The JKR-type adhesive contact problems for power-law shaped axisymmetric punches

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Abstract

The JKR (Johnson, Kendall, and Roberts) and Boussinesq-Kendall models describe adhesive frictionless contact between two isotropic elastic spheres, and between a flat-ended axisymmetric punch and an elastic half-space respectively. However, the shapes of contacting solids may be more general than spherical or flat ones. In addition, the derivation of the main formulae of these models is based on the assumption that the material points within the contact region can move along the punch surface without any friction. However, it is more natural to assume that a material point that came to contact with the punch sticks to its surface, i.e. to assume that the non-slipping boundary conditions are valid. It is shown that the frictionless JKR model may be generalized to arbitrary convex, blunt axisymmetric body, in particular to the case of the punch shape being described by monomial (power-law) punches of an arbitrary degree $d \geq 1$. The JKR and Boussinesq-Kendall models are particular cases of the problems for monomial punches, when the degree of the punch d is equal to two or it goes to infinity respectively. The generalized problems for monomial punches are studied under both frictionless and non-slipping (or no-slip) boundary conditions. It is shown that regardless of the boundary conditions, the solution to the problems is reduced to the same dimensionless relations between the actual force, displacements and contact radius. The explicit expressions are derived for the values of the pull-off force and for the corresponding critical contact radius. Connections of the results obtained to problems of nanoindentation in the case of the indenter shape near the tip has some deviation from its nominal shape and the shape function can be approximated by a monomial function of radius, are discussed.
Keywords: JKR theory, adhesive contact, non-slipping, power-law punches, the Boussinesq-Kendall model

1 Introduction

Adhesion and adhesive contact problems have been studied for a long period of time. Adhesion is a universal physical phenomenon that has usually a negligible effect on surface interactions at the macro-scale, whereas it becomes increasingly significant as the contact size decreases (Kendall 2001). The term "adhesion" may have rather different meanings. It may be used to

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denote both the strong chemical bonds between surfaces and weak connections due to van der Waals (vdW) forces. In addition, contact problems with non-slipping (or no-slip) boundary conditions are often called adhesive contact problems (Mossakovskii 1963, Spence 1968). Here forces of chemical bonding are not studied and only molecular adhesion caused by vdW forces is considered. The distinction of these forces is somewhat artificial, because all of these forces are electrical in nature (Deryagin *et al.* 1978, Parsegian 2005), however, this distinction is very convenient because the values of energy of interactions are rather different. The same distinction is usually introduced for studying phenomena of adsorption of a single molecule to a surface, where it is customary to divide adsorption into physical adsorption (physisorption) and chemisorption. The binding forces for physisorption are relatively weak, while the term "chemisorption" is used if the adsorption energy is large enough to be comparable to chemical bond energies. To study contact problems with molecular adhesion one needs to know the work of adhesion, w that is equal to the energy needed to separate two dissimilar surfaces from contact to infinity.

Apparently the first scientific discussion of the adhesion phenomenon is due to Robert Hooke. Observing liquors, syrups and other "*tenacious and glutinous bodies*", he wrote (Hooke 1667) " *it is evident, that the Parts of the tenacious body, as I may so call it, do stick and adhere so closely together, that though drawn out into long and very slender Cylinders, yet they will not easily relinquish one another ... And this Congruity (that I may here a little further explain it) is both a Tenaceous and an Attractive power; for the Congruity, in the Vibrative motions, may be the cause of all kind of attraction, not only Electrical, but Magnetical also, and therefore it may be also of Tenacity and Glutinousness.*"

In 1873 van der Waals discovered a property of molecules to attract each other and "*come to the conclusion that attraction of the molecules decreases extremely quickly with distance, indeed that the attraction only has an appreciable value at distances close to the size of the molecules*" (van der Waals 1910). Maxwell (1874) gave a very high appraisal of the van der Waals results and agreed that attraction is considered at short distances, however molecules repel each other at a closer approach. Peter Lebedev gave the first electromagnetic explanation to vdW forces (Lebedew 1894). However, only after the introduction of quantum mechanics by M. Planck, modern descriptions of the various kinds of attractive forces were given by Debye, London, and Keesom (Parsegian, 2005). The attractive forces are collectively called van der Waals forces. The term includes attraction between: two permanent dipoles (Keesom force), a permanent dipole and a corresponding induced dipole (Debye force), and two instantaneously induced dipoles (London dispersion force).

Nowadays there are several well-established classic models of adhesive contact that include the JKR (Johnson, Kendall, and Roberts) model, the DMT (Derjaguin-Muller-Toporov) model, and the Maugis transition solution between the JKR and DMT models. These models propose methodologies to predict the adhesion force between contacting spherical surfaces (Johnson *et al.* 1971, Derjaguin *et al.* 1975, Maugis 1992). These classic models are very helpful for studying various phenomena that involve molecular adhesion. For example, these models of adhesive contact of spheres are fundamental for the experimental determination of the work of adhesion and elastic contact modulus of materials by the non-direct method introduced by Borodich and Galanov (2008); it has been shown recently that this non-direct method is fast and robust (Borodich *et al.* 2012b, 2013). However, the shapes of contacting solids may be more general than spherical or flat ones.

Let us use both the Cartesian and cylindrical coordinate frames, namely $x_1 = x, x_2 = y, x_3 = z$ and r, ϕ, z, where $r = \sqrt{x^2 + y^2}$ and $x = r\cos\phi, y = r\sin\phi$. In a geometrically linear formulation of the contact problem, the material sample is modelled as a positive half-space $x_3 \geq 0$.

The non-adhesive Hertz (1882) formulation assumes that initially there is only one point of contact between the punch and the half-space. Let the origin (O) of Cartesian x_1, x_2, x_3 coordinates be at the point of initial contact between the punch and the half-space $x_3 \geq 0$. The boundary plane $x_3 = 0$ is denoted by $\mathbb{R}^2$. Hence, the equation of the surface given by a function f, can be written as $x_3 = -f(x_1, x_2), \quad f \geq 0$. After the punch contacts with the half–space, displacements u_i and stresses σ_{ij} are generated.

The Hertz contact problem for two elastic bodies is mathematically equivalent to the problem of contact between a half-space and a curved body whose shape function f is equal to the initial distance between the surfaces, i.e. $f = f_1 + f_2$, where f_1 and f_2 are the shape functions of the solids. In turn, this problem can be reduced to the problem of contact between a rigid indenter (a punch) and an isotropic elastic half-space with the reduced elastic modulus E^* (Galin 1961, Johnson 1985)

$$\frac{1}{E^*} = \frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}.$$

Here E_i and ν_i $(i = 1, 2)$ are the Young's modulus and the Poisson ratio of the first and the second solid respectively. Further in this paper only rigid axisymmetric indenters are considered and, therefore, $E_2 = \infty$ and $E^* = E/(1-\nu^2)$ where E and ν are the elastic modulus and the Poisson ratio of the half-space, respectively. Formally speaking, to solve the contact problem one needs to find the contact region, displacements u_i, and stresses σ_{ij}. However, the most interesting characteristics for testing of materials are the contact radius a and the depth of indentation δ.

In this paper it will be assumed that the distances between the contacting solids may be described as axisymmetric monomial functions of arbitrary degrees d, $d \geq 1$

$$f(r) = B_d r^d, \tag{1}$$

where B_d is the constant of the shape of the monomial function of degree d. Both the frictionless and non-slipping contact boundary conditions will be considered. We will try in the present work to follow the original JKR approach as closely as possible.

It is known that if the shape function is described by monomial functions then even in the non-axisymmetric case, i.e. when $B_d \neq const$ but it is a function of polar angle ϕ, the non-adhesive Hertz-type contact problems are self-similar for both frictionless (Galanov 1981, Borodich 1983, 1989, Borodich and Galanov 2002) and frictional (Borodich 1993, 2008) boundary conditions. Although the problems lack self-similarity in the presence of molecular adhesion (Kendall 2001, Maugis 2000), one can still obtain explicit results for the axisymmetric monomial punches.

The paper is organized as follows:

In §2 we give some preliminary information concerning adhesion and the JKR model of adhesive contact.

In §3 the frictionless JKR model is considered in the case when the punch shape is described by monomial (power-law) function of radius of an arbitrary degree $d \geq 1$. Finally using the

Galin solution in the representation by Borodich and Keer (2004b), the general expressions are given for the relations among the actual force, the contact radius and the relative approach of the bodies in the framework of the frictionless JKR adhesive contact for arbitrary convex, blunt axisymmetric bodies.

As it has been mentioned above, the derivation of the main formulae of the classic JKR and DMT models is based on the assumption that the material points within the contact region can move along the punch surface without any friction. However, it is more natural to assume that a material point that came to contact with the punch sticks to its surface, i.e. to assume that the non-slipping boundary conditions are valid. In §4 the Boussinesq–Kendall problem of an adhesive contact for a flat ended punch is extended to the case of non-slipping contact. Then the JKR approach is extended to non-slipping adhesive contact problems for an arbitrary convex punch whose shape is described by a monomial function of an arbitrary degree $d \geq 1$.

In §5 connections between the obtained results and problems of nanoindentation are discussed. Problems of adhesive contact for conical and spherical indenters are studied analytically. For compressible materials, it is shown that the critical radius of the contact region and the corresponding critical load in the case of non-slipping contact are slightly less than the values obtained by the frictionless JKR approach. The obtained equations for general monomial punches are written in dimensionless form. It is shown that the dimensionless relations between the actual force, displacements and contact radius are the same regardless of the contact boundary conditions. The graphs of the relations are presented for some values of d, $1 \leq d \leq 2$.

