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# On the spectrum of waveguides in planar photonic bandgap structures

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We study a Helmholtz-type spectral problem related to the propagation of electromagnetic waves in photonic crystal waveguides. The waveguide is created by introducing a linear defect into a two-dimensional periodic medium; the defect is infinitely extended and aligned with one of the coordinate axes. This perturbation introduces guided mode spectrum inside the band gaps of the fully periodic, unperturbed spectral problem. In the first part of the paper, we prove that guided mode spectrum can be created by arbitrarily “small” perturbations. Secondly we show that, after performing a Floquet decomposition in the axial direction of the waveguide, for any fixed value of the quasi-momentum  $k_x$  the perturbation generates at most finitely many new eigenvalues inside the gap.

## 1. Introduction

The concept of a photonic crystal (also called a photonic band-gap material) was suggested in 1987 (see e.g. [22] for a textbook introduction) and has received significant attention from both the theoretical and experimental viewpoint. In practice, photonic crystals are often manufactured using periodic crystalline structures, a feature of which is their ability to allow, or deny the propagation of electromagnetic radiation which lies in a well defined range of the frequency spectrum. We call ranges where the electromagnetic radiation can propagate spectral bands and ranges where propagation is prevented spectral gaps. These concepts will be made

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precise in Section 2.

One possible application of photonic crystals is their use for manufacturing highly efficient optical waveguides which allow propagation of electromagnetic radiation only in a very narrow range of frequencies. Briefly, these are created by taking some photonic band-gap material and introducing a linear defect which breaches the periodicity. This may have the effect of allowing propagation of electromagnetic radiation in a range of the frequency spectrum in which propagation is not possible without the defect. Waves in this narrow frequency band, sometimes called guided modes, should then be almost entirely confined to the defect.

An appropriate mathematical model for such materials is given by the spectral problem for the Maxwell equations, which in the context of polarised waves in two dimensions reduce to the Helmholtz equation. We shall study the spectral problem for

$$-\varepsilon^{-1}\Delta, \quad (1.1)$$

in  $\mathbb{R}^2$ , where  $\varepsilon$  is the dielectric function (or, equivalently, the square of the refractive index of the material). The background problem without a waveguide is modelled by choosing  $\varepsilon$  to be periodic. To model a waveguide in a periodic crystal, we take

$$\varepsilon = \varepsilon_0 + \varepsilon_1$$

where  $\varepsilon_0$  is periodic and  $\varepsilon_1$  is a perturbation supported only in the waveguide, which we choose as a strip in the  $x$ -direction. The precise assumptions on  $\varepsilon_0$  and  $\varepsilon_1$  will be stated in Section 2.

To establish the existence of guided modes in a band gap of the unperturbed medium (modelled by  $-\varepsilon_0^{-1}\Delta$ ), we must prove that this gap of the unperturbed problem now contains spectrum of  $-\varepsilon^{-1}\Delta$  induced by the perturbation  $\varepsilon_1$ . We first note that existence of gaps in the spectrum of some problems with periodic background media was proved in [10,16,19] and in [17] for the full Maxwell case. Using layer potential techniques this question had been studied in [1–3]. For the physical motivation of linear waveguides in two dimensional photonic crystals, we refer to [22, Chapter 5]. The mathematical investigation of guided modes in waveguides produced by linear defects was begun in [4], with further contributions in [7,28,29,32]. The latter two articles work with the 3D Maxwell equations, rather than the Helmholtz equation. In particular, [28,29,32] give sufficient conditions for the existence of guided modes in the spectral gap of the unperturbed problem; these require either the gap to be wide enough or the perturbation to be sufficiently large.

In this paper, we ask the following question: is there a certain threshold strength of the perturbation or width of the gap needed to produce spectrum inside the gap? Our analysis shows that spectrum appears in all gaps of the unperturbed problem under arbitrarily small signed perturbations (Theorem 4.2). We then proceed to study the structure of the induced spectrum for fixed quasimomentum  $k_x$  in the direction of propagation of the guided wave. In particular, we show that for fixed  $k_x$ , the eigenvalues introduced into the gap by the perturbation do not accumulate at the ends of the gap.

It is interesting to compare our results to those available in the literature. In [14,15] the authors show generation of a finite number of eigenvalues in the gaps by localized perturbations of the periodic medium. Another of their results in dimension  $d = 3$  states that perturbations with  $\varepsilon_1 < 0$  (positive defects in their terminology) need a certain threshold strength to create eigenvalues in gaps. This is in sharp contrast to our result, where the defect is an infinitely extended waveguide. In [4, Corollary 1], a result concerning the existence of a threshold for a two-dimensional waveguide analogous to that in [15] is claimed. This seems to be at odds with the result of Theorem 4.1 presented here.

Our result is close in spirit to the result in [34] for the periodic Schrödinger equation. There, the authors use a rather different method to prove that weak perturbations create spectrum in band gaps of a periodic Schrödinger operator. We refer to [34] for a discussion of weak localization in the context of quantum mechanics.

In three dimensions, vector-valued equations of the form (1.1), coupled by a curl-condition (see, e.g. [13]), arise in the study of elastic wave propagation in phononic crystals. As well as having applications to photonic bandgap structures, the results in the present paper are also a contribution to the study of the effects of perturbations on the spectrum of linear periodic differential operators in Hilbert space.

Our paper is structured as follows: in section 2, we introduce the operators to be studied and remind the reader briefly of the Floquet transform. Section 3 contains some preparatory material on the band functions and Bloch functions, which will play an important role later on. In section 4, we use variational arguments to prove the existence of guided mode spectrum. Finally, in section 5, we consider the question of non-accumulation of eigenvalues at the ends of the gap.

We remark that the existence part of our results carry over to the three-dimensional Helmholtz equation, where certain non-degeneracy assumptions on the band structure have to be imposed, see [8].

## 2. Formulation of the problem

We first consider the fully periodic background problem in  $\mathbb{R}^2$ . We will make the following assumptions throughout the paper.

**Assumption 2.1.** Let  $\varepsilon_0 \in L^\infty(\mathbb{R}^2)$  be a positive bounded function such that  $\varepsilon_0^{-1} \in L^\infty(\mathbb{R}^2)$  and which is periodic in both  $x$  and  $y$ . For simplicity, we will assume that the basic cell of periodicity is  $[0, 1]^2$ , i.e. that  $\varepsilon_0(x+1, y) = \varepsilon_0(x, y) = \varepsilon_0(x, y+1)$  for all  $(x, y) \in \mathbb{R}^2$ .

For any open set  $O \subseteq \mathbb{R}^2$  we denote by  $L_{\varepsilon_0}^2(O)$  the weighted  $L^2$ -space with norm given by

$$\|u\|_{\varepsilon_0}^2 = \int_O \varepsilon_0(x) |u(x)|^2 dx.$$

A 2D-periodic crystalline structure may be modeled by an operator  $L_0$  in  $L_{\varepsilon_0}^2(\mathbb{R}^2)$  given by

$$L_0 u = -\frac{1}{\varepsilon_0(x, y)} \Delta u \quad \text{with domain } D(L_0) = H^2(\mathbb{R}^2).$$

This is a self-adjoint operator. A standard tool in the analysis of periodic problems is the *Floquet-Bloch* transform. We will state here some of the results used in this paper and refer the reader to [26,36] for proofs and more background on the theory. The Floquet-Bloch transform  $U_x$  associated with the periodicity in the  $x$ -direction is

$$(U_x f)(x, y, k) = (2\pi)^{-1/2} \sum_{m \in \mathbb{Z}} e^{ikm} f(x - m, y), \quad (2.1)$$

where  $(x, y) \in \Omega := (0, 1) \times \mathbb{R}$ ,  $k \in [-\pi, \pi]$  and  $f \in L_{\varepsilon_0}^2(\mathbb{R}^2)$  is a function of compact support.  $U_x$  is then extended to a map  $U_x : L_{\varepsilon_0}^2(\mathbb{R}^2) \rightarrow L_{\varepsilon_0}^2(\Omega \times (-\pi, \pi))$  by continuity.

We now consider an operator family  $L_0(k_x)$  on the strip  $\Omega$  parametrised by  $k_x \in [-\pi, \pi]$ .  $L_0(k_x)$  is the self-adjoint operator in  $L_{\varepsilon_0}^2(\Omega)$  given by

$$L_0(k_x)u = -\frac{1}{\varepsilon_0(x, y)} \Delta u \quad (2.2)$$

defined on the space of all functions  $u \in H^2(\Omega)$  which satisfy the quasi-periodic boundary conditions

$$u(1, y) = e^{ik_x} u(0, y) \quad \text{and} \quad \frac{\partial u}{\partial x}(1, y) = e^{ik_x} \frac{\partial u}{\partial x}(0, y). \quad (2.3)$$

It follows from the general theory [26,36] that  $L_0$  is the direct integral of the operators  $L_0(k_x)$ :

$$L_0 = \int_{[-\pi, \pi]}^{\oplus} L_0(k_x) dk_x. \quad (2.4)$$

As a consequence, the spectrum of the problem in the plane is

$$\sigma(L_0) = \overline{\bigcup_{k_x \in [-\pi, \pi]} \sigma(L_0(k_x))}. \quad (2.5)$$

Moreover, in view of periodicity in the  $y$ -direction, similar arguments apply for each operator  $L_0(k_x)$  and the spectrum of the operator  $L_0(k_x)$  is

$$\sigma(L_0(k_x)) = \overline{\bigcup_{k \in [-\pi, \pi]} \sigma(L_0(k_x, k))},$$

where  $L_0(k_x, k)$  is the operator  $-\frac{1}{\varepsilon_0(x, y)} \Delta$  in  $L^2_{\varepsilon_0}((0, 1)^2)$  whose domain consists of those  $H^2((0, 1)^2)$ -functions satisfying quasi-periodic boundary conditions in both the  $x$ - and  $y$ -directions:

$$\begin{cases} u(1, y) = e^{ik_x} u(0, y), & \frac{\partial u}{\partial x}(1, y) = e^{ik_x} \frac{\partial u}{\partial x}(0, y), \\ u(x, 1) = e^{ik} u(x, 0), & \frac{\partial u}{\partial y}(x, 1) = e^{ik} \frac{\partial u}{\partial y}(x, 0). \end{cases} \quad (2.6)$$

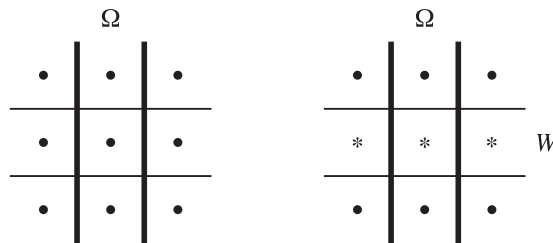
For notational simplicity we shall refer to the additional parameter as  $k$  rather than  $k_y$ .

