b}{px_i + (1-p)\, y_i}\left\{\frac{y_i}{y_j}\left(px_j + (1-p)\, y_j \sum_{k=1}^{j-1} y_k\right) - \left(px_i + (1-p)\, y_i \sum_{k=1}^{j-1} y_k\right)\right\} + \Phi$$

$$= \frac{pb}{px_i + (1-p)\, y_i}\left(\frac{y_i}{y_j} x_j - x_i\right) + \Phi,$$

where the rest of the terms that do not contain b are collected in the parameter Φ. Since $x_j/y_j > x_i/y_i$ for all $j < i$, it follows that the payoff of type i is increasing in b for $b < \bar{b}_{i-1}$ and therefore bidder 1 of type i does not want to deviate by bidding below $\bar{b}_{i-1}$. Similarly, since $x_j/y_j < x_i/y_i$ for all $j > i$, it follows that the payoff of type i is decreasing in b for $b > \bar{b}_i$ and therefore bidder 1 of type i does not want to deviate by bidding above $\bar{b}_i$ either. The same argument establishes that there is no equilibrium, in which the supports of equilibrium strategies are arranged in a different order. That is, for any two types i and j such that $i < j$ it must be the case that the support of type i's mixed strategy must lie to the left of the support of type j's mixed strategy.

It remains to prove that supports cannot overlap in an equilibrium. Suppose on the contrary that an interval $\left[\underline{b}, \bar{b}\right]$ belongs to the support of equilibrium mixed strategies of more than one type. Let the set of these types be denoted by S. The expected payoff of type $i \in S$ from bidding in the interval $\left[\underline{b}, \bar{b}\right]$ is

$$\pi_i = \frac{px_i + (1-p)\, y_i \sum_{k \in N} y_k \tilde{F}_k(b)}{px_i + (1-p)\, y_i}(1-b),$$

where $\tilde{F}_k(b)$ denotes the distribution of bids for type k in this equilibrium. If we multiply both sides with

$$\frac{y_j}{px_j + (1-p)\, y_j}$$

where $j \in S \backslash \{i\}$, we obtain

$$\frac{y_j \pi_i}{px_j + (1-p)\, y_j} = \frac{px_i y_j + (1-p)\, y_i y_j \sum_{k \in N} y_k \tilde{F}_k(b)}{(px_i + (1-p)\, y_i)(px_j + (1-p)\, y_j)} (1-b).$$

If we subtract the analogous expression, in which the roles of types i and j are reversed, from the above expression we have that

$$\frac{y_j \pi_i}{px_j + (1-p)\, y_j} - \frac{y_i \pi_j}{px_i + (1-p)\, y_i} = \frac{px_i y_j - px_j y_i}{(px_i + (1-p)\, y_i)(px_j + (1-p)\, y_j)} (1-b).$$

Given that $x_i y_j \neq x_j y_i$, the above expression is satisfied only for a single value of b. Therefore, the supports of mixed strategies of types i and j cannot overlap in the equilibrium. This completes the proof that the symmetric equilibrium, described in Proposition 3, is unique. If $y_1 = 0$, using the tie breaking rule, the proof is basically the same. ■

Proof of Theorem 3. Using (5)-(7), the expected payoff of a bidder can be expressed as

$$P(x, y) = (1-p)(z_1 + z_1 z_2 + z_1 z_2 z_3 + ... + z_1 z_2 ... z_{n-1} z_n),$$

where

$$z_i = \frac{px_i + (1-p)\, y_i \sum_{j=1}^{i-1} y_j}{px_{i-1} + (1-p)\, y_{i-1} \sum_{j=1}^{i-1} y_j}$$

for $i > 1$ and $z_1 = px_1$. The proof consists of two parts. First, we show that a signal structure $(x, y)_n$ with $x_n > 0$ cannot be optimal. Next, we show for any signal structure $(x, y)_n$ with $x_n = 0$ how we can increase bidder's payoff by introducing an additional signal.