In §6 some questions related to incompatibility of formulations of problems of adhesive contact and methods of solving these problems are discussed.

2 Preliminaries

2.1 Models of adhesive contact

Bradley (1932) was the first who considered attraction between two absolutely rigid spheres. Taking into account only one of components of the vdW forces, namely the London dispersion force, he calculated pointwise the attraction of each point of one sphere to another one. Assuming additivity of the London forces, he calculated the total force of adhesion between the spheres P_c. Although strictly speaking the London forces are not additive (Derjaguin *et al.* 1958), the assumption of additivity of the forces is usually considered as acceptable (Deryagin *et al.* 1978).

Derjaguin (1934) pointed out that to calculate adhesive interactions between solids, one needs to take into account their deformations. He presented the first attempt to consider the problem of adhesion between elastic spheres or between an elastic sphere and an elastic half-space. He assumed that the deformed shape of the sphere can be calculated by solving the Hertz contact problem and suggested to calculate the adhesive interaction using only attraction between points at the surfaces of the solids and by introduction of the work of adhesion (this is the so-called Derjaguin approximation). In fact, his approach can be formulated as follows: (i) it reduces the volume molecular attractions to surface interactions, and it does not employ the pairwise summation of the interactions between all elements of solids as did Bradley (1932); (ii)

the surface interactions are taken into account only between closest elements of the surfaces, and (iii) the interaction energy per unit area between small elements of curved surfaces is the same as this energy between two parallel infinite planar surfaces. As Greenwood (1997) noted the expression for adhesion between rigid spheres P_c that was obtained by Bradley (1932) after rather lengthy calculations, can be derived just in one line using the Derjaguin approximation. Indeed, using the Hertz approximation, one can replace a sphere of radius R by a paraboloid of revolution $z = f(r) = B_2 r^2$, where $B_2 = 1/(2R)$. Then applying the Derjaguin approximation, one obtains

$$P_c = \int_0^{2\pi} \int_0^{\infty} p_a[z(r)] r dr d\phi = 2\pi R \int_0^{\infty} p_a(z) \cdot dz = 2\pi R w, \qquad w = \int_0^{\infty} p_a(z) dz. \quad (2)$$

Here $p_a(z)$ is the adhesive force per unit area between flat surfaces separated by a distance z, and w is the work of adhesion that is *equal to the tensile force integrated through the distance necessary to pull the two surfaces completely apart* (Harkins 1919). Although, Derjaguin's assumption about the shape of deformed solids and some of his calculations were in error (he was not consistent in application of his approach), his approximation is very useful. Later Sperling (1964) discussed the adhesion between solid particles. He used both the Derjaguin approximation and Derjaguin's idea that the virtual work done by the external load is equal to the sum of the virtual change of the potential elastic energy and the virtual work that will be consumed by the increase of the surface attractions (see (21) in Derjaguin 1934).

Johnson (1958) made an attempt to solve the adhesive contact problem for spheres by adding two simple stress distributions, namely the Hertz stress field to a rigid flat-ended punch tensile stress distribution. Johnson argued that the infinite tension at the periphery of the contact would ensure that the spheres would peel apart when the compressive load was removed. Although Johnson's conclusion about impossibility of adhesive contact was not correct, his suggestion to superpose the stress fields is very fruitful.

According to Kendall (2001, pages 185-186), Johnson *et al.* (1971) applied Derjaguin's idea to equate *the work done by the surface attractions against the work of deformation in the elastic spheres*, to Johnson's stress superposition, and created the famous JKR theory of adhesive contact. Nowadays, two other models of adhesion of elastic spheres are also in common use: the DMT model (Derjaguin *et al.* 1975) and the Maugis (1992) theory (the JKR-DMT transition). A detailed description of the theories is given by Maugis (2000).

Thus, the adhesive forces can be taken into account by various methods, e.g. (i) by pointwise integration of the interaction forces between points of the bodies, whose interaction energy is proportional to ρ^{-6} of the distance ρ between the points; (ii) by using the Derjaguin approximation; (iii) by introducing an interaction potential between points on the surfaces, for example, a Lennard-Jones potential (see, for example, Muller *et al.* 1980, Borodich and Galanov 2004) or (iv)by using piecewise-constant approximations of these potentials (Maugis 1992, Johnson 1997, Goryacheva and Makhovskaya 2001, Zheng and Yu 2007).

Using the Galin (1946, 1961) expressions, an extension of the JKR adhesive frictionless contact problem to monomial punches was first obtained by Galanov (1993) (see also Galanov and Grigor'ev 1994). The same year Borodich gave another derivation of Galanov's solution, however it was published much later (Borodich and Galanov 2004, Borodich 2008). For the sake of completeness, this solution and also the further analysis of the problem will be presented below. Solutions to particular cases of the problem were independently presented by Carpick

et al. (1996) when d is an even integer, and Maugis (2000) for a conical punch ($d = 1$). In fact, the solution presented by Carpick *et al.* (1996) may be obtained by application of the JKR approach to the Shtaerman (1939) expressions (see also Eq. (5.20) by Johnson 1985), while the solution presented by Maugis (2000) may be obtained by application of the JKR approach to the Love (1939) result (see a discussion by Borodich *et al.* 2012a).

The adhesive contact problems with non-slipping boundary conditions were studied mainly in the two-dimensional case (see, e.g. Leng *et al.* 2000, Chen and Gao 2006a, Zhupanska 2012). However, there were also attempts to consider non-slipping adhesive contact between spheres (Yang *et al.* 2001, Chen and Gao 2006b, Waters and Guduru 2010, Guo *et al.* 2011). The non-slipping adhesive contact problems for a flat ended punch and a cone have been recently discussed by Borodich (2011) and Borodich *et al.* (2012a).

2.2 The JKR approach to adhesive contact

It is assumed that the state of the contact process can be completely characterized by the current value of an external parameter ($\mathcal{P}$), e.g., the force (P), the relative approach of the bodies (δ) or the contact radius (a). For a rigid punch, δ is the depth of indentation. The original JKR approach assumes that the problems are frictionless, i.e. the following conditions hold within the contact region G

$$\sigma_{31}(\mathbf{x};\mathcal{P}) = \sigma_{32}(\mathbf{x};\mathcal{P}) = 0, \quad \mathbf{x} \in G(\mathcal{P}) \subset \mathbb{R}^2. \tag{3}$$

The JKR approach is based on the use of a geometrically linear formulation of the contact problem, and a combination of both the Hertz contact problem for two elastic spheres and the Boussinesq relation for a flat ended cylindrical indenter. The Boussinesq relation for a flat ended cylindrical indenter of radius a is

$$P = \frac{2E}{1-\nu^2} a\delta \equiv 2E^* a\delta \tag{4}$$

If there were no surface forces of attraction, the radius of the contact area under a punch subjected to the external load P_0 would be a_0 and it could be found by solving the Hertz-type contact problem. However, in the presence of the forces of molecular adhesion, the equilibrium contact radius a_1 would be greater than a_0 under the same force P_0.

Johnson *et al.* (1971) suggested to consider the total energy of the contact system U_T as made up of three terms, the stored elastic energy U_E, the mechanical energy in the applied load U_M and the surface energy U_S. It is assumed that the contact system has come to its real state in two steps: (i) first it has got real contact radius a_1 and an apparent depth of indentation δ_1 under some apparent Hertz load P_1, then (ii) it is unloaded from P_1 to a real value of the external load P_0 keeping the contact radius a_1 constant (Fig. 1). The Boussinesq solution for contact between an elastic half-space and a flat punch of radius a_1 may be used on the latter step.

In this case, one can calculate U_E as the difference between the stored elastic energies $(U_E)_1$ and $(U_E)_2$ on loading and unloading branches respectively. Therefore,

$$(U_E)_1 = P_1\delta_1 - \int_0^{P_1} \delta dP. \tag{5}$$

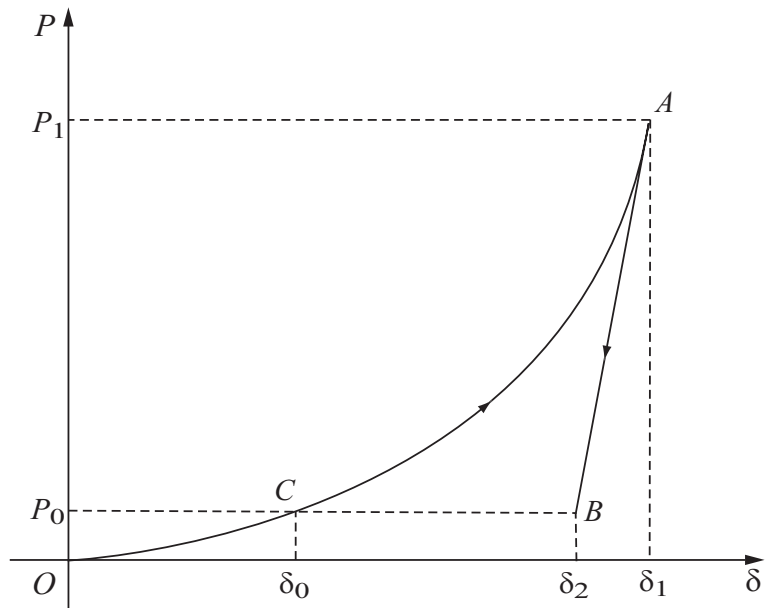

Figure 1: Loading diagram explaining the JKR model of adhesive contact. At branch OA the loading curve $P - \delta$ follows the Hertz-type $P \sim \delta^{(d+1)/d}$ contact relation, while the relation at the branch AB is linear.

