GENERALIZED FROBENIUS NUMBERS: BOUNDS AND AVERAGE BEHAVIOR

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ABSTRACT. Let $n \geq 2$ and $s \geq 1$ be integers and $a = (a_1, \ldots, a_n)$ be a relatively prime integer n-tuple. The s-Frobenius number of this n-tuple, $\mathrm{F}_s(a)$, is defined to be the largest positive integer that cannot be represented as $\sum_{i=1}^{n} a_i x_i$ in at least s different ways, where $x_1, ..., x_n$ are non-negative integers. This natural generalization of the classical Frobenius number, $\mathrm{F}_1(a)$, has been studied recently by a number of authors. We produce new upper and lower bounds for the s-Frobenius number by relating it to the so called s-covering radius of a certain convex body with respect to a certain lattice; this generalizes a well-known theorem of R. Kannan for the classical Frobenius number. Using these bounds, we obtain results on the average behavior of the s-Frobenius number, extending analogous recent investigations for the classical Frobenius number by a variety of authors. We also derive bounds on the s-covering radius, an interesting geometric quantity in its own right.

1. Introduction

Let a be a positive integral n-dimensional primitive vector, i.e., $a = (a_1, \ldots, a_n)^\intercal \in \mathbb{Z}^n_{>0}$ with $\gcd(a) := \gcd(a_1, \ldots, a_n) = 1$, so that $a_1 < a_2 < \cdots < a_n$. For a positive integer s the *s-Frobenius number* $\mathrm{F}_s(a)$, is the largest number which cannot be represented in at least s different ways as a non-negative integral combination of the a_i's, i.e.,

$$\mathrm{F}_s(a) = \max\{b \in \mathbb{Z} : \#\{z \in \mathbb{Z}^n_{\geq 0} : \langle a, z \rangle = b\} < s\},$$

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$.

This generalized Frobenius number has been introduced and studied by Beck and Robins [7], who showed, among other results, that for $n = 2$

$$\mathrm{F}_s(a) = s\, a_1\, a_2 - (a_1 + a_2). \tag{1.1}$$

In particular, this identity generalizes the well-known result in the setting of the (classical) Frobenius number which corresponds to $s = 1$. The origin of this classical result is unclear, it was most likely known already to Sylvester, see e.g. [23]. The literature on the Frobenius number $\mathrm{F}_1(a)$ is vast; for a comprehensive and extensive survey we refer the reader to the book of Ramirez Alfonsin [19]. Despite the exact formula in the case $n = 2$, for

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general n only bounds on the Frobenius number $\mathrm{F}_1(a)$ are available. For instance, for $n \geq 3$

$$(1.2)\quad ((n-1)!\, a_1 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}} - (a_1 + \cdots + a_n) < \mathrm{F}_1(a) \leq 2\, a_n \left[\frac{a_1}{n}\right] - a_1.$$

Here the lower bound follows from a sharp lower bound due to Aliev and Gruber [1], and the upper bound is due to Erdős and Graham [9]. Hence, in the worst case scenario we have an upper bound of the order $|a|_\infty^2$ on the Frobenius number with respect to the maximum norm of the input vector a. It is worth a mention that an upper bound on $\mathrm{F}_1(a)$, which is symmetric in all of the a_i's has recently been produced by Fukshansky and Robins [10]. The quadratic order of the upper bound is known to be optimal (see, e.g., [9]) and in view of the lower bound which is at most of size $|a|_\infty^{\frac{n}{n-1}}$ it is quite natural to study the average behavior of $\mathrm{F}_1(a)$. This research was initiated and strongly influenced by Arnold [4]–[5], and due to recent results of Bourgain and Sinai [8], Aliev and Henk [2], Aliev, Henk and Hinrichs [3], Marklof [18], Li [17], Shur, Sinai and Ustinov [20], Strömbergsson [22] and Ustinov [24] we have a pretty clear picture of "the average Frobenius number". In order to describe some of these results, which are going to extend to the s-Frobenius number $\mathrm{F}_s(a)$, we need a bit more notation. Let