We now turn to the problem which is our main interest in this paper. We perturb the original periodic problem in the plane to consider a wave-guide  $W = \mathbb{R} \times (0, 1)$  inside the 2D-periodic crystalline structure. This new problem is modeled by an operator  $L$  acting on  $L^2_{\varepsilon}(\mathbb{R}^2)$  with domain  $H^2(\mathbb{R}^2)$  given by

$$Lu = -\frac{1}{\varepsilon(x, y)} \Delta u$$

with  $\varepsilon(x, y) = \varepsilon_0(x, y) + \varepsilon_1(x, y)$ .

**Assumption 2.2.** *The perturbation  $\varepsilon_1 \in L^\infty(\mathbb{R}^2)$  is supported in  $W$ , in the  $x$ -direction it is periodic with period 1 and it is such that  $\inf \varepsilon_0 + \varepsilon_1 > 0$ .*



**Figure 1.** The background problem and the vertical strip  $\Omega$  on the left. The right shows the problem with the horizontal waveguide  $W$  added.

In view of the periodicity in the  $x$ -direction we can use the Floquet-Bloch transform (2.1) to generate, as in the case of  $L_0$ , a selfadjoint operator family  $L(k_x)$  on the strip  $\Omega$ , acting in the space  $L^2_{\varepsilon}(\Omega)$ , parametrized by  $k_x \in [-\pi, \pi]$  and given by

$$L(k_x)u := -\frac{1}{\varepsilon_0 + \varepsilon_1} \Delta u$$

with domain in  $H^2(\Omega)$ , subject to the quasi-periodic boundary conditions (2.3). As before, the spectrum of the waveguide problem  $L$  is

$$\sigma(L) = \overline{\bigcup_{k_x \in [-\pi, \pi]} \sigma(L(k_x))}. \quad (2.7)$$



The chief goal of this paper is to compare the spectra of  $L_0$  and  $L$ . In view of (2.5) and (2.7), this amounts to comparing the spectra of  $L_0(k_x)$  and  $L(k_x)$ . In [4, Lemma 10] it is shown that the essential spectra of these two operators coincide. Therefore,  $\sigma(L(k_x))$  can differ from  $\sigma(L_0(k_x))$  only through the introduction of extra eigenvalues inside or at the endpoints of the spectral gaps of  $L_0(k_x)$ . In particular, the eigenvalues of  $L(k_x)$  cannot accumulate at any point inside the spectral gaps.

In this paper, we will assume that there is a gap in the spectrum of  $L_0(k_x)$ , which we denote by  $(\mu_0, \mu_1)$ . As  $L_0(k_x)$  is a non-negative operator, we will assume throughout that  $\mu_0 > 0$ . We will then study the eigenvalues of the perturbed problem

$$-\Delta u = \lambda(\varepsilon_0 + \varepsilon_1)u \quad \text{on } \Omega \quad (2.8)$$

where  $\lambda \in (\mu_0, \mu_1)$ , i.e.  $\lambda$  lies in a gap of the spectrum of the operator  $L_0(k_x)$ . It is understood that from now on all functions satisfy the quasi-periodic boundary conditions (2.3). From the equation (2.8) we get

$$-\frac{1}{\varepsilon_0}\Delta u - \lambda u = \lambda \frac{\varepsilon_1}{\varepsilon_0} u.$$

It then follows that  $\lambda$  is an eigenvalue in the gap iff

$$u = \lambda (L_0(k_x) - \lambda)^{-1} \left( \frac{\varepsilon_1}{\varepsilon_0} u \right) \quad (2.9)$$

holds for some non-zero  $u$ . Our strategy will be based on finding solutions of (2.9) using information on the resolvent of the unperturbed operator  $L_0(k_x)$ .

As mentioned in the introduction, we shall show that small signed perturbations of the operator  $L_0(k_x)$  create extra spectrum in an arbitrary fixed gap  $(\mu_0, \mu_1)$ . This will imply that these perturbations create guided mode spectrum of the operator  $L$ . Our second aim in this paper is to prove that the additional eigenvalues do not accumulate anywhere on the closure of the spectral gap, in particular not at the endpoints  $\mu_0$  and  $\mu_1$ .

We consider the non-accumulation result as a first step towards a better understanding of the structure of the guided mode spectrum. One could surmise that the guided mode spectrum  $\sigma(L(k_x)) \setminus \sigma(L_0(k_x))$  can be written in terms of continuous band functions  $\{\theta_j(k_x)\}$  depending on  $k_x \in [-\pi, \pi]$ . However, since eigenvalues may be emitted and absorbed by the essential spectrum of the unperturbed operator  $L_0(k_x)$  as  $k_x$  varies, the functions  $\theta_j$  may possibly be defined only on subintervals of  $[-\pi, \pi]$ . An open question is whether a finite total number of band functions  $\theta_j$  (possibly defined on subintervals of  $[-\pi, \pi]$ ) is sufficient to describe the guided mode spectrum.

### 3. Eigenvalues at band edges and their eigenfunctions

Our analysis of the waveguide problem will be via a study of the resolvent of the unperturbed operator  $L_0(k_x)$ . This will be performed using the Floquet-Bloch transform. In this section we introduce a representation of the resolvent in terms of so-called Bloch functions and show some results which will be crucial in our later analysis. We note that the perturbation plays no role in this section.

For fixed  $k_x$  we consider the operator  $L_0(k_x, k)$  in  $L^2_{\varepsilon_0}((0, 1)^2)$  introduced in the previous section. The operator depends on  $k$  via the quasi-periodic boundary conditions and thus has a  $k$ -dependent domain. We can transform the eigenvalue problem for  $L_0(k_x, k)$  into an eigenvalue problem for a  $k$ -dependent operator with periodic boundary conditions in the  $y$ -direction: Define  $\Delta_k := \nabla_k^2$  in  $L^2_{\varepsilon_0}((0, 1)^2)$  with domain consisting of those  $H^2((0, 1)^2)$ -functions satisfying the boundary conditions

$$\begin{cases} u(1, y) = e^{ik_x} u(0, y), & \frac{\partial u}{\partial x}(1, y) = e^{ik_x} \frac{\partial u}{\partial x}(0, y), \\ u(x, 1) = u(x, 0), & \frac{\partial u}{\partial y}(x, 1) = \frac{\partial u}{\partial y}(x, 0), \end{cases} \quad (3.1)$$

where  $\nabla_k = \nabla + i \begin{pmatrix} 0 \\ k \end{pmatrix}$ .

We wish to apply [23, Theorem VII.3.9] to the analytic operator family  $-\frac{1}{\varepsilon_0} \Delta_k$ . This requires the operator family to be analytic of type (A). A family of closed operators  $T(z)$  on a Hilbert space, defined for  $z$  in a domain  $D_0 \subseteq \mathbb{C}$ , is called analytic of type (A), if the domains  $D(T(z))$  are independent of  $z$  and  $T(z)u$  is analytic in  $z \in D_0$  for all  $u$  in the domain of the operator (cf. [23]).

Clearly,  $-\frac{1}{\varepsilon_0} \Delta_k$  in  $L^2_{\varepsilon_0}((0, 1)^2)$  defined above for  $k \in \mathbb{C}$  is analytic of type (A). It is selfadjoint for  $k \in [-\pi, \pi]$  and has compact resolvent. Hence, by [23, Theorem VII.3.9 and Remark VII.3.10], there exist collections of functions  $\{\lambda_s(k)\}_{s \in \mathbb{N}}$  in  $\mathbb{R}$  and  $\{\phi_s(k)\}_{s \in \mathbb{N}}$ , normalized in  $L^2_{\varepsilon_0}((0, 1)^2)$  satisfying (3.1), which are real-analytic functions in the variable  $k$  on  $[-\pi, \pi]$  and are such that

$$-\frac{1}{\varepsilon_0} \Delta_k \phi_s(k) = \lambda_s(k) \phi_s(k).$$

Moreover, for each  $s \in \mathbb{N}$ , there exist  $\delta_s, \eta_s > 0$  such that  $\lambda_s(\cdot)$  and  $\phi_s(\cdot)$  can be continued analytically to the open set

$$\{z \in \mathbb{C} : \operatorname{Re} z \in (-\pi - \delta_s, \pi + \delta_s), |\operatorname{Im} z| < \eta_s\}$$

containing the interval  $[-\pi, \pi]$ . We note that the eigenvalues are not necessarily ordered by magnitude. Moreover, the analyticity results depend critically on the fact that we only let the scalar parameter  $k$  vary. The normalized eigenfunctions  $\{\psi_s(x, y, k)\}_{s \in \mathbb{N}}$  of  $L_0(kx, k)$  are then given by  $\psi_s(x, y, k) = e^{iky} \phi_s(x, y, k)$ , where  $\phi_s(x, y, k)$  corresponds to the eigenvalue  $\lambda_s(k)$ . We call the functions  $\lambda_s$  the band functions and  $\psi_s(x, y, k)$  the Bloch functions.