Using the Boussinesq solution (4), we obtain for the unloading branch

$$(U_E)_2 = \int_{P_0}^{P_1} \frac{P}{2E^* a_1} dP = \frac{P_1^2 - P_0^2}{4E^* a_1}. \tag{6}$$

Thus, the stored elastic energy U_E is

$$U_E = (U_E)_1 - (U_E)_2. \tag{7}$$

The mechanical energy in the applied load

$$U_M = -P_0 \delta_2 = -P_0(\delta_1 - \Delta\delta) \tag{8}$$

where $\Delta\delta = \delta_1 - \delta_2$ is the change in the depth of penetration due to unloading.

Since only the surface adhesive interactions within the contact region are taken into account (one neglects the adhesive forces acting outside the contact region), the surface energy can be written as

$$U_S = -w\pi a_1^2. \tag{9}$$

The total energy U_T can be obtained by summation of (7), (8) and (9), i.e.

$$U_T = U_E + U_M + U_S. \tag{10}$$

It is assumed in the JKR model that the equilibrium at contact satisfies the equation

$$\frac{dU_T}{da_1} = 0, \quad \text{or} \quad \frac{dU_T}{dP_1} = 0. \tag{11}$$

The above was applied to the case of the initial distance between contacting solids being described by a paraboloid of revolution $z = r^2/(2R)$ (this is a very good approximation for a sphere). In the framework of the JKR theory the following relation between the external load P_0 acting on the spheres and the adhesive contact radius a_1 was obtained

$$P_0 = (4E^*/3R)a_1^3 - \sqrt{8\pi w E^* a_1^3} \tag{12}$$

where R is the effective radius of the spheres ($1/R = 1/R_1 + 1/R_2$).

3 Frictionless JKR adhesive contact theory

Let us generalize the JKR model of contact with molecular adhesion and consider the case of the distances between the contacting solids being described as a convex axisymmetric monomial functions (1) of arbitrary degrees d.

3.1 The JKR model for axisymmetric monomial punches.

The non-adhesive frictionless Hertz-type contact problem for punches described by (1) was given by Galin (Galin 1946, 1961). According to this solution the contact radius a_0 under the external load P_0 is given by

$$a_0 = \left(\frac{P_0}{C(d)E^*B_d}\right)^{1/(d+1)}, \quad C(d) = \frac{d^2}{d+1}2^{d-1}\frac{[\Gamma(d/2)]^2}{\Gamma(d)}, \tag{13}$$

where Γ is the Euler gamma function. The contact radius a_1 and depth of indentation δ_1 under some apparent Hertz load P_1, are given by

$$a_1 = \left(\frac{P_1}{C(d)E^*B_d}\right)^{1/(d+1)}, \qquad \delta_1 = \left[\frac{C(d)B_d}{(E^*)^d}\right]^{\frac{1}{d+1}}\left(\frac{d+1}{2d}\right)P_1^{d/(d+1)}. \tag{14}$$

Substituting (14) into (5) and (6) , we obtain

$$(U_E)_1 = \frac{d+1}{2(2d+1)}P_1^{(2d+1)/(d+1)}\left[\frac{C(d)B_d}{(E^*)^d}\right]^{\frac{1}{d+1}},$$

$$(U_E)_2 = \frac{[C(d)B_d]^{1/(d+1)}}{4(E^*)^{d/(d+1)}}\left(P_1^{(2d+1)/(d+1)} - P_0^2P_1^{-1/(d+1)}\right).$$

Using the above expressions and (7), and substituting (14) into (8) and (9), we obtain the following expressions for the components of energy

$$U_E = \frac{1}{4}\left[\frac{C(d)B_d}{(E^*)^d}\right]^{1/(d+1)}\left(\frac{1}{2d+1}P_1^{(2d+1)/(d+1)} + P_0^2P_1^{-1/(d+1)}\right), \tag{15}$$

$$U_M = -P_0\frac{d+1}{2d}\frac{[C(d)B_d]^{1/(d+1)}}{(E^*)^{d/(d+1)}}\left[\frac{P_1^{d/(d+1)}}{d+1} + \frac{P_0P_1^{-1/(d+1)}d}{d+1}\right]. \tag{16}$$

According to the Derjaguin assumptions, the adhesive interactions are reduced to the surface forces acting perpendicularly to the boundary of the half-space. The JKR theory considers only the adhesive forces acting within the contact region that is always a circle and the surface energy can be written as

$$U_S = -w\pi \left(\frac{P_1}{C(d)E^* B_d} \right)^{2/(d+1)}. \tag{17}$$

Thus, the total energy U_T can be obtained by summation of (15), (16) and (17)

$$U_T = \frac{1}{4} \left[\frac{C(d)B_d}{(E^*)^d} \right]^{\frac{1}{d+1}} \left[\frac{1}{2d+1} P_1^{\frac{2d+1}{d+1}} - P_0^2 P_1^{\frac{-1}{d+1}} - \frac{2}{d} P_0 P_1^{\frac{d}{d+1}} \right] - w\pi \left(\frac{P_1}{C(d)E^* B_d} \right)^{\frac{2}{d+1}}. \tag{18}$$

From (18) and (11), one may obtain

$$P_0^2 - 2P_0(C(d)B_d E^*)a_1^{d+1} + (C(d)B_d E^*)^2 a_1^{2(d+1)} - 8w\pi E^* a_1^3 = 0.$$

Solving this equation and taking the stable solution, one obtains an exact formula giving a relation between the real load P_0 and the real radius of the contact region a_1 (Borodich and Galanov 2004)

$$P_0 = P_1 - \sqrt{8\pi w E^* a_1^3} = C(d)B_d E^* a_1^{d+1} - \sqrt{8\pi w E^* a_1^3}. \tag{19}$$

The real displacement of the punch is $\delta_2 = (\delta_1 - \Delta\delta)$, i.e.

$$\delta_2 = B_d C(d) \frac{d+1}{2d} a_1^d - \left(\frac{2\pi w a_1}{E^*} \right)^{1/2}. \tag{20}$$

It is convenient to write the formula for the real displacement δ_2 in the case of frictionless boundary condition as

$$\delta_2 = \frac{B_d C(d)}{2d} a_1^d \left[1 + d\frac{P_0}{P_1} \right].$$

Zheng and Yu (2007) suggested to write the relations (19) and (20) using the Euler beta function $\mathbf{B}(x, y)$ of variables x and y. Indeed, the expression (13) for $C(d)$ can be written as

$$C(d) = \frac{d^2}{d+1} \mathbf{B}\left(\frac{d}{2}, \frac{1}{2} \right) = d\mathbf{B}\left(1 + \frac{d}{2}, \frac{1}{2} \right).$$

Then one can write

$$P_0 = dB_d \mathbf{B}\left(1 + \frac{d}{2}, \frac{1}{2} \right) E^* a_1^{d+1} - \sqrt{8\pi w E^* a_1^3}; \tag{21}$$

$$\delta_2 = \frac{dB_d}{2} \mathbf{B}\left(\frac{d}{2}, \frac{1}{2} \right) a_1^d - \left(\frac{2\pi w a_1}{E^*} \right)^{1/2}. \tag{22}$$

3.2 General expressions for the frictionless JKR adhesive contact

It is well known that the displacement components within an elastic half-space can be expressed in terms of harmonic functions. This is the so-called Papkovich-Neuber representation. Further if all tangential stresses on the boundary plane of the half-space are equal to zero then the boundary-value contact problem for a linear elastic half-space can be formulated in terms of just a single harmonic function Φ. In particular, for $x_3 = 0$ one has (see, e.g. Galin 1961, Borodich 1983)

$$u_3 = 2(1-\nu)\Phi(x_1, x_2, 0), \quad \sigma_{33} = \frac{E}{1+\nu}\frac{\partial \Phi(x_1, x_2, 0)}{\partial x_3}.$$

Using the harmonic function presented by Kochin (1940), Galin (1946) obtained expressions for the contacting force P, the depth of penetration δ and the pressure distribution under a convex, smooth in $\mathbb{R}^2 \setminus \{0\}$ punch of the arbitrary shape for axisymmetric frictionless Hertz-type contact problems for an elastic isotropic half-space. The expressions (13) and (14) are corollaries of this solution. The Galin solution can be expressed in various forms (see e.g. Sneddon 1965, Borodich and Keer 2004b).

The Papkovich-Neuber formalism and the Galin solution were used in application to mechanics of adhesive contact. In particular, Zheng and Yu (2007) and Zhou *et al.* (2011) considered the JKR and Maugis-Dugdale contact problems for power-law shaped solids. As Zheng and Yu (2007) noted, their solution to the JKR problem for power-law shaped solids coincides with the solution by Borodich and Galanov (2004). Indeed, if one denotes $Q = dB_d$ and $\Delta\gamma = w$ then the formulae (33) and (34) by Zheng and Yu (2007) in dimensional form coincide with (21) and (22). Naturally Zhou *et al.* (2011) solution for JKR theory coincides with solutions by Zheng and Yu (2007) and by the authors (Galanov 1993, Borodich and Galanov 2004). Although just the formula for the contact load was announced in the short abstract by Borodich and Galanov (2004), both formulae were presented by Galanov (1993). However, Galanov (1993) and Galanov and Grigor'ev (1994) used a different way for normalization of the variables.