$$\mathrm{G}(T) = \{a \in \mathbb{Z}^n_{>0} : \gcd(a) = 1,\, |a|_\infty \leq T\},$$

be the set of all possible input vectors of the Frobenius problem of size (in maximum norm) at most T. Aliev, Henk and Hinrichs [3] showed that

$$(1.3)\qquad \sup_T \frac{\sum_{a\in \mathrm{G}(T)} \mathrm{F}_1(a)/\,(a_1\, a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}}}{\#\mathrm{G}(T)} \ll\gg_n 1,$$

i.e., the expected size of $\mathrm{F}_1(a)$ is "close" to the size of its lower bound in (1.2); here and below $\ll_n$ and $\gg_n$ denote the Vinogradov symbols with the constant depending on n only. Recently, Li [17] gave the bound

$$(1.4)\qquad \mathrm{Prob}\left(\mathrm{F}_1(a)/\,(a_1\, a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}} \geq D\right) \ll_n D^{-(n-1)},$$

where $\mathrm{Prob}\,(\cdot)$ is meant with respect to the uniform distribution among all points in the set $\mathrm{G}(T)$. The bound (1.4) is best possible due to an unpublished result of Marklof, and clearly implies (1.3).

The main purpose of this paper is to extend the results stated above, i.e., (1.2), (1.3) and (1.4), to the generalized Frobenius number $\mathrm{F}_s(a)$ in the following way:

Theorem 1.1. *Let $n \geq 2$, $s \geq 1$. Then*

$$\mathrm{F}_s(a) \geq s^{\frac{1}{n-1}}\, ((n-1)!\, a_1 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}} - (a_1 + \cdots + a_n)\,,$$
$$\mathrm{F}_s(a) \leq \mathrm{F}_1(a) + (s-1)^{\frac{1}{n-1}}\, ((n-1)!\, a_1 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}}\,.$$

Bounds with almost the same dependencies on s were recently obtained by Fukshansky and Schürmann [11]. Their lower bound, however, is only valid for sufficiently large s. Aliev and Gruber [1] applied the results of Schinzel [21] to obtain a sharp lower bound for the Frobenius number in terms of the covering radius of a simplex. The same approach can be used to obtain a sharp lower bound for the s-Frobenius number as well. We postpone a detailed discussion of these matters to a future paper.

As an almost immediate consequence of Theorem 1.1 we obtain:

Corollary 1.2. *Let $n \geq 3$, $s \geq 1$. Then*

$$\text{i)} \quad \operatorname{Prob}\left(\mathrm{F}_s(a)/\left(s \cdot a_1\, a_2 \cdot \ldots \cdot a_n\right)^{\frac{1}{n-1}} \geq D\right) \ll_n D^{-(n-1)},$$

$$\text{ii)} \quad \sup_T \frac{\sum_{a \in \mathrm{G}(T)} \mathrm{F}_s(a)/\left(s \cdot a_1\, a_2 \cdot \ldots \cdot a_n\right)^{\frac{1}{n-1}}}{\#\mathrm{G}(T)} \ll\gg_n 1.$$

Hence in this generalized setting the average s-Frobenius number is of the size $(s \cdot a_1\, a_2 \cdot \ldots \cdot a_n)^{\frac{1}{n-1}}$, which again is the size of its lower bound as stated in Theorem 1.1.

The proof of Theorem 1.1 is based on a generalization of a result of Kannan which relates the classical Frobenius number to the covering radius of a certain simplex with respect to a certain lattice. In our setting we need a kind of generalized covering radius, whose definition as well as some properties and background information from the Geometry of Numbers will be given in Section 2. In Section 3 we will prove, analogously to the mentioned result of Kannan, an identity between $\mathrm{F}_s(a)$ and this generalized covering radius and will present a proof of Theorem 1.1. The last section contains a proof of Corollary 1.2.