**Lemma 3.1.** *Let  $\|u\|_2 = \left(\int_{(0,1)^2} |u|^2\right)^{1/2}$  denote the unweighted  $L^2$ -norm of a function over  $(0, 1)^2$ . We have the following gradient estimates:*

$$\|\nabla \phi_s\|_2 \leq C(\sqrt{\lambda_s(k)} + 1) \quad \text{and} \quad \|\nabla \psi_s\|_2 \leq \sqrt{\lambda_s(k)}. \quad (3.2)$$

*Proof.* As  $-\frac{1}{\varepsilon_0} \Delta_k \phi_s = \lambda_s(k) \phi_s(k)$ , testing with  $\phi_s$ , gives

$$\int_{(0,1)^2} |\nabla_k \phi_s|^2 = \int_{(0,1)^2} \left(-\frac{1}{\varepsilon_0} \Delta_k \phi_s\right) \overline{\phi_s} \varepsilon_0 = \int_{(0,1)^2} \lambda_s(k) \phi_s \overline{\phi_s} \varepsilon_0 = \lambda_s(k).$$

Hence,

$$\|\nabla \phi_s\|_2 = \left\| \nabla_k \phi_s - i \begin{pmatrix} 0 \\ k \end{pmatrix} \phi_s \right\|_2 \leq \|\nabla_k \phi_s\|_2 + |k| \|\phi_s\|_2 \leq C(\sqrt{\lambda_s(k)} + 1).$$

Note that, as  $k$  runs through a bounded set, the constant  $C$  can be chosen independently of  $k$ . However, it is dependent on  $\|1/\varepsilon_0\|_{\infty}$ . The second statement follows from integration by parts:

$$\int_{(0,1)^2} |\nabla \psi_s|^2 = \int_{(0,1)^2} \left(-\frac{1}{\varepsilon_0} \Delta \psi_s\right) \overline{\psi_s} \varepsilon_0 = \int_{(0,1)^2} \lambda_s(k) \psi_s \overline{\psi_s} \varepsilon_0 = \lambda_s(k).$$

□

We remind the reader that  $\mu_1$  is the lowest point of a spectral band and lies at the top end of a gap. The next result shows that only finitely many spectral bands can touch  $\mu_1$  and that each  $\lambda_s(k)$  can touch  $\mu_1$  only finitely many times.

**Proposition 3.1.** *The equation*

$$\lambda_s(k) = \mu_1 \quad (3.3)$$

*has a solution  $(s, k) \in \mathbb{N} \times [-\pi, \pi]$ . Moreover, there are only finitely many pairs  $(s_p, k_p)$  satisfying (3.3).*

*Proof.* We first prove that the solution set is finite. Let  $k, k_0 \in [-\pi, \pi]$ . Arguing as in [23, VII.3.6], we find that there exist  $s$ -independent constants  $C_1, C_2$  such that

$$|\lambda_s(k) - \lambda_s(k_0)| \leq (C_1 + |\lambda_s(k_0)|)(e^{C_2|k-k_0|} - 1).$$

Next, choose  $|k - k_0| < \delta$ , where  $\delta$  is such that  $e^{C_2\delta} - 1 < 1/2$ . Then

$$|\lambda_s(k) - \lambda_s(k_0)| \leq \frac{1}{2}(C_1 + |\lambda_s(k_0)|)$$

and so whenever  $\lambda_s(k_0) \geq 0$  we get

$$\lambda_s(k) \geq \frac{1}{2}\lambda_s(k_0) - C.$$

For fixed  $k_0$ , we have  $\lambda_s(k_0) \rightarrow \infty$  as  $s \rightarrow \infty$ . Hence,

$$\inf_{\{k: |k-k_0| < \delta\}} \lambda_s(k) \geq \frac{1}{2}\lambda_s(k_0) - C \rightarrow \infty, \quad \text{as } s \rightarrow \infty.$$

Since  $\delta$  can be chosen independently of  $s$  and  $k_0$ , we can cover  $[-\pi, \pi]$  with finitely many intervals of length  $\delta$  and so

$$\inf_{k \in [-\pi, \pi]} \lambda_s(k) \rightarrow \infty \quad \text{as } s \rightarrow \infty. \quad (3.4)$$

Therefore, there are only finitely many values of  $s$  such that  $\lambda_s(k) = \mu_1$  for some  $k$ .

On the other hand, if for any fixed  $s$  we have  $\lambda_s(k_p) = \mu_1$  for infinitely many  $k_p$ , then these must accumulate in  $[-\pi, \pi]$  and the analyticity of  $\lambda_s$  would imply  $\lambda_s(k) \equiv \mu_1$ . This leads to a contradiction in the following way. A classical argument by Thomas (see [26,37]) shows that the periodic Schrödinger operator  $-\Delta + V$  with  $V \in L^\infty$  cannot have constant band functions, i.e. if

$$(-\Delta_k + V)\phi(k) = \nu(k)\phi(k) \quad \text{with } \phi(k) \not\equiv 0, \quad (3.5)$$

then  $\nu(k)$  is not constant. Since  $\lambda_s(k) \equiv \mu_1$  implies

$$(-\Delta_k - \varepsilon_0\mu_1)\phi_s(k) = 0,$$

i.e. (3.5) holds with  $V = -\varepsilon_0\mu_1$  and  $\nu = 0$ , this leads to a contradiction. Thus the set of solutions to (3.3) is finite.

Now, as  $(\mu_0, \mu_1)$  is a spectral gap, we can order the continuous band functions such that for some  $s' \in \mathbb{N}$  we have  $\lambda_s(k) \leq \mu_0$  for  $s < s'$  and  $\lambda_s(k) \geq \mu_1$  for  $s \geq s'$  and all  $k \in [-\pi, \pi]$ . Therefore,  $\mu_1 = \inf\{\lambda_s(k) : s \geq s', k \in [-\pi, \pi]\}$ . By (3.4), there exists  $s'' \in \mathbb{N}$  such that in fact  $\mu_1 = \inf\{\lambda_s(k) : s' \leq s \leq s'', k \in [-\pi, \pi]\}$ . By continuity of the band functions, there then exists a solution to (3.3).  $\square$

We now discuss the expansion of functions in  $L^2_{\varepsilon_0}((0, 1)^2)$  in terms of the Bloch functions  $\psi_s$ . Since for each  $k$ , the eigenfunctions  $\psi_s(\cdot, k)$  form a complete orthonormal system in  $L^2_{\varepsilon_0}((0, 1)^2)$ , we have for  $r \in L^2_{\varepsilon_0}((0, 1)^2)$  and  $k \in (-\pi, \pi)$

$$r = \sum_{s \in \mathbb{N}} \langle r, \psi_s(\cdot, k) \rangle_{L^2_{\varepsilon_0}((0,1)^2)} \psi_s(\cdot, k)$$

and hence by using Parseval's identity and integrating over  $k$ , we get

$$\|r\|_{L^2_{\varepsilon_0}((0,1)^2)}^2 = \frac{1}{2\pi} \sum_{s \in \mathbb{N}} \int_{-\pi}^{\pi} |\langle r, \psi_s(\cdot, k) \rangle_{L^2_{\varepsilon_0}((0,1)^2)}|^2 dk. \quad (3.6)$$

We now derive a representation for  $(L_0(k_x) - \lambda)^{-1}r$  with  $r \in L^2_{\varepsilon_0}((0, 1)^2)$  in terms of Bloch functions. Whenever we apply an operator with domain  $L^2(\Omega)$  to a function  $r \in L^2_{\varepsilon_0}((0, 1)^2)$ , it is to be understood as the application of the operator to the function  $r$  extended to all of  $\Omega$  by zero. For notational simplicity, we will use the same letter  $r$  for the extended function.



Let  $U$  denote the Floquet transform in the  $y$ -variable defined analogously to (2.1) and set

$$P_s(k, r) := \langle Ur(\cdot, k), \psi_s(\cdot, \bar{k}) \rangle_{L^2_{\varepsilon_0}((0,1)^2)} \psi_s(\cdot, k). \quad (3.7)$$

We note that since  $r$  is supported in  $[0, 1]^2$ , we have

$$P_s(k, r) = \frac{1}{\sqrt{2\pi}} \langle r, \psi_s(\cdot, \bar{k}) \rangle_{L^2_{\varepsilon_0}((0,1)^2)} \psi_s(\cdot, k). \quad (3.8)$$

In view of the analyticity of the Bloch functions  $\psi_s$ , it follows that also  $P_s$  depends analytically on  $k$  in a small complex neighborhood of  $[-\pi, \pi]$  (note that  $k \mapsto \overline{\psi_s(\cdot, \bar{k})}$  is analytic in  $k$ , which can be seen by an expansion into power series).

Using this notation we have the following resolvent formula for the operator  $L_0(k_x)$ , see [26, 33,36]:

$$(L_0(k_x) - \lambda)^{-1} r = \frac{1}{\sqrt{2\pi}} \sum_{s \in \mathbb{N}} \int_{-\pi}^{\pi} (\lambda_s(k) - \lambda)^{-1} P_s(k, r) dk \quad (3.9)$$

for  $\lambda$  outside the spectrum of  $L_0(k_x)$  and  $r \in L^2_{\varepsilon_0}((0,1)^2)$ .

## 4. Generation of spectrum in the gap

In this section we use the representation of the resolvent by Bloch functions and the variational principle to show that if the perturbation  $\varepsilon_1$  is of fixed sign it will lead to the generation of extra spectrum in the gap  $(\mu_0, \mu_1)$  of the spectrum of the operator  $L_0(k_x)$ .

For definiteness, throughout this section we make the following additional assumption on  $\varepsilon_1$ .

**Assumption 4.1.**  $\varepsilon_1$  is a non-negative function and there exist  $\alpha > 0$  and a nonempty open set  $D$  such that  $\inf_D \varepsilon_1|_{\Omega} = \alpha$ .

The first step consists of performing a Birman-Schwinger-like reformulation of the problem. We note that the Birman-Schwinger principle is a classical tool in handling perturbations of Schrödinger operators (the reader might consult e.g. [36], or [25] for an application of the principle to eigenvalue counting in spectral gaps).