Using the Galin (1946) solution in the representation by Borodich and Keer (2004b), one can show that for an arbitrary convex body of revolution $f(r)$, $f(0) = 0$, the JKR theory leads to the following expressions

$$P_1 = P_0 + \sqrt{8\pi w E^* a_1^3}, \quad \delta_2 = \delta_1 - \sqrt{\frac{2\pi w a_1}{E^*}}$$

or

$$P_0 = P_1 - \sqrt{8\pi w E^* a_1^3} = 2E^* \int_0^{a_1} \frac{r^2 f'(r) dr}{\sqrt{a_1^2 - r^2}} - \sqrt{8\pi w E^* a_1^3} \tag{23}$$

and

$$\delta_2 = \int_0^{a_1} \frac{f'(r)}{\sqrt{1 - r^2/a_1^2}} dr - \left(\frac{2\pi w a_1}{E^*}\right)^{1/2}. \tag{24}$$

However, we concentrated here on the solution to the JKR problems for monomial solids, while a discussion of problems for arbitrary solids of revolution is out the scope of the paper.

4 Non-slipping adhesive contact problems

The non-slipping contact problems were discussed by many researchers, see a discussion by Borodich and Keer (2004b), Zhupanska (2009) and Guo *et al.*(2011). The analysis of the non-slipping contact problems was performed first incrementally for a growth in the contact radius a (Mossakovskii 1954, 1963). However, Spence (1968) pointed out that for punches described by monomial functions (1), the solution can be obtained directly without application of the incremental techniques. Borodich and Keer (2004a) noted that it is convenient to present the results obtained in non-slipping formulation of the contact problems using the following parameter

$$C_{NS} = \frac{(1-\nu)\ln(3-4\nu)}{1-2\nu}.$$

In the case of non-compressible materials, i.e. for $\nu = 0.5$, one obtains $\lim\limits_{\nu\to 0.5} C_{NS} = 1$.

4.1 Non-slipping Boussinesq–Kendall adhesive problem

Consider an axisymmetric flat ended punch of radius a_1 that is vertically pressed into an elastic half-space. The frictionless case of this problem was considered by Boussinesq (see, e.g. Galin 1961), non-slipping contact was studied by Mossakovskii (1954), and frictionless contact with molecular adhesion was studied by Kendall (1971). Let us consider the problem with non-slipping boundary conditions and taking into account molecular adhesion. Then the arguments by Kendall (1971) have to be slightly modified.

In this problem the boundary conditions for the radial displacements within the contact region $0 \le r \le a_1$ have the following form

$$u_r(r) = 0, \quad 0 \le r \le a_1. \tag{25}$$

The elastic material deforms according to the Mossakovskii (1954) equation

$$\delta = \frac{P}{2E^* C_{NS} a_1}. \tag{26}$$

As one can see from (4), the equation has the same form as the one for frictionless case (the Boussinesq solution) with C_{NS} equal to unity.

The surface energy is given as above by (9). Using (26), one obtains that the stored elastic energy U_E and the mechanical energy of the applied load U_M are respectively

$$U_E = \int P d\delta = \frac{P^2}{4E^* C_{NS} a_1} + A, \quad U_M = -P\delta + B = -\frac{P^2}{2E^* C_{NS} a_1} + B \tag{27}$$

where A and B are arbitrary constants.

The total energy U_T can be obtained by summation of all components given by (9) and (27)

$$U_T = -w\pi a_1^2 - \frac{P^2}{4E^* C_{NS} a_1} + A + B. \tag{28}$$

From the equilibrium equation (11), one has

$$\frac{dU_T}{da_1} = 0 = -2w\pi a_1 + \frac{P_c^2}{4E^* C_{NS} a_1^2} \tag{29}$$

and, hence, one may obtain the adherence force (the pull-off force) of a flat ended circular punch of radius a_1 at the non-slipping boundary conditions

$$P_c = \sqrt{8\pi w E^* C_{NS} a_1^3}. \tag{30}$$

Thus, one can see from (30) that the adherence force is proportional neither to the energy of adhesion nor to the area of the contact. Maugis (2000) came to the same conclusion about the frictionless Boussinesq–Kendall problem.

Comment. The indefinite integrals were used in (27) to calculate the energies U_E and U_M because we are trying in the present work to follow the original Kendall (1971) approach as closely as possible. Of course, one could obtain the same result as above if the expressions were written as

$$U_E = \int_0^{P_0} P d\delta = \frac{P_0^2}{4E^* C_{NS} a_1}, \quad U_M = -P_0 \delta = -\frac{P_0^2}{2E^* C_{NS} a_1}.$$

4.2 Non-slipping JKR contact problem for monomial punches

In Hertz-type contact problems, the load P can be taken as the external parameter of the contact problem. If the parameter P is gradually increased then the surface displacements $u_r(r, 0, P)$ and $u_z(r, 0, P)$ will be functions of both r and the parameter of the problem P.

4.2.1 Formulations of non-slipping non-adhesive contact problems.

Axisymmetric mixed boundary value contact problems can be formulated in various ways: (i) as a general formulation; and (ii) as a Hertz-type contact problem the non-slipping boundary conditions.

In the general formulation (see, e.g. Popov 1973, Guo *et al.* 2011), it is assumed that in the system subjected to a normal contact force P, the displacements $u_r(r, 0, P)$ and $u_z(r, 0, P)$ are known within the contact region, and the solids are not loaded outside the contact region, i.e.

$$u_r(r, 0, P) = s(r), \quad u_z(r, 0, P) = g(r), \quad \text{for} \quad r \le a; \tag{31}$$

$$\sigma_{rz}(r, 0, P) = \sigma_{zz}(r, 0, P) = 0, \quad \text{for} \quad r > a; \tag{32}$$

where $s(r)$ and $g(r)$ are known functions of the radial and normal displacements, respectively. The condition for the given radial displacements $u_r(r, 0, P)$ can be reformulated as the condition for mismatch strain distributions $\epsilon(r)$ between the contact surfaces (Guo *et al.* 2011).

In the framework of the axisymmetric Hertz-type contact problem, the non-slipping boundary conditions mean that once the point of the surface contacts with the indenter, its radial displacement does not change further with P. Hence, in addition to the boundary condition (32), instead of the boundary conditions (3), one can write the condition that the values of the radial displacements within the contact region do not change with augmentation of the external parameter of the problem

$$\frac{\partial u_r}{\partial P}(r, 0, P) = 0, \quad dP > 0. \tag{33}$$

In this formulation, the normal and radial displacements are consistent with the punch shape and, therefore $g(r) = \delta - f(r)$ and the radial displacements $u_r(r, 0, P)$ cannot be arbitrary.

4.2.2 Self-similarity of the $3D$ Hertz-type contact problems.

The approach by Spence (1968) was based on self-similarity of the axisymmetric Hertz-type contact problem. In fact, these contact problems are self-similar even in three-dimensional ($3D$) case (see, e.g. Galanov 1981, Borodich 1983, 1989). The self-similarity of the $3D$ Hertz-type contact problem holds not only in the frictionless problems but also under non-slipping or frictional conditions (Borodich 1993, 2008). In general case, the similarity in the Hertz-type contact problem can be found for solids whose operators of constitutive relations are homogeneous functions of degree κ with respect to the components of the strain tensor ϵ_{ij}, i.e., for each positive k one has

$$F(k\epsilon_{ij}) = k^{\kappa} F(\epsilon_{ij}), \tag{34}$$

in which F is the operator of constitutive relations. The material behaviour of the medium may be anisotropic or isotropic, depending of the form of the operator F. For these materials the following theorem is valid (see, e.g. Borodich 2008).

Theorem. Let the shape of a blunt punch be determined by a positive, homogeneous function of degree $d > 0$. In addition let the operator of the constitutive relations F satisfy (34).

Assume further that for an initial value of the compressing force P_I the solution of the Hertz-type contact problem with frictionless (3) or the conditions (33) within the contact region is given by the functions $\sigma_{ij}(\mathbf{x}, P_I)$, $\epsilon_{ij}(\mathbf{x}, P_I)$, $u_i(\mathbf{x}, P_I)$, quantity $\delta(P_I)$ and the contact region $G(P_I)$.