2. The s-covering radius

In what follows, let $\mathcal{K}^n$ be the space of all full-dimensional convex bodies, i.e., closed bounded convex sets with non-empty interior in the n-dimensional Euclidean space $\mathbb{R}^n$. The volume of a set $X \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure, is denoted by $\operatorname{vol} X$. Moreover, we denote by $\mathcal{L}^n$ the set of all n-dimensional lattices in $\mathbb{R}^n$, i.e., $\mathcal{L}^n = \{B\,\mathbb{Z}^n : B \in \mathbb{R}^{n\times n},\ \det B \neq 0\}$. For $\Lambda = B\,\mathbb{Z}^n \in \mathcal{L}^n$, $\det \Lambda = |\det B|$ is called the determinant of the lattice Λ. Here we are interested in the following quantity:

Definition 2.1. *Let $s \in \mathbb{N}$, $s \geq 1$. For $K \in \mathcal{K}^n$ and $\Lambda \in \mathcal{L}^n$ let*

$$\mu_s(K, \Lambda) = \min\{\mu > 0 : \textit{for all } t \in \mathbb{R}^n \textit{ there exist } b_1, \ldots, b_s \in \Lambda \\ \textit{such that } t \in b_i + \mu K\ \forall\ 1 \leq i \leq s\}$$

be the smallest positive number μ such that any $t \in \mathbb{R}^n$ is covered by at least s lattice translates of $\mu\, K$. $\mu_s(K, \Lambda)$ is called the s-covering radius *of K with respect to Λ.*

For $s = 1$ we get the well-known covering radius, for the information about which we refer the reader to Gruber [12] and Gruber and Lekkerkerker [13]. These books also serve as excellent sources for more information on lattices and convex bodies in the context of Geometry of Numbers.

Note that the s-covering radius is different from the jth covering minimum introduced by Kannan and Lovász [16]. We also remark that $\mu_s(K,\Lambda)$ may be described equivalently as the smallest positive number μ such that any translate of μK contains at least s lattice points, i.e.,

$$\mu_s(K,\Lambda) = \min\{\mu > 0 : \#\{(t+\mu K)\cap\Lambda\} \geq s \text{ for all } t \in \mathbb{R}^n\}. \tag{2.1}$$

Lemma 2.2. *Let $s \in \mathbb{N}$, $s \geq 1$, $K \in \mathcal{K}^n$ and let $\Lambda \in \mathcal{L}^n$. Then*

$$s^{\frac{1}{n}}\left(\frac{\det\Lambda}{\operatorname{vol}K}\right)^{\frac{1}{n}} \leq \mu_s(K,\Lambda) \leq \mu_1(K,\Lambda) + (s-1)^{\frac{1}{n}}\left(\frac{\det\Lambda}{\operatorname{vol}K}\right)^{\frac{1}{n}}.$$

Proof. It suffices to prove these inequalities for the standard lattice $\mathbb{Z}^n$ of determinant 1; for brevity, we will just write μ_s instead of $\mu_s(K,\mathbb{Z}^n)$. The lower bound just reflects the fact that each point of $\mathbb{R}^n$ is covered at least s times by the lattice translates of $\mathbb{Z}^n + \mu_s K$. A standard argument to see this in a more precise way is the following. Let $P = [0,1)^n$ be the half open cube of edge length 1, and for $L \subseteq \mathbb{R}^n$ let $\chi_L : \mathbb{R}^n \to \{0,1\}$ be its characteristic function, i.e., $\chi_L(x) = 1$ if $x \in L$, otherwise it is 0. Then with $L = \mu_s K$ we get

$$\begin{aligned}
\operatorname{vol}(L) &= \int_{\mathbb{R}^n} \chi_L(x)\,\mathrm{d}x = \int_{\mathbb{Z}^n+P} \chi_L(x)\,\mathrm{d}x = \sum_{z\in\mathbb{Z}^n}\int_{z+P}\chi_L(x)\,\mathrm{d}x \\
&= \sum_{z\in\mathbb{Z}^n}\int_P \chi_{-z+L}(x)\,\mathrm{d}x = \int_P\left(\sum_{z\in\mathbb{Z}^n}\chi_{-z+L}(x)\right)\mathrm{d}x \\
&\geq \int_P s\,\mathrm{d}x = s.
\end{aligned} \tag{2.2}$$