In (2.9), set

$$v := \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u,$$

then  $v$  is supported in  $[0, 1]^2$ , as  $\varepsilon_1|_{\Omega}$  is, and  $v$  satisfies

$$v = \lambda \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} (L_0(k_x) - \lambda)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} v. \quad (4.1)$$

Note that conversely, if  $v$  satisfies (4.1), then

$$u := \lambda (L_0(k_x) - \lambda)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} v$$

satisfies (2.9) and lies in  $L^2_{\varepsilon_0}(\Omega)$ . It is therefore sufficient for our purposes to study (4.1).

We now define the operator  $A_{\lambda}$  on  $L^2_{\varepsilon_0}((0,1)^2)$  by

$$A_{\lambda} v = \left( \lambda \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} (L_0(k_x) - \lambda)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} v \right) \Big|_{(0,1)^2} \quad (4.2)$$

and note that (4.1) has a non-trivial solution if and only if 1 is an eigenvalue of the operator  $A_{\lambda}$ .

**Lemma 4.1.** Let  $\lambda \in (\mu_0, \mu_1)$ . Then  $A_{\lambda} : L^2_{\varepsilon_0}((0,1)^2) \rightarrow L^2_{\varepsilon_0}((0,1)^2)$  is symmetric and compact.

*Proof.* Let  $u, v \in L^2_{\varepsilon_0}((0, 1)^2)$ . Then

$$\begin{aligned} \langle \varepsilon_0 A_\lambda u, v \rangle_{L^2((0,1)^2)} &= \left\langle \varepsilon_0 \lambda \left( -\frac{1}{\varepsilon_0} \Delta - \lambda \right)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} v \right\rangle_{L^2(\Omega)} \\ &= \left\langle \varepsilon_0 \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \lambda \left( -\frac{1}{\varepsilon_0} \Delta - \lambda \right)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} v \right\rangle_{L^2(\Omega)} \\ &= \left\langle \varepsilon_0 u, \lambda \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} \left( \left( -\frac{1}{\varepsilon_0} \Delta - \lambda \right)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} v \right) \Big|_{(0,1)^2} \right\rangle_{L^2((0,1)^2)} \\ &= \langle \varepsilon_0 u, A_\lambda v \rangle_{L^2((0,1)^2)}, \end{aligned}$$

so the operator is symmetric. Moreover, by standard estimates (see e.g. [12]) for the elliptic operator on the strip,

$$\begin{aligned} \left\| (L_0(k_x) - \lambda)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u \right\|_{H^1((0,1)^2)} &\leq \left\| (L_0(k_x) - \lambda)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u \right\|_{H^1(\Omega)} \\ &\leq C_\lambda \|u\|_{L^2(\Omega)} = C_\lambda \|u\|_{L^2((0,1)^2)}. \end{aligned}$$

Thus  $A_\lambda$  is the composition of a compact map with the continuous map of multiplication by the function  $\sqrt{\frac{\varepsilon_1}{\varepsilon_0}}$  and is therefore compact as a map from  $L^2((0, 1)^2)$  to  $L^2((0, 1)^2)$ . Multiplication by the bounded and boundedly invertible weight  $\varepsilon_0$  does not affect this.  $\square$

We next investigate the dependence of the maximum eigenvalue of  $A_\lambda$  on  $\lambda$ . To do this, we will use the standard method of estimating eigenvalues using the variational characterisation via the Rayleigh quotient (see, e.g. [36]). This technique has previously been extensively used in the study of waveguides, see for example [5,6,31,34].

Define

$$\kappa_{max}(\lambda) := \sup_{\|u\| \neq 0} \frac{\langle A_\lambda u, u \rangle_{\varepsilon_0}}{\langle u, u \rangle_{\varepsilon_0}}.$$

It follows from the standard variational characterisation of the spectrum for compact operators that if  $\kappa_{max}(\lambda) > 0$ , it is the maximum eigenvalue of  $A_\lambda$ .

- Lemma 4.2.** (i) On the interval  $(\mu_0, \mu_1)$  the map  $\lambda \mapsto \kappa_{max}(\lambda)$  is continuous and increasing.  
(ii) If there exists  $\lambda' \in (\mu_0, \mu_1)$  such that  $\kappa_{max}(\lambda') > 0$ , then  $\lambda \mapsto \kappa_{max}(\lambda)$  is strictly increasing on  $(\lambda', \mu_1)$ .

*Proof.* (1) From (4.2), we see that  $\lambda \mapsto A_\lambda$  is norm continuous as a map from  $(\mu_0, \mu_1)$  to  $\mathcal{L}(L^2_{\varepsilon_0}((0, 1)^2))$ , the space of bounded linear operators on  $L^2_{\varepsilon_0}((0, 1)^2)$ . Moreover, for any  $\lambda \in (\mu_0, \mu_1)$  and any  $\tilde{\varepsilon} > 0$  there exists  $\delta > 0$  such that for  $|\lambda - \tilde{\lambda}| < \delta$

$$\left| \langle A_{\tilde{\lambda}} u, u \rangle_{\varepsilon_0} - \langle A_\lambda u, u \rangle_{\varepsilon_0} \right| \leq \|A_\lambda - A_{\tilde{\lambda}}\| \|u\|_{\varepsilon_0}^2 \leq \tilde{\varepsilon} \|u\|_{\varepsilon_0}^2.$$

This implies  $|\kappa_{max}(\tilde{\lambda}) - \kappa_{max}(\lambda)| \leq \tilde{\varepsilon}$ , so  $\lambda \mapsto \kappa_{max}(\lambda)$  is continuous.

Let  $\mu_0 < \lambda < \tilde{\lambda} < \mu_1$ . Then

$$\frac{\tilde{\lambda}}{\lambda_s(k) - \tilde{\lambda}} - \frac{\lambda}{\lambda_s(k) - \lambda} = \frac{(\tilde{\lambda} - \lambda)\lambda_s(k)}{(\lambda_s(k) - \tilde{\lambda})(\lambda_s(k) - \lambda)} \geq 0 \quad (4.3)$$

since  $(\lambda_s(k) - \tilde{\lambda})(\lambda_s(k) - \lambda) > 0$  and  $\lambda_s(k) \geq 0$  for all  $s$  and all  $k$ . We have the following expression for  $\langle A_\lambda u, u \rangle_{\varepsilon_0}$  (see the calculation preceding (4.5)):

$$\frac{\lambda}{2\pi} \int_{-\pi}^{\pi} \sum_{s \in \mathbb{N}} (\lambda_s(k) - \lambda)^{-1} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \psi_s(\cdot, k) \right\rangle_{L^2_{\varepsilon_0}((0,1)^2)} \right|^2 dk.$$

In view of (4.3), we see that  $\lambda \mapsto \kappa_{max}(\lambda)$  is monotonically increasing proving the first statement of the lemma.

(2) We now investigate strict monotonicity of  $\lambda \mapsto \kappa_{max}(\lambda)$ . We note that  $\lambda \mapsto \kappa_{max}(\lambda)$  need not be differentiable, as it is possible for eigenvalue branches to cross. However,  $\kappa_{max}$  is piecewise analytic on the interval  $[\lambda', \mu_1)$ . Note that for  $\lambda > \lambda'$ ,  $\kappa_{max}(\lambda) \geq \kappa_{max}(\lambda') > 0$  by the monotonicity shown in the first part of the proof, so the point 0 in the spectrum of  $A_\lambda$  does not cause difficulties for the analytic continuation of the eigenvalues (the reader may refer to the discussion of these matters in Kato's book, [23, Theorem VII.3.9 and Remark VII.3.11]).

Consider now some subinterval of  $[\lambda', \mu_1)$  where  $\kappa_{max}$  is analytic and let  $u_\lambda$  be the corresponding normalised eigenfunction of  $A_\lambda$ , depending analytically on  $\lambda$ . Then

$$\kappa_{max}(\lambda) = \langle A_\lambda u_\lambda, u_\lambda \rangle_{\varepsilon_0},$$

so using the symmetry of  $A_\lambda$  we have

$$\begin{aligned} \frac{\partial \kappa_{max}(\lambda)}{\partial \lambda} &= \left\langle \frac{\partial A_\lambda}{\partial \lambda} u_\lambda, u_\lambda \right\rangle_{\varepsilon_0} + \left\langle A_\lambda \frac{\partial u_\lambda}{\partial \lambda}, u_\lambda \right\rangle_{\varepsilon_0} + \left\langle A_\lambda u_\lambda, \frac{\partial u_\lambda}{\partial \lambda} \right\rangle_{\varepsilon_0} \\ &= \left\langle \frac{\partial A_\lambda}{\partial \lambda} u_\lambda, u_\lambda \right\rangle_{\varepsilon_0} + \kappa_{max}(\lambda) \underbrace{\left( \left\langle \frac{\partial u_\lambda}{\partial \lambda}, u_\lambda \right\rangle_{\varepsilon_0} + \left\langle u_\lambda, \frac{\partial u_\lambda}{\partial \lambda} \right\rangle_{\varepsilon_0} \right)}_{= \frac{\partial}{\partial \lambda} \langle u_\lambda, u_\lambda \rangle_{\varepsilon_0} = 0} \\ &= \left\langle \frac{\partial A_\lambda}{\partial \lambda} u_\lambda, u_\lambda \right\rangle_{\varepsilon_0} = \frac{1}{2\pi} \sum_s \int_{-\pi}^{\pi} \frac{\lambda_s(k)}{(\lambda_s(k) - \lambda)^2} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u_\lambda, \psi_s \right\rangle_{\varepsilon_0} \right|^2 dk, \end{aligned}$$

where in the last step we have used the representation of the resolvent via Bloch functions (3.9). Now,  $\lambda_s(k) \geq 0$  and  $\lambda_s(k) = 0$  holds at most for a finite number of  $k$ , else the analytic band function  $\lambda_s$  would be constant. Assume for a contradiction that  $\frac{\partial \kappa_{max}(\lambda)}{\partial \lambda} = 0$ . Then, from the above calculation  $\left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u_\lambda, \psi_s \right\rangle = 0$  for a.e.  $k$  and all  $s \in \mathbb{N}$ . By (3.6), we have  $\sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u_\lambda = 0$ , which implies  $A_\lambda u_\lambda = 0$ . However, this cannot be the case, as  $\kappa_{max}(\lambda) > 0$ . Therefore,  $\kappa_{max}(\lambda)$  is strictly increasing on  $(\lambda', \mu_1)$ .  $\square$

We are now ready to prove our main result which shows that small perturbations of  $\varepsilon_0$  create eigenvalues in the spectral gap.