Assume first that $x_n > 0$. Let us define a new signal structure $(x', y')_n$ as follows: $x'_i = x_i$ for $i = 1, ..., n-2$, $x'_{n-1} = x_{n-1} + \epsilon$, $x'_n = x_n - \epsilon \geq 0$, and $y'_i = y_i$ for $i = 1, ..., n$. ϵ must be sufficiently small to ensure that $x'_{n-2}/y'_{n-2} > x'_{n-1}/y'_{n-1}$ holds.[22] Given the assumptions about $(x', y')_{n+1}$, we only need to show that

$$z_{n-1}(1 + z_n) < z'_{n-1}(1 + z'_n)$$

[22] It can also be shown that it is always possible to increase bidder's payoff by introducing an extra signal when $x_n > 0$. It can be done by defining the new probabilities in the following way: $x'_i = x_i$ and $y'_i = y_i$ for $i = 1, ..., n-1$, and $x'_n = x_n/2 + \epsilon$, $x'_{n+1} = x_n/2 - \epsilon \geq 0$, $y'_n = y_n/2$, and $y'_{n+1} = y_n/2$, where ϵ is such that $x'_{n-1}/y'_{n-1} > x'_n/y'_n$ holds.

or

$$\frac{px_{n-1}+(1-p)\,y_{n-1}\sum y_j}{px_{n-2}+(1-p)\,y_{n-2}\sum y_j}\times\left(1+\frac{px_n+(1-p)\,y_n\left(\sum y_j+y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}\right) <$$

$$\frac{p\left(x_{n-1}+\epsilon\right)+(1-p)\,y_{n-1}\sum y_j}{px_{n-2}+(1-p)\,y_{n-2}\sum y_j}\times\left(1+\frac{p\left(x_n-\epsilon\right)+(1-p)\,y_n\left(\sum y_j+y_{n-1}\right)}{p\left(x_{n-1}+\epsilon\right)+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}\right)$$

where we write $\sum y_j$ instead of $\sum_{j=1}^{n-2} y_j$. The above expression is equivalent to

$$\frac{px_{n-1}+(1-p)\,y_{n-1}\sum y_j}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)} < \frac{p\left(x_{n-1}+\epsilon\right)+(1-p)\,y_{n-1}\sum y_j}{p\left(x_{n-1}+\epsilon\right)+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}.$$

The right hand side is increasing in ϵ and since both sides are equivalent for $\epsilon=0$, it follows that we can always increase the payoff by shifting some probability away from x_n to x_{n-1}. Thus, $x_n>0$ cannot hold in the optimum.

Suppose now that $x_n=0$. We introduce an additional signal in the following way: $x_i'=x_i$ and $y_i'=y_i$ for $i=1,...,n-2$, $x_{n-1}'+x_n'=x_{n-1}$ and $y_{n-1}'+y_n'+y_{n+1}'=y_{n-1}+y_n$, and let $x_{n+1}'=0$. Now we need to compare $z_{n-1}\left(1+z_n\right)$ with $z_{n-1}'\left(1+z_n'\left(1+z_{n+1}'\right)\right)$.

Now

$$z_{n-1}\left(1+z_n\right)=\frac{px_{n-1}+(1-p)\,y_{n-1}\sum y_j}{px_{n-2}+(1-p)\,y_{n-2}\sum y_j}\left(1+\frac{(1-p)\,y_n\left(\sum y_j+y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}\right)$$

where we write $\sum y_j$ instead of $\sum_{j=1}^{n-2} y_j$, while

$$\begin{aligned} z_{n-1}'\left(1+z_n'\left(1+z_{n+1}'\right)\right) &= \frac{px_{n-1}'+(1-p)\,y_{n-1}'\sum y_j}{px_{n-2}+(1-p)\,y_{n-2}\sum y_j}\times \\ &\times\left(1+\frac{px_n'+(1-p)\,y_n'\left(\sum y_j+y_{n-1}'\right)}{px_{n-1}'+(1-p)\,y_{n-1}'\left(\sum y_j+y_{n-1}'\right)}\times\right. \\ &\times\left.\left(1+\frac{(1-p)\,y_{n+1}'\left(\sum y_j+y_{n-1}'+y_n'\right)}{px_n'+(1-p)\,y_n'\left(\sum y_j+y_{n-1}'+y_n'\right)}\right)\right). \end{aligned}$$

Desired inequality

$$z_{n-1}\left(1+z_n\right)<z_{n-1}'\left(1+z_n'\left(1+z_{n+1}'\right)\right)$$

is satisfied if

$$\frac{px_{n-1}+(1-p)\,y_{n-1}\sum y_j}{px'_{n-1}+(1-p)\,y'_{n-1}\sum y_j}\left(1+\frac{(1-p)\,y_n\left(\sum y_j+y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}\right)$$
$$< \quad 1+\frac{px'_n+(1-p)\,y'_n\left(\sum y_j+y'_{n-1}\right)}{px'_{n-1}+(1-p)\,y'_{n-1}\left(\sum y_j+y'_{n-1}\right)}\left(1+\frac{(1-p)\,y'_{n+1}\left(\sum y_j+y'_{n-1}+y'_n\right)}{px'_n+(1-p)\,y'_n\left(\sum y_j+y'_{n-1}+y'_n\right)}\right)$$