Then, for the any positive force P the solution of the contact boundary-value problem will be given by

$$\begin{aligned} u_i(\mathbf{x}, P) &= \lambda^{-d} u_i(\lambda\mathbf{x}, P_I), \\ \epsilon_{ij}(\mathbf{x}, P) &= \lambda^{(1-d)} \epsilon_{ij}(\lambda\mathbf{x}, P_I), \\ \sigma_{ij}(\mathbf{x}, P) &= \lambda^{\kappa(1-d)} \sigma_{ij}(\lambda\mathbf{x}, P_I) \\ \delta(P) &= \lambda^{-d} \delta(P_I), \end{aligned}$$

where $\lambda = (P_I/P)^{1/[2+\kappa(d-1)]}$, i.e., $P_I = \lambda^{[2+\kappa(d-1)]} P$ and the contact region $G(P_I)$ changes according to the transformation of homothety, i.e.,

$$[(x_1, x_2) \in G(P)] \Longleftrightarrow [(\lambda x_1, \lambda x_2) \in G(P_I)].$$

It follows from Theorem in axisymmetric case for linear materials ($\kappa = 1$) and non-slipping conditions (33), i.e. for the Mossakovskii-Spence type contact problems, that the following rescaling formulae are valid

$$u_r(r, 0, P) = \lambda^{-d} u_r(\lambda r, 0, P_I), \quad \delta(P) = \lambda^{-d} \delta(P_I),$$

$$[(r \in G(P)] \Longleftrightarrow [\lambda r \in G(P_I)].$$

Let P_* be such a value of the external compressing force that $a(P_*) = r_*$ then for $P \geq P_*$,

$$[r = a(P)] \Longleftrightarrow [\lambda r = \lambda a = r_* = a(P_*)], \quad \lambda = (P_*/P)^{1/(d+1)} = a_*/a.$$

and on the boundary plane ($z = 0$) one has

$$u_r(r, 0, P) = (r/a_*)^d u_r(a_*, 0, P_*) = C_0 r^d, \quad C_0 = u_r(a_*, 0, P_*) a_*^{-d}.$$

Hence, if it is assumed that there are non-zero radial displacements in the self-similar Mossakovskii-Spence type contact problems then $s(r) = C_0 r^d$ within the contact region. These conditions were considered by Spence (1968) and for $d = 2$ by Zhupanska (2009).

Thus, in the framework of the Mossakovskii-Spence formulation, the radial displacements $u_r(r, 0, P)$ arise initially outside the contact region due to bounded contact stresses (see Fig. 1 in Spence 1968). Then the radial displacements can be treated as the *frozen-in* displacements (Zhupanska 2009) because the constant C_0 ensures that the radial strain at any given point of the contact zone does not change when the size of the contact region increases due to increase of the external parameter of the contact problem.

4.2.3 The total energy of the system with non-slipping conditions.

As it has been explaining above, it is attempted here to follow the original JKR approach as closely as possible avoiding the resolution of interfacial tractions. However, one needs to provide the clear rationale to the extension of the frictionless JKR approach to the case of non-slipping contact conditions. The work W of the external forces that include the surface tractions, the body forces and the applied load, can be written as

$$W = \int_S T_i u_i dS + \int_V X_i u_i dV + U_M$$

and according to Clapeyron's theorem, it is stored in the linear-elastic body in the form of the strain energy (see, e.g. Lurie 2005). Here T_i are the surface tractions, and X_i are the body forces and S and V are the surface and the body volume, respectively. The body forces in the problem under consideration are the adhesive forces. Because in the no-slipping case, both the normal and radial tractions exist over the contact region, formally the work of radial surface tractions $(U_E)_3$ should be added to the expression for the stored elastic energy.

$$\int_S T_i u_i dS = (U_E)_1 - (U_E)_2 + (U_E)_3, \quad (U_E)_3 = \int_S T_r u_r dS.$$

Although the contact problems with an unknown contact region are non-linear, one can use the superposition of two contact solutions for linear elastic materials if the contact region $0 \leq r \leq a_1$ is fixed. Hence, the tangential stresses in no-slipping contact problem can be obtained as the difference between the tangential stress field $\tau_M(r)$ of the Mossakovskii (or Mossakovskii-Spence) type problem (this is the Hertz-type contact problem with non-slipping boundary conditions) for the punch loaded by P_1 and the tangential stress field $\tau_B(r)$ of the Boussinesq-Mossakovskii contact problem after the unloading from P_1 to P_0, i.e. $\tau(r) = \tau_M(r) - \tau_B(r)$. Here the subscripts M and B denote variables associated with the Mossakovskii-Spence type and the Boussinesq-Mossakovskii contact problems respectively. Due to the conditions (33), the differentials of the work done by the tangential tractions during increasing P from 0 to P_1 and then decreasing from P_1 to P_0 are zero and, hence, the work of the tangential surface tractions $(U_E)_3 = 0$.

As it has been mentioned, the work of the external body forces in the problem under consideration, is the work done by the forces of adhesion. In this paper, it is assumed that the Derjaguin approximation is valid, i.e. the adhesive interactions are reduced to the surface forces acting perpendicularly to the boundary of the half-space, and therefore, the work of the

surface adhesive forces on radial displacements is equal to zero. Thus, in the framework of the above assumptions the JKR expression for the total energy is

$$U_T = (U_E)_1 - (U_E)_2 + U_M + U_S \tag{35}$$

and the problem is reduced to the classic JKR approach, however, the expressions for the components in (35) should be found from corresponding problems using the Mossakovskii-Spence formulation.

Comment. There are other approaches to problems of adhesive contact where the above assumptions are not accepted and the work of the surface adhesive forces on radial displacements is not equal to zero. These approaches (the mode-mixity approaches) will be considered below in Discussion.

4.2.4 The JKR approach to non-slipping contact.

Let us consider as above the axisymmetric monomial punches (1) in the case of non-slipping contact conditions. If there were no surface forces then the contact radius a_0 of a punch under the external load P_0 could be found from the solution given by Borodich and Keer (2004b)

$$a_0 = \left(\frac{I^*(d)P_0 d}{E^* C_{NS} B_d C(d)}\right)^{1/(d+1)}, \quad I^*(d) = \int_0^1 t^{d-1} \cos\left[\frac{\ln(3-4\nu)}{2\pi} \ln\frac{1-t}{1+t}\right] dt. \tag{36}$$

The non-adhesive contact radius a_1 and depth of indentation δ_1 under some apparent load P_1, are given by

$$a_1 = \left(\frac{I^*(d)P_1 d}{E^* C_{NS} B_d C(d)}\right)^{1/(d+1)}, \qquad \delta_1 = \left[\frac{B_d C(d)}{d I^*(d)(E^* C_{NS})^d}\right]^{\frac{1}{d+1}} \left(\frac{d+1}{2d}\right) P_1^{d/(d+1)}. \tag{37}$$

In the case of $\nu = 0.5$, one has $I^*(d) = 1/d$ and $C_{NS} = 1$. Hence, for incompressible materials, the Borodich-Keer formulae (36) and (37) are identical to the corresponding formulae of the Galin solution (13) and (14).

Applying the above described assumptions and the JKR approach, one can obtain

$$(U_E)_1 = \left[\frac{B_d C(d)}{d I^*(d)(E^* C_{NS})^d}\right]^{\frac{1}{d+1}} \left(\frac{d+1}{2(2d+1)}\right) P_1^{(2d+1)/(d+1)}.$$

Using the Mossakovskii solution (26) and (37), one obtains for the unloading branch

$$(U_E)_2 = \frac{P_1^2 - P_0^2}{4E^* C_{NS} a_1} = \frac{1}{4}\left(\frac{B_d C(d)}{d I^*(d)(E^* C_{NS})^d}\right)^{1/(d+1)} \left(P_1^{(2d+1)/(d+1)} - P_0^2 P_1^{-1/(d+1)}\right).$$

Hence, the stored elastic energy U_E is

$$U_E = \frac{1}{4}\left(\frac{B_d C(d)}{d I^*(d)(E^* C_{NS})^d}\right)^{1/(d+1)} \left(\frac{1}{2d+1} P_1^{(2d+1)/(d+1)} + P_0^2 P_1^{-1/(d+1)}\right). \tag{38}$$

The mechanical energy in the applied load U_M can be found using (8). Taking into account that $\Delta\delta = (P_1 - P_0)/(2C_{NS}E^* a_1)$, one obtains

$$U_M = -P_0 \frac{1}{2d} \left(\frac{B_d C(d)}{dI^*(d)(E^* C_{NS})^d} \right)^{1/(d+1)} \left[P_1^{d/(d+1)} + P_0 P_1^{-1/(d+1)} d \right]. \tag{39}$$

The surface energy U_S is

$$U_S = -w\pi \left(\frac{P_1 d I^*(d)}{E^* C_{NS} C(d) B_d} \right)^{2/(d+1)}. \tag{40}$$

Thus, the total energy U_T is

$$U_T = \frac{1}{4} \left(\frac{B_d C(d)}{dI^*(d)(E^* C_{NS})^d} \right)^{\frac{1}{d+1}} \left[\frac{P_1^{\frac{2d+1}{d+1}}}{2d+1} - P_0^2 P_1^{\frac{-1}{d+1}} - \frac{2}{d} P_0 P_1^{\frac{d}{d+1}} \right] - w\pi \left(\frac{P_1 d I^*(d)}{E^* C_{NS} C(d) B_d} \right)^{\frac{2}{d+1}}. \tag{41}$$

Using (11), one may obtain from (41)

$$P_0^2 - 2P_0 \frac{E^* C_{NS} C(d) B_d}{dI^*(d)} a_1^{d+1} + \left[\frac{E^* C_{NS} C(d) B_d}{dI^*(d)} \right]^2 a_1^{2(d+1)} - 8\pi w E^* C_{NS} a_1^3 = 0.$$

Solving this equation and taking the stable solution, one obtains an exact formula giving relation between the load P and the radius of the contact region a

$$P_0 = P_1 - \sqrt{8\pi w E^* C_{NS} a_1^3} = \frac{E^* C_{NS} C(d) B_d}{dI^*(d)} a_1^{d+1} - \sqrt{8\pi w E^* C_{NS} a_1^3}. \tag{42}$$