**Theorem 4.1.** *Let  $\varepsilon_0$  and  $\varepsilon_1$  satisfy Assumptions 2.1, 2.2, 4.1 and that*

$$\|\varepsilon_1\|_\infty < \frac{(\mu_1 - \tilde{\mu}_0) \inf \varepsilon_0}{\tilde{\mu}_0} \quad (4.4)$$

for some  $\tilde{\mu}_0 \in [\mu_0, \mu_1)$ . Then there exists an eigenvalue of the operator  $L(k_x)$  in  $(\tilde{\mu}_0, \mu_1)$ .

*Proof.* We shall estimate the function  $\kappa_{max}$  to obtain information on the largest eigenvalue of  $A_\lambda$ . Let  $u \in L^2((0, 1)^2)$ , then using (3.9) we have

$$\begin{aligned} &\langle \varepsilon_0 A_\lambda u, u \rangle_{L^2((0,1)^2)} \\ &= \lambda \left\langle \varepsilon_0 \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} \left[ \left( -\frac{1}{\varepsilon_0} \Delta - \lambda \right)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u \right]_{(0,1)^2}, u \right\rangle \\ &= \lambda \left\langle \varepsilon_0 \left( -\frac{1}{\varepsilon_0} \Delta - \lambda \right)^{-1} \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u \right\rangle_{L^2(\Omega)} \\ &= \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} \sum_{s \in \mathbb{N}} (\lambda_s(k) - \lambda)^{-1} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \psi_s(\cdot, k) \right\rangle_{L^2_{\varepsilon_0}((0,1)^2)} \right|^2 dk. \end{aligned} \quad (4.5)$$

To first get an upper estimate on  $\kappa_{\max}$ , let  $s'$  be such that  $\mu_1$  is the lowest point of the band function  $\lambda_{s'}$  and  $\mu_0$  is the highest point of  $\lambda_{s'-1}$ . We note that such an  $s'$  must exist for there to be a gap. Let  $\lambda \in (\tilde{\mu}_0, \mu_1)$ . Then since  $\lambda_s(k) - \lambda \leq 0$  for  $s < s'$  we have

$$\begin{aligned} \langle \varepsilon_0 A_\lambda u, u \rangle_{L^2((0,1)^2)} &\leq \frac{\lambda}{2\pi} \int_{-\pi}^{\pi} \sum_{s \geq s'} (\lambda_s(k) - \lambda)^{-1} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \psi_s(\cdot, k) \right\rangle_{\varepsilon_0} \right|^2 dk \\ &\leq \frac{\lambda}{2\pi(\mu_1 - \lambda)} \int_{-\pi}^{\pi} \sum_{s \geq s'} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \psi_s(\cdot, k) \right\rangle_{\varepsilon_0} \right|^2 dk \\ &\leq \frac{\lambda}{2\pi(\mu_1 - \lambda)} \int_{-\pi}^{\pi} \sum_{s \in \mathbb{N}} \left| \left\langle \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u, \psi_s(\cdot, k) \right\rangle_{\varepsilon_0} \right|^2 dk \\ &= \frac{\lambda}{(\mu_1 - \lambda)} \left\| \sqrt{\frac{\varepsilon_1}{\varepsilon_0}} u \right\|_{\varepsilon_0}^2 \leq \frac{\lambda \|\varepsilon_1\|_\infty}{(\mu_1 - \lambda) \inf \varepsilon_0} \|u\|_{\varepsilon_0}^2. \end{aligned} \quad (4.6)$$

Therefore, if the perturbation  $\varepsilon_1$  is sufficiently small such that (4.4) holds, we can find  $\lambda' \in (\tilde{\mu}_0, \mu_1)$  such that

$$\kappa_{\max}(\lambda') = \sup_{\|u\| \neq 0} \frac{\langle A_{\lambda'} u, u \rangle_{\varepsilon_0}}{\|u\|_{\varepsilon_0}^2} < 1. \quad (4.7)$$

We next want to find a lower bound on  $\kappa_{\max}$ . Let  $k_0$  be such that  $\lambda_{s'}(k_0) = \mu_1$ .

We remind the reader that in this section we are assuming  $\varepsilon_1 \geq \alpha > 0$  on a nonempty open set  $D$ . It follows from unique continuation (see e.g. [35]) that for any Bloch function  $\psi_s(\cdot, k)$  we have

$$\int_D |\psi_s(\cdot, k)|^2 \varepsilon_0 > 0.$$

Therefore, and by continuity of the Bloch functions there exist  $\delta > 0$  and  $a > 0$  such that

$$|\langle \psi_{s'}(\cdot, k_0), \psi_{s'}(\cdot, k) \rangle_{L^2_\varepsilon(D)}|^2 \geq a$$

for  $k \in (k_0 - \delta, k_0 + \delta)$ . As test function, we choose  $u = \sqrt{\frac{\varepsilon_0}{\varepsilon_1}} \psi_{s'}(\cdot, k_0)$  on  $D$  and extend  $u$  by zero to  $(0, 1)^2$ . Then from (4.5) we get that for  $\lambda \in (\tilde{\mu}_0, \mu_1)$

$$\begin{aligned} \frac{\langle \varepsilon_0 A_\lambda u, u \rangle}{\|u\|_{\varepsilon_0}^2} &= \frac{\lambda}{2\pi \|u\|_{\varepsilon_0}^2} \int_{-\pi}^{\pi} \sum_{s \in \mathbb{N}} (\lambda_s(k) - \lambda)^{-1} \left| \langle \psi_{s'}(\cdot, k_0), \psi_s(\cdot, k) \rangle_{L^2_\varepsilon(D)} \right|^2 dk \\ &\geq \frac{\lambda}{2\pi \|u\|_{\varepsilon_0}^2} \int_{-\pi}^{\pi} \sum_{s \leq s'} (\lambda_s(k) - \lambda)^{-1} \left| \langle \psi_{s'}(\cdot, k_0), \psi_s(\cdot, k) \rangle_{L^2_\varepsilon(D)} \right|^2 dk \\ &\geq \frac{a\lambda}{2\pi \|u\|_{\varepsilon_0}^2} \int_{k_0-\delta}^{k_0+\delta} \frac{dk}{\lambda_{s'}(k) - \lambda} - \\ &\quad \frac{1}{2\pi \|u\|_{\varepsilon_0}^2} \int_{-\pi}^{\pi} \sum_{s < s'} \frac{\lambda}{\lambda - \lambda_s(k)} \left| \langle \psi_{s'}(\cdot, k_0), \psi_s(\cdot, k) \rangle_{L^2_\varepsilon(D)} \right|^2 dk \\ &\geq \frac{a\lambda}{2\pi \|u\|_{\varepsilon_0}^2} \int_{k_0-\delta}^{k_0+\delta} \frac{dk}{\lambda_{s'}(k) - \lambda} - C_\lambda, \end{aligned}$$

where  $C_\lambda$  is uniformly bounded in  $\lambda$  in the interval  $[\mu_0 + \eta, \mu_1]$  for any  $0 < \eta < \mu_1 - \mu_0$ . Using the Taylor expansion of the analytic function  $\lambda_{s'}$ , which has a minimum at  $k_0$ , we see that

$$\lambda_{s'}(k) \leq \mu_1 + \alpha_n (k - k_0)^{2n}$$

for  $k \in (k_0 - \delta, k_0 + \delta)$  and for some  $n \in \mathbb{N}$  with

$$\alpha_n = \frac{1}{(2n)!} \max_{\xi \in [k_0 - \delta, k_0 + \delta]} \lambda_{s'}^{(2n)}(\xi) > 0.$$

Thus for  $\delta < 1$

$$\begin{aligned} \int_{k_0-\delta}^{k_0+\delta} \frac{dk}{\lambda_{s'}(k) - \lambda} &\geq \int_{k_0-\delta}^{k_0+\delta} \frac{dk}{\mu_1 - \lambda + \alpha_n(k - k_0)^{2n}} \geq \int_{k_0-\delta}^{k_0+\delta} \frac{dk}{\mu_1 - \lambda + \alpha_n(k - k_0)^2} \\ &= \frac{1}{\mu_1 - \lambda} \arctan \left( \sqrt{\frac{\alpha_n}{\mu_1 - \lambda}} X \right) \Big|_{X=-\delta}^{X=\delta} \\ &= \frac{2}{\sqrt{\alpha_n(\mu_1 - \lambda)}} \arctan \left( \sqrt{\frac{\alpha_n}{\mu_1 - \lambda}} \delta \right) \rightarrow \infty \text{ as } \lambda \nearrow \mu_1. \end{aligned}$$

Combining our estimates, we have

$$\kappa_{max}(\lambda) \geq \frac{\langle \varepsilon_0 A_\lambda u, u \rangle}{\|u\|_{\varepsilon_0}^2} \rightarrow \infty \text{ as } \lambda \nearrow \mu_1.$$

Since by (4.7) we have some  $\lambda' \in (\tilde{\mu}_0, \mu_1)$  with  $\kappa_{max}(\lambda') < 1$  and  $\kappa_{max}(\cdot)$  is continuous, we can use the Intermediate Value Theorem to find  $\lambda \in (\lambda', \mu_1)$  such that  $\kappa_{max}(\lambda) = 1$ . From (2.9) and (4.1) we see that this gives an eigenvalue of the perturbed strip operator  $L(k_x)$  between  $\lambda'$  and  $\mu_1$  and thus in  $(\tilde{\mu}_0, \mu_1)$ .  $\square$

**Remark 4.1.** (i) Our result only holds for small perturbations. One would of course generally expect that larger perturbations lead to the creation of more spectrum in the gap as lower eigenvalues of  $A_\lambda$  cross 1, but our method of proof does not cover those cases. However, existence of guided modes for sufficiently large perturbations is shown for example in [14,32].