Hence $\operatorname{vol}(\mu_s K) \geq s$. Combining this observation with the homogeneity of the volume we obtain the lower bound. For the upper bound we may assume $s \geq 2$, since there is nothing to prove for $s = 1$. The first two lines of (2.2) also prove a well-known result of van der Corput [13, pp. 47], which in our setting of a convex body says: if $L \in \mathcal{K}^n$ with $\operatorname{vol}(L) \geq s-1$ then there exists a $t \in P$ such that t is covered by at least s lattice translates of L. Hence for $\overline{\mu} = ((s-1)/\operatorname{vol}(K))^{1/n}$ we know that there exist $z_1,\ldots,z_s \in \mathbb{Z}^n$ and a $\overline{t} \in P$ such that $\overline{t} \in z_i + \overline{\mu}K$, $1 \leq i \leq s$. Now given an arbitrary $t \in \mathbb{R}^n$ we know by the definition of the covering radius μ_1 that there exists a $z \in \mathbb{Z}^n$ such that $t - \overline{t} \in z + \mu_1 K$. Hence

$$t \in (z+z_i) + (\mu_1 + \overline{\mu})\,K, \quad 1 \leq i \leq s,$$

and so $\mu_s \leq \mu_1 + \overline{\mu}$ which gives the upper bound. $\square$

It is also worth a mention that, as an immediate corollary of Lemma 2.2 and tools from the Geometry of Numbers, we can obtain upper bounds on $\mu_s(K,\Lambda)$ for any $s \geq 1$ in terms of successive minima of K with respect to Λ. Recall that successive minima $\lambda_i(K,\Lambda)$ of a convex body $K \in \mathcal{K}^n$ with respect to a lattice $\Lambda \in \mathcal{L}^n$ are defined by

$$\lambda_i(K,\Lambda) = \min\{\lambda > 0 : \dim(\lambda(K-K)\cap\Lambda) \geq i\}, 1 \leq i \leq n.$$

Proposition 2.3. *Let $s \in \mathbb{N}$, $s \geq 1$, $K \in \mathcal{K}^n$ and let $\Lambda \in \mathcal{L}^n$. Then*

$$\mu_s(K,\Lambda) \leq \left(1 + (n!^{1/n}/n)(s-1)^{1/n}\right)\sum_{i=1}^{n}\lambda_i(K,\Lambda).$$

Proof. It was pointed out by Kannan and Lovasz [16, Lemma 2.4] that Jarnik's inequalities, relating the covering radius and the successive minima of 0-symmetric convex bodies, are also valid for arbitrary bodies. Hence we have

$$\mu_1(K,\Lambda) \leq \sum_{i=1}^{n}\lambda_i(K,\Lambda). \tag{2.3}$$

On the other hand it is also well known that Minkowski's theorems on successive minima can also be extended to the family of arbitrary convex bodies [13, pp. 59], [14], and, in particular, we have

$$\mathrm{vol}\,(K)\prod_{i=1}^{n}\lambda_i(K,\Lambda) \geq \frac{1}{n!}\,\det\Lambda. \tag{2.4}$$

Applying (2.3) and (2.4) to the upper bound on $\mu_s(K,\Lambda)$ in Lemma 2.2 leads to

$$\begin{aligned}\mu_s(K,\Lambda) &\leq \mu_1(K,\Lambda) + (s-1)^{\frac{1}{n}}\left(\frac{\det\Lambda}{\mathrm{vol}\,K}\right)^{\frac{1}{n}}\\ &\leq \sum_{i=1}^{n}\lambda_i(K,\Lambda) + (s-1)^{\frac{1}{n}}\left(n!\prod_{i=1}^{n}\lambda_i(K,\Lambda)\right)^{\frac{1}{n}}\\ &\leq \left(1 + (n!^{1/n}/n)(s-1)^{1/n}\right)\sum_{i=1}^{n}\lambda_i(K,\Lambda),\end{aligned}$$

by the arithmetic-geometric mean inequality. □

Unfortunately, we are not aware of a nice generalization of Jarnik's lower bound (cf. [16, Lemma 2.4]) $\mu_1(K,\Lambda) \geq \lambda_n(K,\Lambda)$ to the s-covering radius.