As in the above frictionless problem, the real displacement of the punch is $\delta_2 = (\delta_1 - \Delta\delta)$

$$\delta_2 = B_d C(d) \frac{d+1}{2d} \frac{1}{dI^*(d)} a_1^d - \left(\frac{2\pi w a_1}{E^* C_{NS}} \right)^{1/2}. \tag{43}$$

It is convenient to write the formula for the real displacement δ_2 in the case of non-slipping boundary condition as

$$\delta_2 = \frac{B_d C(d)}{2d^2 I^*(d)} a_1^d \left[1 + d \frac{P_0}{P_1} \right].$$

5 Adhesive indentation by non-ideal shaped indenters

The depth-sensing indentation (DSI) is the continuously monitoring of the $P - \delta$ diagram where P is the applied load and δ is the displacement (the approach of the distant points of the indenter and the sample). DSI techniques are especially important when mechanical properties of materials are studied using very small volumes of materials. The $P - \delta$ diagrams for material characterization are so important that these diagrams are often considered in materials science community as "finger-prints" of materials. The DSI analysis of materials

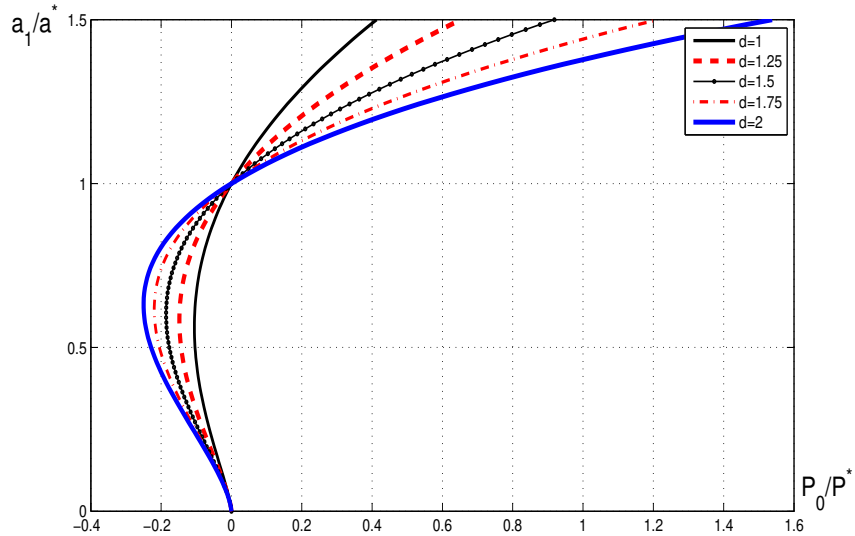

Figure 2: The JKR dimensionless $a_1/a^* - P_0/P^*$ relation for monomial indenters.

was proposed in the pioneering paper by Kalei (1968) where he noted that adhesion to a solid surface (substrate) may affect the measurements obtained by DSI.

It is usually assumed that the indenter is a sharp pyramid or a cone. However, the nominally sharp indenters are in fact not ideal. Let us apply the results obtained above to problems of nanoindentation when the indenter shape near the tip has some deviation from its nominal shape. It will be assumed further that the indenter shape function can be approximated by a monomial function of radius.

5.1 Frictionless adhesive indentation

It follows from (19) that the radius a_1 of the contact region at $P_0 = 0$ is

$$a_1(0) = \left[\frac{8\pi w}{E^* C^2(d) B_d^2}\right]^{1/(2d-1)}.$$

This value can be used as a characteristic size of the contact region in order to write dimensionless parameters. As it is known, the choice of the characteristic parameters of the adhesive contact problem is rather arbitrary (Borodich and Galanov 2008). For example, the characteristic parameters of the classic JKR model ($d = 2$) can be taken as (Johnson 1997)

$$a_J^* = \left(\frac{3\pi w R^2}{4E^*}\right)^{1/3}, \quad P_J^* = \pi w R, \quad \delta_J^* = \left(\frac{9\pi^2 w^2 R}{16(E^*)^2}\right)^{1/3}. \tag{44}$$

or as the following ones (Maugis 2000, Johnson and Sridhar 2001)

$$a_M^* = \left(\frac{9\pi w R^2}{4E^*}\right)^{1/3}, \quad P_M^* = 3\pi w R, \quad \delta_M^* = \left(\frac{9\pi^2 w^2 R}{16(E^*)^2}\right)^{1/3}. \tag{45}$$

Let us write the characteristic parameters of the adhesive contact problems as

$$a^* = a_1(0), \quad P^* = \left\{ \frac{(8\pi w)^{d+1}(E^*)^{d-2}}{[C(d)B_d]^3} \right\}^{\frac{1}{2d-1}}, \quad \delta^* = \left[\frac{2^{d+1}}{C(d)B_d} \left(\frac{\pi w}{E^*} \right)^d \right]^{\frac{1}{2d-1}}. \tag{46}$$

Then (19) and (20) have the following form

$$P_0/P^* = (a_1/a^*)^{d+1} - (a_1/a^*)^{3/2} \tag{47}$$

and

$$\frac{\delta_2}{\delta^*} = \frac{d+1}{d} \left(\frac{a_1}{a^*} \right)^d - \left(\frac{a_1}{a^*} \right)^{1/2}. \tag{48}$$

5.2 Non-slipping adhesive indentation

In this case, the radius a_1 of the contact region at $P_0 = 0$ can be obtained from (42)

$$a_1(0) = \left[\sqrt{\frac{8\pi w}{E^* C_{NS}}} \frac{dI^*(d)}{C(d)B_d} \right]^{\frac{2}{2d-1}}.$$

Let us write the characteristic parameters of the non-slipping adhesive contact problems as

$$a^* = a_1(0), \quad P^* = \left\{ \frac{(8\pi w)^{d+1}(E^* C_{NS})^{d-2}}{[C(d)B_d/(dI^*(d))]^3} \right\}^{\frac{1}{2d-1}}, \quad \delta^* = \left[\frac{2^{d+1} dI^*(d)}{C(d)B_d} \left(\frac{\pi w}{E^* C_{NS}} \right)^d \right]^{\frac{1}{2d-1}}. \tag{49}$$

Then (42) and (43) will have the same dimensionless form as the frictionless case. Hence, the equations (47) and (48) are also valid for the non-slipping adhesive JKR contact case.

5.3 Dimensionless relations for adhesive indentation

Let us denote $\bar{P} = P_0/P^*$, $\bar{a} = a_1/a^*$ and $\bar{\delta} = \delta_2/\delta^*$. Then (47) and (48) can be written as the following dimensionless relations

$$\bar{P} = \bar{a}^{d+1} - \bar{a}^{3/2} \tag{50}$$

and

$$\bar{\delta} = \frac{d+1}{d} (\bar{a})^d - (\bar{a})^{1/2} \tag{51}$$

that are valid for arbitrary axisymmetric monomial punch of degree $d \geq 1$ regardless of the contact boundary conditions.

The graphs of the dimensionless relations (50) and (51) for several values of degree d of the indenter shape monoms are shown respectively in Figure 2 and Figure 3.

The instability point of a $\bar{P} - \bar{\delta}$ curve is at the point where $d\bar{P}/d\bar{\delta} = 0$. Taken into account that $d\bar{P}/d\bar{\delta} = d\bar{P}/d\bar{a} \cdot d\bar{a}/d\bar{\delta}$, one obtains from (50) at the instability point

$$d\bar{P}/d\bar{a} = (d+1)\bar{a}^d - (3/2)\bar{a}^{1/2} = 0.$$

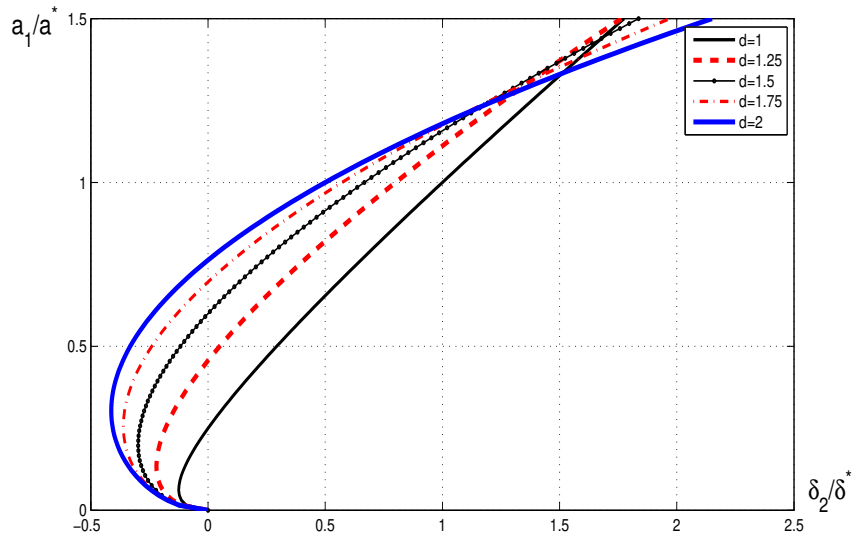

Figure 3: The JKR dimensionless $a_1/a^* - \delta_2/\delta^*$ relation for monomial indenters.