(ii) We consider a special case: Assume that  $\varepsilon_1(x) = \alpha \tilde{\varepsilon}(x)$  for some function  $\tilde{\varepsilon}(x) \geq 0$  and  $\alpha \in \mathbb{R}^+$ . This enables us to study the spectrum in terms of the scalar parameter  $\alpha$ . We can interpret moving from  $\alpha = 0$  to  $\alpha > 0$  as switching on the perturbation and higher values of  $\alpha$  as increasing the perturbation. Let  $\tilde{A}_\lambda = A_\lambda/\alpha$  and  $\nu_{max}(\lambda)$  be the maximum eigenvalue of  $\tilde{A}_\lambda$ . Then we have  $\lambda \in \sigma(L(k_x))$  iff  $\alpha \nu_{max}(\lambda) = 1$ . This gives the following results:

- (a) Monotonicity of  $\lambda \mapsto \nu_{max}(\lambda)$  and the fact that the maximal eigenvalue  $\nu_{max}(\lambda)$  of  $\tilde{A}_\lambda$  tends to  $+\infty$  as  $\lambda \rightarrow \mu_1$  implies that for any given small  $\alpha$ , we can find  $\lambda$  such that  $\nu_{max}(\lambda) = \frac{1}{\alpha}$ , so an additional spectral point is produced as soon as the perturbation is switched on.
- (b) Strict monotonicity of  $\lambda \mapsto \nu_{max}(\lambda)$  implies that there exist  $\lambda' > \mu_0$  and  $\beta > 0$  such that  $\nu_{max} : (\lambda', \mu_1) \rightarrow (\beta, \infty)$  is invertible. Given  $\alpha < \frac{1}{\beta}$ , the function  $g(\alpha) := \nu_{max}^{-1}(\frac{1}{\alpha})$  always gives an eigenvalue and  $g(\alpha) \rightarrow \mu_1$  when  $\alpha \rightarrow 0$ .
- (c) Let  $\lambda \in (\mu_0, \mu_1)$ . The estimate (4.6) shows that the maximum eigenvalue  $\nu_{max}(\lambda)$  of  $\tilde{A}_\lambda$  satisfies

$$\nu_{max}(\lambda) \leq \frac{\lambda \|\tilde{\varepsilon}\|_\infty}{(\mu_1 - \lambda) \inf \varepsilon_0}.$$

Hence, if  $\alpha < \frac{(\mu_1 - \lambda) \inf \varepsilon_0}{\lambda \|\tilde{\varepsilon}\|_\infty}$  then  $\alpha \nu_{max}(\lambda) < 1$ , so  $\lambda$  is not an eigenvalue. This implies that the perturbation needs to have a certain size before the spectrum can appear at any given point in the gap away from  $\mu_1$ .

(iii) In the case when  $\varepsilon_1$  is a non-positive function analogous results can be shown by considering the operator

$$A'_\lambda v = -\lambda \sqrt{\frac{-\varepsilon_1}{\varepsilon_0}} (L_0(k_x) - \lambda)^{-1} \sqrt{\frac{-\varepsilon_1}{\varepsilon_0}} v.$$

In this case, the extra eigenvalues created for small perturbations will appear at the bottom end of the spectral gap near  $\mu_0$ .

We end the section by considering the consequences of this result for the spectrum of the operator  $L$  in  $L_\varepsilon^2(\mathbb{R}^2)$ .

**Theorem 4.2.** Let  $(M_0, M_1)$  be a gap in  $\sigma(L_0)$ . Assume that  $\varepsilon_1$  is non-negative, is strictly positive on a non-empty open set and that

$$\|\varepsilon_1\|_\infty < \frac{(M_1 - M_0) \inf \varepsilon_0}{M_0}. \quad (4.8)$$

Then  $\sigma(L) \cap (M_0, M_1)$  contains a non-empty interval.

*Proof.* We fix  $\hat{k}_x \in [-\pi, \pi]$  such that  $M_1 \in \sigma(L_0(\hat{k}_x))$ . Such a  $\hat{k}_x$  must exist by a similar argument to that in the proof of Proposition 3.1. Applying Theorem 4.1 with  $\mu_1 = M_1$  and  $\tilde{\mu}_0 = M_0$  gives an eigenvalue  $\lambda(\hat{k}_x)$  of  $L(\hat{k}_x)$  in  $(M_0, M_1)$ . By standard perturbation arguments, there exists  $\delta > 0$  such that for  $k_x \in (\hat{k}_x - \delta, \hat{k}_x + \delta)$  the operator  $L(k_x)$  has an eigenvalue  $\lambda(k_x)$  such that  $k_x \mapsto \lambda(k_x)$  is continuous. Moreover, by [20], it's non-constant. Therefore,  $\{\lambda(k_x) : k_x \in (\hat{k}_x - \delta, \hat{k}_x + \delta)\}$  is a non-empty interval and by (2.7), it is contained in  $\sigma(L)$ .  $\square$

**Remark 4.2.** Since by [20] the spectrum  $\sigma(L)$  does not contain eigenvalues, the additional spectrum induced in the gap  $(M_0, M_1)$  is purely continuous. Thus the perturbed problem really deserves the name waveguide since the spectrum of  $L$  within  $(M_0, M_1)$  corresponds to frequency ranges in which light is transmitted through the structure, which were not present in the original unperturbed bulk.

## 5. Non-accumulation of the eigenvalues

In this section we wish to show that new eigenvalues generated by the perturbation cannot accumulate at the band edges. Unlike in section 4, we no longer make any assumptions on the sign of the perturbation.

In (2.9), set

$$v = \frac{\varepsilon_1}{\varepsilon_0} u.$$

Note that  $v$  is supported in  $[0, 1]^2$ , as  $\varepsilon_1|_\Omega$  is, and  $v$  satisfies

$$v = \lambda \frac{\varepsilon_1}{\varepsilon_0} (L_0(k_x) - \lambda)^{-1} v. \quad (5.1)$$

Therefore, we shall study the spectrum of the operator  $\lambda \frac{\varepsilon_1}{\varepsilon_0} (L_0(k_x) - \lambda)^{-1}$  acting on functions supported in  $[0, 1]^2$ .

### (a) Analytic continuation of the resolvent

Recall the resolvent formula (3.9):

$$(L_0(k_x) - \lambda)^{-1} r = \frac{1}{\sqrt{2\pi}} \sum_{s \in \mathbb{N}} \int_{-\pi}^{\pi} (\lambda_s(k) - \lambda)^{-1} P_s(k, r) dk \quad (5.2)$$

for  $\lambda$  outside the spectrum of  $L_0(k_x)$  and  $r \in L_{\varepsilon_0}^2((0, 1)^2)$ . Considered as a map in  $\mathcal{L}(L_{\varepsilon_0}^2((0, 1)^2))$  the resolvent  $(L_0(k_x) - \lambda)^{-1}$  is compact and we would like to use meromorphic Fredholm theory (see Theorem 5.1) to analyze the point spectrum near the band edges. Fredholm theory is a useful tool for studying boundary value problems for partial differential equations, and is particularly efficient when one has to deal with an operator equation depending analytically on a parameter. This situation arises frequently in scattering theory, e.g. see Jakšić and Poulin [21] and Colton, Päiväranta and Sylvester [9] or Lakshtanov and Vainberg [30] for applications to transmission eigenvalue problems.

However, in our situation the resolvent is not well-defined in a neighbourhood of the band edge. To overcome this difficulty, we transform the critical integrals in (5.2) by integration over a suitable contour in the complex plane (see Figure (a)) instead of the real interval  $[-\pi, \pi]$ . In this way, we obtain an analytic operator family, see (5.3), that coincides with the resolvent on a sector of the complex plane. Analytic continuation techniques are very useful for periodic problems, when there is a distinguished spatial direction (i.e. the waveguide axis), although to the authors'



knowledge, this has not been exploited too often in the literature. We refer the reader to [11,24] for applications to scattering from a half-space for the Schrödinger equation. More recently, the usefulness of analytic continuation in optical waveguides and for more general elliptic operators was noted by one of the authors and used in [18,20].

We will do this construction for the lower edge of a band near  $\mu_1$ . A similar construction is possible near the top end of a band near  $\mu_0$ .

Note that by Proposition 3.1 there are only finitely many pairs  $(s_p, k_p), p = 1, \dots, N$  such that for  $\lambda = \mu_1$  the integrand  $(\lambda_{s_p}(k) - \lambda)^{-1} P_{s_p}(k, r)$  in (5.2) is singular at  $k_p$ , since  $\lambda_{s_p}(k_p) = \mu_1$ . As none of the analytic functions  $\lambda_s$  can have a zero of infinite multiplicity by the Thomas argument (see [26,37]), we have for each  $p$  that  $\lambda_{s_p}^{(j)}(k_p) = 0$  for  $j = 1, \dots, m_p - 1$  and  $\lambda_{s_p}^{(m_p)}(k_p) \neq 0$  for some even  $m_p \geq 2$ . Since  $\mu_1$  lies at the bottom end of the band, we can write  $\lambda_{s_p}(k) = \mu_1 + (k - k_p)^{m_p} g_p(k)$  with  $g_p$  an analytic function of  $k, g_p(k_p) > 0$  and  $g_p(\mathbb{R}) \subseteq \mathbb{R}$ .