3. Frobenius number and covering radius

For a given primitive positive vector $a = (a_1, \dots, a_n)^\intercal \in \mathbb{Z}^n_{>0}$ let

$$S_a = \left\{ x \in \mathbb{R}^{n-1}_{\geq 0} : a_1\, x_1 + \cdots + a_{n-1}\, x_{n-1} \leq 1 \right\}$$

be the $(n-1)$-dimensional simplex with vertices $0, \frac{1}{a_i}\, e_i$ where e_i is the i-th unit vector in $\mathbb{R}^{n-1}$, $1 \leq i \leq n-1$. Furthermore, we consider the following sublattice of $\mathbb{Z}^{n-1}$

$$\Lambda_a = \left\{ z \in \mathbb{Z}^{n-1} : a_1\, z_1 + \cdots + a_{n-1}\, z_{n-1} \equiv 0 \bmod a_n \right\}.$$

This simplex and lattice were introduced by Kannan in his studies of the Frobenius number [15], where he proved the following beautiful identity:

$$\mu_1(S_a, \Lambda_a) = \mathrm{F}_1(a) + a_1 + \cdots + a_n.$$

Here we just extend his arguments to the s-Frobenius number. We start with the following lemma about an "integral version" of $\mu_s(S_a, \Lambda_a)$.

Lemma 3.1. *Let $n \geq 2$, $s \geq 1$, and let*

$$\mu_s(S_a, \Lambda_a; \mathbb{Z}^{n-1}) = \min\left\{ \rho > 0 : \#\left\{ (z + \rho\, S_a) \cap \Lambda_a \right\} \geq s \ \forall\ z \in \mathbb{Z}^{n-1} \right\}.$$

Then

$$\mu_s(S_a, \Lambda_a; \mathbb{Z}^{n-1}) = \mathrm{F}_s(a) + a_n.$$

Proof. To simplify the notation, for each $y \in \mathbb{R}^n$ let $\tilde{y} = (y_1, \dots, y_{n-1})^\intercal$ be the vector consisting of the first $(n-1)$ coordinates of y. Further, let $\overline{\mu_s} = \mu_s(S_a, \Lambda_a; \mathbb{Z}^{n-1})$ and $\mathrm{F}_s = \mathrm{F}_s(a)$.

First we show that $\overline{\mu_s} \leq \mathrm{F}_s + a_n$. To this end, let $z \in \mathbb{Z}^{n-1}$ and let $k \in \{1, \dots, a_n\}$ be such that $\tilde{a}^\intercal\, z \equiv -(\mathrm{F}_s + k) \bmod a_n$. By the definition of F_s we can find $b_1, \dots, b_s \in \mathbb{Z}^n_{\geq 0}$ with $a^\intercal b_i = \mathrm{F}_s + k$, $1 \leq i \leq s$. Hence we have found s different lattice vectors $z + \tilde{b_i} \in \Lambda_a$, $1 \leq i \leq s$, and since $\tilde{b_i} \in (\mathrm{F}_s + k)\, S_a$ we obtain

$$z + \tilde{b_i} \in z + (\mathrm{F}_s + a_n)\, S_a, \quad 1 \leq i \leq s.$$

Hence $\overline{\mu_s} \leq \mathrm{F}_s + a_n$, and it remains to show the reverse inequality.