or

$$\begin{aligned} &\frac{px_{n-1}+(1-p)\,y_{n-1}\sum y_j}{px'_{n-1}+(1-p)\,y'_{n-1}\sum y_j}\times \\ &\times\frac{px_{n-1}+(1-p)\,(y_{n-1}+y_n)\left(\sum y_j+y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}-1 \\ < \quad &\frac{px'_n+(1-p)\,y'_n\left(\sum y_j+y'_{n-1}\right)}{px'_{n-1}+(1-p)\,y'_{n-1}\left(\sum y_j+y'_{n-1}\right)}\times \\ &\times\frac{px'_n+(1-p)\left(y'_n+y'_{n+1}\right)\left(\sum y_j+y'_{n-1}+y'_n\right)}{px'_n+(1-p)\,y'_n\left(\sum y_j+y'_{n-1}+y'_n\right)}. \end{aligned} \tag{32}$$

Let us define the new probabilities as follows:

$$\begin{aligned} x'_{n-1} &= (1-\delta)\,x_{n-1}, \\ x'_n &= \delta x_{n-1}, \\ x'_{n+1} &= 0, \\ y'_{n-1} &= (1-\delta)\,y_{n-1}, \\ y'_n &= \delta y_{n-1}+\epsilon, \\ y'_{n+1} &= y_n-\epsilon, \end{aligned}$$

where $\delta \in (0,1)$ and $\epsilon \in (0, y_n)$.

Using that

$$\frac{x_{n-1}}{y_{n-1}}=\frac{x'_{n-1}}{y'_{n-1}},$$

the left hand side of (32) becomes

$$\frac{px_{n-1}+(1-p)\,(y_{n-1}+y_n)\left(\sum y_j+y_{n-1}\right)}{px'_{n-1}+(1-p)\,y'_{n-1}\left(\sum y_j+y_{n-1}\right)}-1.$$

Further, using that $x_{n-1} = x'_{n-1} + x'_n$ and $y_{n-1} + y_n = y'_{n-1} + y'_n + y'_{n+1}$, it can be written as

$$\frac{px'_n + (1-p)\left(y'_n + y'_{n+1}\right)\left(\sum y_j + y_{n-1}\right)}{px'_{n-1} + (1-p)\,y'_{n-1}\left(\sum y_j + y_{n-1}\right)}.$$

Thus, it remains to check whether the following inequality

$$\begin{aligned} & \frac{px'_n + (1-p)\left(y'_n + y'_{n+1}\right)\left(\sum y_j + y_{n-1}\right)}{px'_{n-1} + (1-p)\,y'_{n-1}\left(\sum y_j + y_{n-1}\right)} \\ < \; & \frac{px'_n + (1-p)\,y'_n\left(\sum y_j + y'_{n-1}\right)}{px'_{n-1} + (1-p)\,y'_{n-1}\left(\sum y_j + y'_{n-1}\right)} \times \\ & \times \frac{px'_n + (1-p)\left(y'_n + y'_{n+1}\right)\left(\sum y_j + y'_{n-1} + y'_n\right)}{px'_n + (1-p)\,y'_n\left(\sum y_j + y'_{n-1} + y'_n\right)} \end{aligned}$$

or, after substituting for x'_{n-1}, x'_n, y'_{n-1}, and y'_n, and re-arranging, whether the following inequality

$$\begin{aligned} & \frac{px_{n-1} + (1-p)\,y_{n-1}\left(\sum y_j + y_{n-1}\right)}{px_{n-1} + (1-p)\,y_{n-1}\left(\sum y_j + (1-\delta)\,y_{n-1}\right)} \times \\ & \times \frac{p\delta x_{n-1} + (1-p)\left(\delta y_{n-1} + \epsilon\right)\left(\sum y_j + (1-\delta)\,y_{n-1}\right)}{p\delta x_{n-1} + (1-p)\left(\delta y_{n-1} + \epsilon\right)\left(\sum y_j + y_{n-1} + \epsilon\right)} \\ & - \frac{p\delta x_{n-1} + (1-p)\left(\delta y_{n-1} + y_n\right)\left(\sum y_j + y_{n-1}\right)}{p\delta x_{n-1} + (1-p)\left(\delta y_{n-1} + y_n\right)\left(\sum y_j + y_{n-1} + \epsilon\right)} \;>\; 0 \end{aligned}$$

is true.