We set  $h_{s_p, k_p}(k) := (k - k_p)^{m_p} \sqrt[m_p]{g_p(k)}$ . Here, we choose the branch cut of the root away from the positive real axis, so that it does not intersect with the set  $\{g_p(k)\}$  for  $k$  in a small neighbourhood of  $k_p$ . Since  $h'_{s_p, k_p}(k_p) = m_p \sqrt[m_p]{g_p(k_p)} \neq 0$ , we can locally invert  $h_{s_p, k_p}$  to obtain  $k = h_{s_p, k_p}^{-1}(\nu)$ , and  $k$  will depend analytically on  $\nu$  for  $\nu$  in a ball  $\mathcal{N}$  centered at 0. Let  $\mathcal{N}^+ = \mathcal{N} \cap \{\text{Im}(\nu) > 0\}$  and  $\mathcal{N}^- = \mathcal{N} \cap \{\text{Im}(\nu) < 0\}$ . Clearly,  $h_{s_p, k_p} : \mathbb{R} \rightarrow \mathbb{R}$ , so  $h_{s_p, k_p}^{-1} : \mathcal{N}^\pm \rightarrow \mathbb{C} \setminus \mathbb{R}$ . As  $\mathcal{N}^\pm$  are connected,  $h_{s_p, k_p}^{-1}(\mathcal{N}^\pm)$  are connected and are therefore completely contained in either the lower or upper half-plane. Take  $\nu = \pm is \in \mathcal{N}^\pm$ . Then, as

$$\left(h_{s_p, k_p}^{-1}\right)'(0) = 1 / \sqrt[m_p]{g_p(k_p)} > 0,$$

by Taylor series expansion we get

$$h_{s_p, k_p}^{-1}(\pm is) = k_p + \frac{\pm is}{\sqrt[m_p]{g_p(k_p)}} + r_\pm(is)$$

with  $r_\pm(is)/s \rightarrow 0$  as  $s \rightarrow 0$ . Hence,  $\text{Im}(h_{s_p, k_p}^{-1}(is)) > 0$  and  $\text{Im}(h_{s_p, k_p}^{-1}(-is)) < 0$ , so  $h_{s_p, k_p}^{-1}$  maps  $\mathcal{N}^+$  to the upper and  $\mathcal{N}^-$  to the lower half-plane.

**Proposition 5.1.** All solutions of  $(k - k_p)^{m_p} g_p(k) = \nu^{m_p}$  in a neighbourhood of  $k_p$  are given by

$$k = h_{s_p, k_p}^{-1}(e^{2\pi ip/m_p} \nu)$$

with  $p = 0, \dots, m_p - 1$ .

*Proof.* In a punctured neighbourhood of  $k_p$  we have the following equivalences:

$$\begin{aligned} (k - k_p)^{m_p} g_p(k) = \nu^{m_p} &\Leftrightarrow \left(\frac{\nu}{(k - k_p)^{m_p} \sqrt[m_p]{g_p(k)}}\right)^{m_p} = 1 \\ &\Leftrightarrow \frac{\nu}{(k - k_p)^{m_p} \sqrt[m_p]{g_p(k)}} = e^{-2\pi ip/m_p}, p \in \{0, \dots, m_p - 1\} \\ &\Leftrightarrow k = h_{s_p, k_p}^{-1}(e^{2\pi ip/m_p} \nu), p \in \{0, \dots, m_p - 1\}. \end{aligned}$$

□

For simplicity of notation, from now on we restrict ourselves to the case when only one band, which we call the  $s_0$ -band, touches  $\mu_1$ , i.e.  $\lambda_s(k) = \mu_1$  implies  $s = s_0$ . All results generalise in the obvious way in the case when more than one band touches  $\mu_1$ .

Let  $(s_0, k_p), p = 1, \dots, S$  be all pairs described in Proposition 3.1 with  $m_p$  being the order of the first non-vanishing derivative of  $\lambda_{s_0}$  at  $k_p$ . Let  $m$  be the lowest common multiple of the  $m_p$  and  $q_p = m/m_p$ . Using the Taylor expansion of  $\lambda_{s_0}(k)$  around each of these  $k_p$ , we find a complex neighbourhood  $N(\mu_1)$  of  $\mu_1$  and balls  $B_R(k_1), \dots, B_R(k_S)$  of some radius  $R$  around each of the  $k_p$  such that by Proposition 5.1 the equation  $\lambda_{s_0}(k) = \mu_1 + \mu^{m_p}$  has precisely  $m_p$  solutions in  $B_R(k_p)$



any  $\mu \in \mathcal{O}_1$ . For  $\mu \in \mathcal{O}_1 \setminus \{0\}$  we define the operator family  $B_\mu$  on  $L^2_{\varepsilon_0}((0, 1)^2)$  by

$$\begin{aligned} B_\mu(r) &= \frac{1}{\sqrt{2\pi}} \sum_{s \neq s_0} \int_{-\pi}^{\pi} (\lambda_s(k) - \mu_1 - \mu^m)^{-1} P_s(k, r) dk \\ &\quad + \frac{1}{\sqrt{2\pi}} \int_G (\lambda_{s_0}(k) - \mu_1 - \mu^m)^{-1} P_{s_0}(k, r) dk \\ &\quad + \sqrt{2\pi} i \sum_{p=1}^S \sum_{a \in S_p} \lambda'_{s_0} (h_{s_0, k_p}^{-1} (e^{2\pi i a/m_p} \mu^{q_p}))^{-1} P_{s_0} (h_{s_0, k_p}^{-1} (e^{2\pi i a/m_p} \mu^{q_p}), r), \end{aligned} \quad (5.3)$$

where  $S_p = \{0, \dots, \frac{m_p}{2} - 1\}$  and  $r \in L^2_{\varepsilon_0}((0, 1)^2)$ . The next result shows that  $B_\mu$  does indeed extend the resolvent.

**Proposition 5.2.** For  $\mu \in \mathcal{O}_1 \setminus \{0\}$  with  $\arg \mu \in (0, 2\pi/m)$ , we have

$$B_\mu(r) = (L_0(k_x) - \mu_1 - \mu^m)^{-1} r$$

for  $r \in L^2_{\varepsilon_0}((0, 1)^2)$

*Proof.* Noting the resolvent formula (3.9), we see that  $B_\mu$  is obtained from the resolvent by replacing integration over  $(-\pi, \pi)$  with integration over  $G$  when  $s = s_0$  and adding the final term with the point evaluation. Hence, the proof amounts to a simple application of the residue theorem. This involves finding the solutions to  $\lambda_{s_0}(k) = \mu_1 + \mu^m$ , i.e. to  $(k - k_p)^{m_p} g(k) = \mu^m$ , lying between  $G$  and the real axis. Since  $h_{s_0, k_p}^{-1}$  maps points in the upper half-plane to points in the upper half-plane and points in the lower half-plane to points in the lower half-plane, by Proposition 5.1 and the discussion following it, these are given by  $k = h_{s_0, k_p}^{-1} (e^{2\pi i a/m_p} \mu^{m/m_p})$  for those  $a \in \{0, \dots, m_p - 1\}$  such that  $e^{2\pi i a/m_p} \mu^{m/m_p}$  lies in the upper half plane. Now for  $\arg \mu \in (0, 2\pi/m)$ ,

$$\arg \left( e^{2\pi i a/m_p} \mu^{m/m_p} \right) \in \left( 2\pi \frac{a}{m_p}, 2\pi \frac{a}{m_p} + \frac{2\pi}{m_p} \right).$$

This is in  $(0, \pi)$  if and only if  $a \in S_p$ . Note that in calculating the residue, we have used that  $\text{Res}_z f/g = f(z)/g'(z)$  whenever  $g(z) = 0$ ,  $g'(z) \neq 0$ .  $\square$

## (b) Properties of the operator family $B_\mu$

We next look at properties of the operator family  $B_\mu$  from (5.3) with  $\mu$  in a small neighbourhood  $\mathcal{O}$  of 0. Recall that our aim is to be able to apply Fredholm theory to this operator family.

**Proposition 5.3.** There exists a neighbourhood  $\mathcal{O}$  of 0 such that  $\mu \mapsto B_\mu$  is analytic as a map from  $\mathcal{O} \setminus \{0\}$  to  $\mathcal{L}(L^2_{\varepsilon_0}((0, 1)^2))$ .

*Proof.* We discuss the three terms on the right hand side of (5.3) separately.

(1) We first consider the contour integral

$$\frac{1}{\sqrt{2\pi}} \int_G (\lambda_{s_0}(k) - \mu_1 - \mu^m)^{-1} P_{s_0}(k, r) dk. \quad (5.4)$$

Lemma 5.1 implies that along the contour  $G$  we have that  $|(\lambda_s(k) - \lambda_{s_0} - \mu^m)^{-1}|$  is bounded for  $\mu \in \mathcal{O}_1$ . In particular, this implies that the integrand in (5.4) is analytic in  $\mu$ .

(2) The only problem in the third term, the residue term, arises when

$$\lambda'_{s_0}(h_{s_0, k_p}^{-1}(e^{2\pi i a/m_p} \mu^{q_p})) = 0.$$

Now,  $\mu = 0$  is locally the only solution of  $h_{s_0, k_p}^{-1}(\mu) = k_p$  and, since

$$\lambda'_{s_0}(k) = (k - k_p)^{m_p - 1} \underbrace{[m_p g(k) + (k - k_p)g'(k)]}_{\neq 0 \text{ for } k=k_p},$$

$k_p$  is locally the only zero of  $k \mapsto \lambda'_{s_0}(k)$ . Therefore, there exists a neighbourhood  $\mathcal{O}_2$  of 0 such that the residue term is analytic for  $\mu \in \mathcal{O}_2 \setminus \{0\}$ .