Since $\gcd(a) = 1$, we can find a $z \in \mathbb{Z}^{n-1}$ with $\tilde{a}^\intercal z \equiv \mathrm{F}_s \bmod a_n$. Now suppose that for a $0 < \gamma < \mathrm{F}_s + a_n$ we can find $g_1, \dots, g_s \in \Lambda_a$ such that $g_i \in z + \gamma\, S_a$. Since $\tilde{a}^\intercal (g_i - z) \equiv \mathrm{F}_s \bmod a_n$ and $\tilde{a}^\intercal (g_i - z) \leq \gamma < \mathrm{F}_s + a_n$, we conclude that there exist non-negative integers m_i with

$$\tilde{a}^\intercal (g_i - z) = \mathrm{F}_s - m_i\, a_n, \quad 1 \leq i \leq s.$$

Since $g_i \in z + \gamma\, S_a$, we conclude that $(g_i - z)$ is a vector with non-negative integer coordinates, and so $\tilde{a}^\intercal (g_i - z) + m_i\, a_n$, $1 \leq i \leq s$, are s different non-negative integral representations of F_s, which contradicts the definition of F_s. This proves that $\overline{\mu_s} \geq \mathrm{F}_s + a_n$, and completes the proof of the lemma. $\square$

The next theorem is the canonical extension of Kannan's Theorem 2.5 in [15] for the classical Frobenius number.

Theorem 3.2. *Let $n \geq 2$, $s \geq 1$. Then*

$$\mu_s(S_a, \Lambda_a) = \mathrm{F}_s(a) + a_1 + \cdots + a_n.$$

Proof. We keep the notation of Lemma 3.1 and its proof, and in addition we set $\mu_s = \mu_s(S_a, \Lambda_a)$. In view of Lemma 3.1, we have to show that

$$\mu_s = \overline{\mu_s} + (a_1 + \cdots + a_{n-1}). \tag{3.1}$$

First we verify the inequality $\mu_s \leq \overline{\mu_s} + a_1 + \cdots + a_{n-1}$. Since the $(n-1)$-dimensional closed cube $\overline{P} = [0,1]^{n-1}$ of edge-length 1 is contained in $(a_1 + \cdots + a_{n-1})\, S_a$,

$$\mathbb{R}^{n-1} = \mathbb{Z}^{n-1} + (a_1 + \cdots + a_{n-1})\, S_a.$$

Hence, in view of (2.1), it suffices to verify that for each $z \in \mathbb{Z}^{n-1}$

$$\#\, \{(z + \overline{\mu_s}\, S_a) \cap \Lambda_a\} \geq s,$$

which follows by the definition of $\overline{\mu_s}$.

Now suppose $\mu_s < \overline{\mu_s} + a_1 + \cdots + a_{n-1}$. By Lemma 3.1, there exists a $z \in \mathbb{Z}^{n-1}$ such that for any subset $I_s \subset \Lambda_a$ of cardinality at least s there exists a $b \in I_s$ with $(z - b) \notin \mathrm{int}(\overline{\mu_s} S_a)$, where int( ) denotes the interior of a set. Let $u \in \mathbb{Z}^{n-1}$ be the vector with all coordinates equal to 1. By our assumption, there exists at least s lattice points $b_i \in \Lambda_a$, $1 \leq i \leq s$, such that $(z + u) \in b_i + \mathrm{int}((\overline{\mu_s} + a_1 + \cdots + a_{n-1}) S_a)$. Then this is certainly also true for any sufficiently small positive ϵ and the point $z + (1-\epsilon)u$. Thus, for $1 \leq i \leq s$,

$$\begin{aligned} \overline{\mu_s} + a_1 + \cdots + a_{n-1} &> \tilde{a}^{\intercal}(z + (1-\epsilon)u - b_i) \\ &= \tilde{a}^{\intercal}(z - b_i) + (1-\epsilon)(a_1 + \cdots + a_{n-1}). \end{aligned}$$

Since ϵ is an arbitrary sufficiently small positive real number, we conclude that $\tilde{a}^{\intercal}(z - b_i) < \overline{\mu_s}$, $1 \leq i \leq s$. On the other hand, we have $z + (1-\epsilon)u - b_i \geq 0$, which implies $z - b_i \geq 0$, $1 \leq i \leq s$. In other words, the s lattice points $b_1, \ldots, b_s$ lie in the interior of $z + \overline{\mu_s}\, S_a$ which contradicts the definition of $\overline{\mu_s}$. □

We remark that in the case $n = 2$, S_a is just the segment $[0, 1/a_1]$ and Λ_a is the set of all integral multiples of a_2, i.e., $\Lambda_a = \mathbb{Z}\, a_2$. Hence, in this special case,

$$\mu_s(S_a, \Lambda_a) = s\, a_1\, a_2,$$

which gives, via Theorem 3.2, another proof of (1.1).