We take the first order Taylor expansion of the above expression at $\epsilon = 0$ to see if for $\epsilon > 0$ this expression is strictly positive. If the above expression is represented as $f(\epsilon)$, then $f(\epsilon) = f(0) + f'(0)\,\epsilon + R(\epsilon)$. Notice that if $\epsilon = 0$, then the above expression is equal to 0, that is, $f(0) = 0$, while $f'(0)$

is

$$
\begin{aligned}
&\frac{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+(1-\delta)\,y_{n-1}\right)}\\
&\times\left\{\frac{(1-p)\left(\sum y_j+(1-\delta)\,y_{n-1}\right)}{p\delta x_{n-1}+(1-p)\,\delta y_{n-1}\left(\sum y_j+y_{n-1}\right)}\right.\\
&-\frac{p\delta x_{n-1}+(1-p)\,\delta y_{n-1}\left(\sum y_j+(1-\delta)\,y_{n-1}\right)}{p\delta x_{n-1}+(1-p)\,\delta y_{n-1}\left(\sum y_j+y_{n-1}\right)}\times\\
&\left.\times\frac{(1-p)\left(\sum y_j+(1+\delta)\,y_{n-1}\right)}{p\delta x_{n-1}+(1-p)\,\delta y_{n-1}\left(\sum y_j+y_{n-1}\right)}\right\}\\
&+\frac{p\delta x_{n-1}+(1-p)\left(\delta y_{n-1}+y_n\right)\left(\sum y_j+y_{n-1}\right)}{p\delta x_{n-1}+(1-p)\left(\delta y_{n-1}+y_n\right)\left(\sum y_j+y_{n-1}\right)}\times\\
&\times\frac{(1-p)\left(\delta y_{n-1}+y_n\right)}{p\delta x_{n-1}+(1-p)\left(\delta y_{n-1}+y_n\right)\left(\sum y_j+y_{n-1}\right)},
\end{aligned}
$$

which can be simplified to

$$
\begin{aligned}
&\frac{1}{\delta}\times\frac{(1-p)\left(\sum y_j+(1-\delta)\,y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+(1-\delta)\,y_{n-1}\right)}\\
&-\frac{1}{\delta}\times\frac{(1-p)\left(\sum y_j+(1+\delta)\,y_{n-1}\right)}{px_{n-1}+(1-p)\,y_{n-1}\left(\sum y_j+y_{n-1}\right)}\\
&+\frac{(1-p)\left(\delta y_{n-1}+y_n\right)}{p\delta x_{n-1}+(1-p)\left(\delta y_{n-1}+y_n\right)\left(\sum y_j+y_{n-1}\right)}.
\end{aligned}
$$

To simplify notation, let $\Psi=\dfrac{p}{1-p}x_{n-1}$. Then

$$
\begin{aligned}
&\frac{1}{\delta}\times\frac{\sum y_j+(1-\delta)\,y_{n-1}}{\Psi+y_{n-1}\left(\sum y_j+(1-\delta)\,y_{n-1}\right)}\\
&-\frac{1}{\delta}\times\frac{\sum y_j+(1+\delta)\,y_{n-1}}{\Psi+y_{n-1}\left(\sum y_j+y_{n-1}\right)}+\frac{\delta y_{n-1}+y_n}{\delta\Psi+\left(\delta y_{n-1}+y_n\right)\left(\sum y_j+y_{n-1}\right)}\\
=\;&\frac{\Psi}{\left(\delta\Psi+\left(\delta y_{n-1}+y_n\right)\left(\sum y_j+y_{n-1}\right)\right)}\times\\
&\times\frac{\Psi y_n-\delta y_{n-1}\left(\Psi+y_{n-1}\right)}{\left(\Psi+y_{n-1}\left(\sum y_j+(1-\delta)\,y_{n-1}\right)\right)\left(\Psi+y_{n-1}\left(\sum y_j+y_{n-1}\right)\right)}.
\end{aligned}
$$

It follows that if δ is chosen such that

$$
0<\delta<\min\left(1,\frac{\Psi y_n}{y_{n-1}\left(\Psi+y_{n-1}\right)}\right), \tag{33}
$$

then the derivative $f'(0)$ is positive, which was necessary to prove.

Finally,

$$R(\epsilon) = \frac{f''(\theta)}{2}\epsilon^2$$

where $\theta \in [0, \epsilon]$. $f''(\theta)$ exists and is finite for $\theta \in [0, \epsilon]$. Therefore, for a given δ, satisfying (33), we can always select ϵ satisfying

$$\left(f'(0) + \frac{f''(\theta)}{2}\epsilon \right) \epsilon > 0.$$

■