(3) Next, we consider the first term, the infinite sum component in the resolvent. We note that by an argument as in the proof of Proposition 3.1, there exists  $\eta > 0$  such that for all band functions  $\lambda_s$  which do not touch  $\mu_1$  and for all  $k$ , we have  $|\lambda_s(k) - \mu_1| \geq \eta$ . So there exists a neighbourhood  $\mathcal{O}_3$  of the origin such that for all  $\mu \in \mathcal{O}_3$  and for all non-touching bands  $s$  we have  $|\lambda_s(k) - \mu_1 - \mu^m| \geq \frac{\eta}{2}$  for all  $k$ . Hence,  $|(\lambda_s(k) - \mu_1 - \mu^m)^{-1}|$  is uniformly bounded in  $s$  and  $\mu$ . We need to test with a function  $\phi$  and show that for any  $\phi$  the resulting function is analytic in  $\mu$ . Fubini's Theorem allows us to interchange the integration in  $k$  and the  $L^2$ -scalar products. Then we have for  $\mu \in \mathcal{O}_3$  and  $M \geq N$  sufficiently large,

$$\begin{aligned} & \sum_{s=N}^M \int_{-\pi}^{\pi} \frac{1}{\lambda_s(k) - \mu_1 - \mu^m} \langle Ur(\cdot, k), \psi_s(\cdot, k) \rangle_{\varepsilon_0} \langle \psi_s(\cdot, k), \phi \rangle_{\varepsilon_0} dk \\ & \leq \sum_{s=N}^M \int_{-\pi}^{\pi} \frac{2}{\eta} \left| \langle Ur(\cdot, k), \psi_s(\cdot, k) \rangle_{\varepsilon_0} \right| \left| \langle \psi_s(\cdot, k), \phi \rangle_{\varepsilon_0} \right| dk \\ & \leq \frac{2}{\eta} \sqrt{\sum_{s=N}^M \int_{-\pi}^{\pi} |\langle Ur(\cdot, k), \psi_s(\cdot, k) \rangle_{\varepsilon_0}|^2 dk} \sqrt{\sum_{s=N}^M \int_{-\pi}^{\pi} |\langle \psi_s(\cdot, k), \phi \rangle_{\varepsilon_0}|^2 dk}. \end{aligned}$$

By completeness of the Bloch functions (see (3.6)), this tends to 0 as  $N, M \rightarrow \infty$ . Hence we get uniform convergence of the series in  $\mu$  and the limit function is analytic.

Finally, we choose  $\mathcal{O} = \mathcal{O}_1 \cap \mathcal{O}_2 \cap \mathcal{O}_3$ . □

**Proposition 5.4.**  $B_\mu$  is compact for  $\mu \in \mathcal{O} \setminus \{0\}$ .

*Proof.* Again, we consider the three terms on the r.h.s. of (5.3) separately.

(1) The term from the residue is clearly compact, as it is a finite rank operator.

(2) We next consider the contour integral (5.4). Lemma 5.1 implies that along the contour  $G$  we have that  $|(\lambda_{s_0}(k) - \mu_1 - \mu^m)^{-1}|$  is bounded and the same is true of  $\|\psi_{s_0}(\cdot, k)\|$ . Using (3.8) and taking the norm into the integral, we can now estimate

$$\begin{aligned} & \left\| \int_G (\lambda_{s_0}(k) - \mu_1 - \mu^m)^{-1} P_{s_0}(k, r)(\cdot) dk \right\|_{H^1((0,1)^2)} \\ & \leq C \int_G \frac{\|\psi_{s_0}(\cdot, k)\|_{H^1((0,1)^2)}}{|\lambda_{s_0}(k) - \mu_1 - \mu^m|} |\langle r, \psi_{s_0}(\cdot, k) \rangle_{\varepsilon_0}| dk \\ & \leq C \int_G \frac{\sqrt{\lambda_{s_0}(k)+1}}{|\lambda_{s_0}(k) - \mu_1 - \mu^m|} |\langle r, \psi_{s_0}(\cdot, k) \rangle_{\varepsilon_0}| dk \leq C \|r\|_{\varepsilon_0}, \end{aligned}$$

where the second inequality follows using (3.2). By the compact embedding of  $H^1((0,1)^2)$  into  $L^2((0,1)^2)$ , as an operator in  $L^2_{\varepsilon_0}((0,1)^2)$ , this part of the resolvent is compact.

(3) Finally, we consider the infinite sum component in the resolvent. We note that, as before, there exists  $\eta > 0$  such that for all  $\mu \in \mathcal{O}$  and for all non-touching bands  $s$  we have  $|\lambda_s(k) - \mu_1 - \mu^m| \geq \frac{\eta}{2}$  for all  $k$ .

Hence,  $|(\lambda_s(k) - \mu_1 - \mu^m)^{-1}|$  is uniformly bounded in  $s$  and  $\mu$  and

$$\begin{aligned}
& \left\| \sum_{s=N}^M \int_{-\pi}^{\pi} (\lambda_s(k) - \mu_1 - \mu^m)^{-1} P_s(k, r)(\cdot) dk \right\|_{H^1}^2 \\
&= \left\| \int_{-\pi}^{\pi} \sum_{s=N}^M (\lambda_s(k) - \mu_1 - \mu^m)^{-1} \langle Ur(\cdot, k), \psi_s(\cdot, k) \rangle_{\varepsilon_0} \psi_s(\cdot, k) dk \right\|_{H^1}^2 \\
&\leq C \int_{-\pi}^{\pi} \left\| \sum_{s=N}^M (\lambda_s(k) - \mu_1 - \mu^m)^{-1} \langle Ur(\cdot, k), \psi_s(\cdot, k) \rangle_{\varepsilon_0} \psi_s(\cdot, k) \right\|_{H^1}^2 dk \\
&\leq C \int_{-\pi}^{\pi} \sum_{s=N}^M \frac{\|\psi_s(\cdot, k)\|_{H^1}^2}{(\lambda_s(k) - \mu_1 - \mu^m)^2} \left| \langle Ur(\cdot, k), \psi_s(\cdot, k) \rangle_{\varepsilon_0} \right|^2 dk \\
&\leq C \int_{-\pi}^{\pi} \sum_{s=N}^M \frac{(\sqrt{\lambda_s(k)} + 1)^2}{(\lambda_s(k) - \mu_1 - \mu^m)^2} \left| \langle Ur(\cdot, k), \psi_s(\cdot, k) \rangle_{\varepsilon_0} \right|^2 dk \\
&\leq C \sum_{s=N}^M \int_{-\pi}^{\pi} \left| \langle Ur(\cdot, k), \psi_s(\cdot, k) \rangle_{\varepsilon_0} \right|^2 dk \rightarrow 0
\end{aligned}$$

as  $M, N \rightarrow \infty$ . In the third line we have used the  $H^1$ -orthogonality of the eigenfunctions  $\{\psi_s(\cdot, k)\}_{s \in \mathbb{N}}$  for fixed  $k$ . Thus, we have a Cauchy sequence in  $H^1((0, 1)^2)$ . Moreover, the  $H^1$ -norm of the limit is bounded by  $C \|r\|_{\varepsilon_0}$  which gives compactness as an operator in  $L^2_{\varepsilon_0}((0, 1)^2)$ .  $\square$

**Proposition 5.5.**  $B_\mu$  has a pole of finite rank for  $\mu = 0$ .

*Proof.* We note that the only pole comes from the residue term at  $\mu = 0$ . The operator has a pole of finite order as  $\lambda_{s_0}^l$  only has simple zeroes of order  $m_p - 1$  at the points  $k_p$  and  $h_{s_0, k_p}^{-1}$  is analytic in a neighbourhood of 0. Moreover, the factors  $P_{s_0}(h_{s_0, k_p}^{-1}(e^{2\pi i a/m_p} \mu^{q_p}), r)$  are of rank 1.  $\square$

**Proposition 5.6.** Let  $\mu \in \mathcal{O} \setminus \{0\}$  such that  $\arg \mu \in (0, \pi/m)$ . Let

$$\tilde{B}_\mu = (\mu_1 + \mu^m) \frac{\varepsilon_1}{\varepsilon_0} B_\mu. \quad (5.5)$$

Then  $(I - \tilde{B}_\mu)v = 0$  only has the trivial solution in  $L^2_{\varepsilon_0}((0, 1)^2)$ .

*Proof.* Suppose that there exists  $v \neq 0$  such that  $(I - \tilde{B}_\mu)v = 0$ . Set  $w := (L_0(k_x) - \lambda)^{-1} v$  where  $\lambda = \mu_1 + \mu^m$ . Then Proposition 5.2 and the equation for  $v$  imply that  $v = \lambda \frac{\varepsilon_1}{\varepsilon_0} w$ . This gives  $(-\frac{1}{\varepsilon_0} \Delta - \lambda)w = v = \lambda \frac{\varepsilon_1}{\varepsilon_0} w$ , or  $-\Delta w = \lambda(\varepsilon_0 + \varepsilon_1)w$ . However, this implies that  $\lambda = \mu_1 + \mu^m$  is a non-real eigenvalue of the selfadjoint operator  $L(k_x)$ , yielding a contradiction.  $\square$

## (c) Main result

The abstract result on meromorphic Fredholm theory which we shall need is the following [36, Theorem XIII.13]:

**Theorem 5.1.** Let  $D$  be a domain in  $\mathbb{C}$ ,  $S$  a discrete subset of  $D$ , and  $H$  a Hilbert space. Assume we have a family of operators  $\{A_z : z \in D\}$  such that

- (i)  $z \mapsto A_z$  is analytic as a map from  $D \setminus S$  to  $\mathcal{L}(H)$ ,
- (ii)  $A_z$  is compact for  $z \in D \setminus S$ ,

- (iii)  $A_z$  has poles in  $S$  of finite rank,
- (iv) there exists  $z \in D \setminus S$  such that  $(I - A_z)u = 0$  has only the trivial solution.

Then there exists at most a discrete set  $\tilde{S} \subset D$  such that  $(I - A_z)^{-1} \in \mathcal{L}(H)$  for all  $z \in D \setminus (S \cup \tilde{S})$ .

We can therefore prove our main result of this section:

**Theorem 5.2.** *Let  $\varepsilon_0$  and  $\varepsilon_1$  satisfy Assumptions 2.1 and 2.2. The spectrum of the problem on the strip, i.e. the spectrum of the operator  $L(k_x)$  can not accumulate at the ends of bands of the spectrum of the operator  $L_0(k_x)$ .*

*Proof.* By [4, Lemma 10], the spectrum of the operator  $L(k_x)$  outside the bands can only consist of eigenvalues. By (5.1), it is clear that  $\lambda = \mu_1 + \mu^m \in \sigma_p(L(k_x))$  if and only if  $I - \tilde{B}_\mu$  is not invertible.

We apply Theorem 5.1 to the operator family  $\tilde{B}_\mu$  from (5.5) with  $\mu$  in a small neighbourhood  $\mathcal{O}$  of 0. The properties (1)-(3) follow from the propositions proved for  $B_\mu$  in the previous sub-section, which obviously carry over to  $\tilde{B}_\mu$ , while (4) is shown in Proposition 5.6.  $\square$

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