Solving this equation, one obtains for a dimensionless critical contact radius

$$\bar{a}_c = \left[\frac{3}{2(d+1)}\right]^{\frac{2}{2d-1}} . \tag{52}$$

Substituting this expression into (50), one obtains the explicit dimensionless expression for the critical load $\bar{P}_c$ (the adherence force at fixed load)

$$\bar{P}_c = \left[\frac{3}{2(d+1)}\right]^{\frac{2(d+1)}{2d-1}} - \left[\frac{3}{2(d+1)}\right]^{\frac{3}{2d-1}} . \tag{53}$$

One can compare the critical loads for frictionless (P_c^{FL}) and non-slipping (P_c^{NS}) cases. Taking into account that $P_c^{FL} = \bar{P}_c(P^*)^{FL}$ where $(P^*)^{FL}$ is given by (46) and $P_c^{NS} = \bar{P}_c(P^*)^{NS}$ where $(P^*)^{NS}$ is given by (49), one obtains

$$\frac{P_c^{NS}}{P_c^{FL}} = \frac{(P^*)^{NS}}{(P^*)^{FL}} = \left\{C_{NS}^{d-2}[dI^*(d)]^3\right\}^{\frac{1}{2d-1}} . \tag{54}$$

For frictionless (a_c^{FL}) and non-slipping (a_c^{NS}) cases, one has

$$\frac{a_c^{NS}}{a_c^{FL}} = \frac{(a^*)^{NS}}{(a^*)^{FL}} = \left[\frac{dI^*(d)}{\sqrt{C_{NS}}}\right]^{\frac{2}{2d-1}} . \tag{55}$$

5.4 Nanoindenters of monomial shape

It is known (Borodich *et al.* 2003, Borodich 2011) that at shallow depth, the indenter blunt shapes may be often described by homogeneous functions h_d of degree d with $1 \leq d \leq 2$. The

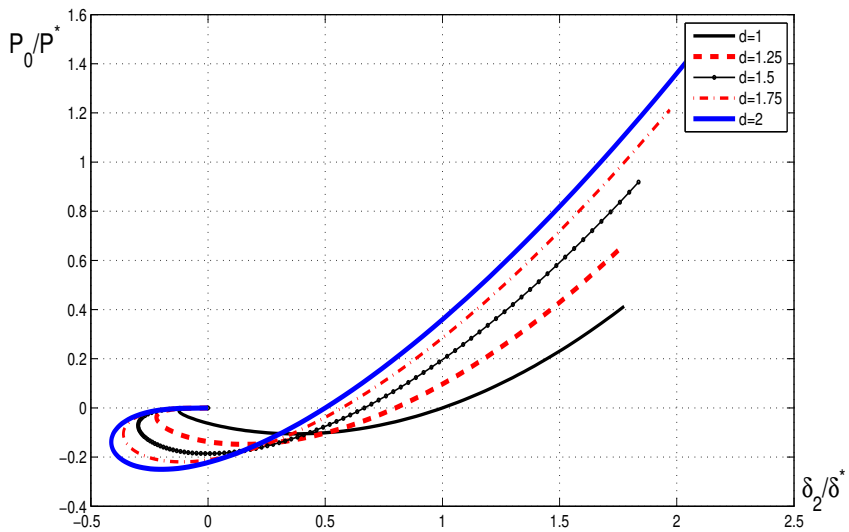

Figure 4: The universal JKR dimensionless $P_0/P^* - \delta_2/\delta^*$ relation for monomial indenters.

graphs of the dimensionless $\bar{P}$ -$\bar{\delta}$ relation for monomial indenters whose degree d are within the $1 \leq d \leq 2$ range are shown in Figure 4. The limiting cases of this range are conical and spherical indenters. Using the above general solution for monomial punches, one can consider analytically these limiting cases.

Spherical punch. For a sphere of radius R, one has $d = 2$, $f(r) = B_2 r^2$, $B_2 = 1/(2R)$ and $C(2) = 8/3$. Comparing (46) with (44) and (45), one obtains in the frictionless case $a^* = 6^{1/3} a_J^* = 2^{1/3} a_M^*$, $P^* = 6P_J^* = 2P_M^*$ and $\delta^* = 2^{5/3} 3^{-1/3} \delta_J^*$. The expression (19) coincides with the classic JKR formula (12). Further one has $\bar{P}_c = -1/4$. In dimensional form one has $P^* = 6\pi R w$ and obtains respectively the classic JKR value $P_c = -(1/4)P^* = -(3/2)\pi R w$. The expression $(\bar{a}_c)_M = 2^{-1/3}$ (Johnson and Sridhar 2001) agrees with $\bar{a}_c = 2^{-2/3}$ obtained from (52). Using (54) for $d = 2$, the non-slipping case can be obtained from the above one $P_c^{NS} = 2I^*(2)P_c^{FL}$.

Spence (1968) suggested to use a decomposition of the integral $I^*(2)$ into a series. Using this decomposition, one obtains

$$\frac{P_c^{NS}}{P_c^{FL}} = 1 - 0.6931 \left(\frac{\ln(3-4\nu)}{\pi} \right)^2 + 0.2254 \left(\frac{\ln(3-4\nu)}{\pi} \right)^4 + \dots .$$

For $\nu = 0$, one has

$$\frac{P_c^{NS}}{P_c^{FL}} = 1 - 0.6931 \left(\frac{\ln 3}{\pi} \right)^2 + 0.2254 \left(\frac{\ln 3}{\pi} \right)^4 + \cdots \approx 0.9186.$$

Hence, the frictionless JKR model slightly overestimates the adherence force for a sphere.

Conical punch. In the case of a cone of semi-vertical angle $\pi/2 - \alpha$, $d = 1$, $f(r) = B_1 r$, $C(1) = \pi/2$, and $B_1 = \tan\alpha$. For a linearized treatment to be possible, α must be small compared with 1 and $\tan\alpha = B_1 \approx \alpha$. It follows from (52) and (53) that the dimensionless

critical contact radius and the adherence force at fixed load are respectively $\bar{a}_c = 9/16$ and $\bar{P}_c = -27/256$.

One can get the dimensional form in the frictionless case from (46). This gives the following values

$$P^* = 512w^2/(\pi E^* B_1^3) \quad \text{and} \quad P_c = -54w^2/(\pi E^* B_1^3). \tag{56}$$

The contact radius and displacement under zero load are respectively

$$a_1(0) = a^* = 32w/(\pi E^* B_1^2) \quad \text{and} \quad \delta_2(0) = \delta^* = 8w/(E^* B_1).$$

These expressions coincide with the formulae presented by Maugis except that the formulae (4.253) for $\delta_2(0)$ in the book by Maugis (2000) has a wrong coefficient 24 (see also a discussion by Borodich *et al.* 2012a).

In the non-slipping case, one can get the dimensional form from (49). This gives the following values

$$P^* = 512w^2[I^*(1)]^3/(\pi E^* C_{NS} B_1^3) \quad \text{and} \quad P_c = -54w^2[I^*(1)]^3/(\pi E^* C_{NS} B_1^3).$$

Because in this case the parameter $I^*(d)$ can be calculated exactly (Spence 1968, Borodich and Keer 2004a,b)

$$I^*(1) = \frac{\ln(3-4\nu)\sqrt{3-4\nu}}{2(1-2\nu)},$$

one obtains

$$P_c = -\frac{27w^2}{4\pi E^* B_1^3} \frac{(3-4\nu)^{3/2}\ln^2(3-4\nu)}{(1-\nu)(1-2\nu)^2}. \tag{57}$$

The contact radius and displacement under zero load are respectively

$$a^* = a_1(0) = 32w[I^*(1)]^2/(\pi E^* C_{NS} B_1^2), \quad \delta_2(0) = \delta^* = 8wI^*(1)/(E^* C_{NS} B_1).$$

Using (54), one obtains in the case $d = 1$

$$\frac{P_c^{NS}}{P_c^{FL}} = \frac{[I^*(1)]^3}{C_{NS}} = \frac{(3-4\nu)^{3/2}\ln^2(3-4\nu)}{8(1-\nu)(1-2\nu)^2}.$$

For $\nu = 0.5$, one has

$$\lim_{\nu\to 0.5} \frac{(3-4\nu)^{3/2}\ln^2(3-4\nu)}{8(1-\nu)(1-2\nu)^2} = 1$$

and hence, as it is expected, (57) coincides with (56). Correspondingly, for $\nu = 0$, one has

$$\frac{P_c^{NS}}{P_c^{FL}} = \frac{3^{3/2}\ln^2 3}{8} \approx 0.784.$$

Using (55), one obtains in the case $d = 1$

$$\frac{a_c^{NS}}{a_c^{FL}} = \frac{[I^*(1)]^2}{C_{NS}} = \frac{(3-4\nu)\ln(3-4\nu)}{4(1-\nu)(1-2\nu)}.$$

For $\nu = 0$, one has

$$\frac{a_c^{NS}}{a_c^{FL}} = \frac{3\ln 3}{4} \approx 0.824.$$

Thus, for compressible materials, the critical radius of the contact region and the corresponding critical load in the case of non-slipping contact are less than the predictions by the frictionless JKR approach.

6 Discussion

Some issues related to problems under consideration should be discussed.

6.1 Incompatibility of formulations of the Hertz-type contact problems.

One needs to be aware that the Hertz formulation of the contact problems leads to incompatibility of displacement fields. If problem is frictionless then the radial displacements are neglected and the points of the surface formally penetrate the punch surface. As an attempt to reduce the degree of this incompatibility, Galanov (1983) considered a refined formulation of the Hertz-type contact problem when radial displacements were taken into account.