Proof of Theorem 1.1. First we observe that $\det \Lambda_a = a_n$. This follows, for instance, from the fact that there are at most a_n residue classes of the sublattice Λ_a with respect to $\mathbb{Z}^{n-1}$, and since $\gcd(a) = 1$ we have exactly a_n

distinct residue classes. Next we note for the ($(n-1)$-dimensional) volume of S_a that

$$\mathrm{vol}\,(S_a) = \frac{1}{(n-1)!}\frac{1}{a_1\cdot\ldots\cdot a_{n-1}},$$

so that $\det\Lambda_a/\mathrm{vol}\,(S_a) = (n-1)!a_1\cdot\ldots\cdot a_n$. Hence Lemma 2.2 and Theorem 3.2 give the desired bounds. □

4. Average behaviour

It will be convenient to define

$$X_s(a) = \frac{\mathrm{F}_s(a)}{(s\cdot a_1\, a_2\cdot\ldots\cdot a_n)^{1/(n-1)}}$$

for each $a\in\mathrm{G}(T)$. We start with the proof of Corollary 1.2.

Proof. For i), we observe that by (1.4) we may assume $s\geq 2$ and by the upper bound of Theorem 1.1 we have

$$X_s(a)\leq s^{-\frac{1}{n-1}}X_1(a)+\mathrm{c_n}, \tag{4.1}$$

for a dimensional constant $\mathrm{c}_n = ((n-1)!)^{\frac{1}{n-1}}$. Hence, (1.4) implies for, say, $D\geq 2\,\mathrm{c}_n$,

$$\mathrm{Prob}\,(X_s(a)\geq D)\ll_n \frac{1}{s}(D-\mathrm{c}_n)^{-(n-1)}\leq D^{-(n-1)}.$$

Now, in view of (1.3), (4.1) also implies that

$$\frac{1}{\#\mathrm{G}(T)}\sum_{a\in\mathrm{G}(T)}X_s(a)\leq s^{-\frac{1}{n-1}}\left(\frac{1}{\#\mathrm{G}(T)}\sum_{a\in\mathrm{G}(T)}X_1(a)\right)+\mathrm{c}_n\ll_n 1.$$

In order to show that the left hand side is also bounded from below by a constant depending only on n, we use the lower bound of Theorem 1.1 and obtain

$$\frac{1}{\#\mathrm{G}(T)}\sum_{a\in\mathrm{G}(T)}X_s(a)>\mathrm{c}_n - s^{-\frac{1}{n-1}}\frac{1}{\#\mathrm{G}(T)}\sum_{a\in\mathrm{G}(T)}\frac{a_1+\cdots+a_n}{(a_1\cdot\ldots\cdot a_n)^{\frac{1}{n-1}}}.$$

The latter sum has already been investigated in [3], where the proof of Proposition 1 shows precisely that

$$\frac{1}{\#\mathrm{G}(T)}\sum_{a\in\mathrm{G}(T)}\frac{a_1+\cdots+a_n}{(a_1\cdot\ldots\cdot a_n)^{\frac{1}{n-1}}}\leq \mathrm{C}_n\,T^{-\frac{1}{n-1}},$$

for another constant C_n depending only on n. Hence, for sufficiently large T we obtain

$$\frac{1}{\#\mathrm{G}(T)}\sum_{a\in\mathrm{G}(T)}X_s(a)\gg_n 1,$$

which completes the proof of ii). □

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