The same type of incompatibility exists in the non-slipping contact problem (see, e.g. Spence 1968, Zhupanska 2009). The boundary conditions of a Hertz-type contact problem in the Mossakovskii-Spence formulation prescribe *a priori* the radial and normal displacement distributions within the contact region

$$u_r(r,0,P) = C_0 r^d \quad \text{and} \quad u_z(r,0,P) = \delta - f(r) = \delta - B_d r^d, \quad \text{for} \quad r \le a. \tag{58}$$

These conditions may be treated as a parametric representation of the indent surface after contact of the punch and the half-space. One can show that if $C_0 < 0$ then the punch cannot be put in the indent because it is too small; and if $C_0 > 0$ then the indent is too large and there is no contact. Hence, the correct solution of the contact problem with boundary conditions (58) gives such stress fields that being applied to the boundary of an elastic half-space, produce the above mentioned incompatibility.

Further one has to realize that the formulation of the contact problem with non-slipping boundary conditions may lead to stress fields having oscillations near the edge of contact region. Indeed, as it was shown by Abramov (1937) (see also, Muskhelishvili 1963, Rvachev and Protsenko, 1977) for the two-dimensional problem of a non-slipping contact between a flat ended punch of width $2l$ loaded by the force P, that the normal p and tangential τ stress distributions are

$$p(x) = \frac{P}{\pi\sqrt{l^2 - x^2}} \frac{4\nu - 2}{\sqrt{3 - 4\nu}} \cos\left[\frac{\ln(4\nu - 2)}{2\pi} \frac{l + x}{l - x}\right],$$
$$\tau(x) = \frac{P}{\pi\sqrt{l^2 - x^2}} \frac{4 - 4\nu}{\sqrt{3 - 4\nu}} \sin\left[\frac{\ln(4\nu - 2)}{2\pi} \frac{l + x}{l - x}\right]$$

Hence, when the coordinate x approaches the edges of the contact zone, both the normal and tangential stresses change their signs infinitely many times and there are tensile normal stresses within the contact region. In the axisymmetric contact problems, the displacement incompatibility is the same type. One can see from a complete analytical solution for a non-slipping contact problem between a flat circular centrally loaded punch and an isotropic elastic half-space presented by Fabrikant (1991) that the field of radial displacements has a jump near the edge of the contact region (see Fig. 5.1.1. of the book) and after the deformation the material points at the edge have to penetrate the punch. Evidently, this has no physical meaning. Discussing the Abramov contact problem, Muskhelishvili (1963) noted *for all real*

solids $1 < 3-4\nu < 3$, and hence the first value $|x|$ such that $p(x) = 0$ is $x = \pm 0.9997l$. Because such oscillations have no physical meaning, Rvachev and Protsenko (1977) referred to the corresponding strains as fictitious strains. They advised not to attach too much importance to the investigation of the behaviour of the solutions within very small regions at singular points, where the solution may be devoid of any physical meaning (see also a recent discussion by Guo *et al.* 2011).

There are two sources of the incompatibility of the contact problems: (i) penetration of the upper material layer into the lower one due to geometrically linear formulation of the problem; (ii) penetration of the material into the punch due to neglecting of the tangential displacements in Hertz-type contact problems. Considering the classic Boussinesq problem for a concentrated load and the Abramov (1937) problem, Rvachev and Protsenko (1977) discussed both types of the incompatibility. To avoid or at least to reduce the incompatibility one needs to employ the geometrically non-linear formulation of the contact problem that includes the Signorini-Fichera conditions of impenetrability of the material points (Signorini 1933, Fichera 1972) along with accounting the boundary tangential displacements. Contact problems with the conditions of impenetrability linearized with respect to boundary tangential displacements were studied by Galanov (1983) and Galanov and Krivonos (1984a). If one takes into account the conditions of impenetrability

$$u_z(r, 0, P) - \delta + f[r + u_r(r, 0, P)] \geq 0$$

then instead of the second condition in (31), the following one has to be written

$$u_z(r, 0, P) - \delta + f[(r + u_r(r, 0, P)] = 0. \tag{59}$$

The above equation is normally non-linear, hence the condition within the contact region can be linearized with respect to u_r and written as

$$u_z(r, 0, P) - \delta + f(r) + L(r)u_r(r, 0, P) = 0,$$

where $L(r)$ is obtained by linearization of f with respect to u_r. It was shown that the use of this more rigorous formulation than the Hertzian one substantially reduces the degree of the displacement incompatibility observed at the contact region and under the region (Galanov 1983).

The non-linear boundary condition (59) was studied by Galanov and Krivonos (1984b). It was shown that this formulation reduces the degree of the displacement incompatibility. However, if one compares these solutions with the relations of Hertz contact problem used in the JKR model then one can see that the influence of the refined solutions is rather small. Hence, the use of the JKR approach is acceptable for the adhesive contact problems under consideration. Of course this does not mean that there is no sense to study the adhesive contact using the improved problem formulations.

6.2 The fracture mechanics approach

It is known that the frictionless JKR results can be obtained by the use of linear fracture mechanics concepts (Maugis and Barquins, 1978, Johnson 1996, Maugis, 2000). In frictionless case the equilibrium is given by $G = w$ where G is the energy release rate at the edge of

the contact. The ideas of fracture mechanics were also used for adhesive contact problems in the presence of the tangential stresses. For example, Johnson (1997) used the mode-mixity fracture mechanics approach to study the Cattaneo-Mindlin type problem, when an elastic sphere subjected to a constant normal load P and a monotonically increasing tangential force T is in contact with a flat surface. We would like to remind that in the problem under consideration there is no external tangential force T acting on the contacting solids.

Various issues related to the use of the fracture mechanics concepts in application to mechanics of adhesive contact between isotropic elastic materials were discussed (see, e.g. Johnson 1996, Chen *et al.* 2009, Waters and Guduru 2010). The mode-mixity and its effects on adhesion were analytically studied by Chen *et al.* (2009) and Waters and Guduru (2010) in application to problem of adhesive contact. In these papers it was argued that the adhesion energy is not a material constant independent of the local failure mode but rather is a function of the mode-mixity. Although the fracture mechanics formalism is out the scope of the paper, the authors would prefer to use Johnson's interpretation (Johnson 1996): the work of adhesion w is a material constant, see (2), while the critical energy release rate G_c is $G_c = w[1 + \alpha(K_{II}^2/K_I^2)]$ where the parameter α can vary from 0 to 1.0 and K_I, and K_{II} are mode I and II stress intensity, respectively.

As it has been discussed above, the Mossakovskii-Spence formulation of the non-slipping contact problem assumes the radial displacements u_r are consistent with the shape of the punch. For monomial punches of degree d, the contact problem is self-similar, the radial displacements are given by the power-law expression $u_r = C_0 r^d$, and the constant of the frozen-in radial displacements ensures that the radial strain at any given point of the contact zone does not change when the size of the contact region increases and both the tangential and normal contact stresses are bounded. The presence of unbounded stresses in the adhesive contact problem are due to superposition of the Boussinesq-Mossakovskii stresses in the framework of the JKR approach. If one accepts the Derjaguin approximation then the surface energy can be calculated by (9), there is no need to consider the mode-mixity, and the classic JKR approach is applicable even in the non-slipping case. If the Derjaguin assumptions are not accepted and/or there is friction at the edge of the contact region (Galin 1945, Spence 1975, Zhupanska 2008) then the adhesive forces can work on tangential displacements and the mode-mixity effects have to be discussed; some interesting experimental results on adhesive axisymmetric contact between a punch and a polymer layer subjected to equi-biaxial stretch have been presented recently by Waters *et al.* (2012). As Johnson (1996) noted, *interaction between adhesion and friction under both static and kinetic conditions is still an open question.*

7 Conclusion

The classic JKR and Boussinesq-Kendall models of adhesive contact have been discussed. As one can see from the above discussion, the classic JKR approach to adhesive contact of linear elastic solids is very elegant. However, the original paper by Johnson *et al.* (1971) considered only a very important case of contact between spheres. The classic JKR approach has been generalized to the case of the punch shape being described by monomial (power-law) punches of an arbitrary degree $d \geq 1$. Although one could extend formally the above calculations to the case $0 < d < 1$, these cases are not discussed. The JKR and Boussinesq-Kendall models can be

considered as two particular cases of contact problems with molecular adhesion for monomial punches, when the degree of the punch d is equal to two or it goes to infinity respectively.

It has been noted that the derivation of the main formulae of both the JKR and Boussinesq-Kendall models is based on the assumption that the material points within the contact region can move along the punch surface without any friction. However, it is more natural to assume that a material point that came to contact with the punch sticks to its surface, i.e. to assume that the non-slipping boundary conditions are valid. Hence, the generalized adhesive contact problems for monomial punches have been studied for both frictionless and non-slipping boundary conditions. The clear rationale has been given to justify the use of the JKR approach to non-slipping contact. It has been shown that for compressible materials, the critical radius of the contact region and the corresponding critical load in the case of non-slipping contact are slightly less than the values predicted by the classic frictionless JKR approach.

Evidently the present paper does not cover all possible extensions of the classic adhesive contact problems. It is possible to extend these models to adhesive contact problems for arbitrary convex solids of revolution (see (23) and (24)), for transversely isotropic, prestressed, elastic-plastic, layered and coated solids (see, e.g., discussions by Kendall 1971, Johnson and Sridhar 2001, Chen and Gao 2006a, Sergici *et al.* 2006, Barthel 2008, Espinasse *et al.* 2010, Zhupanska 2012 and Olsson and Larsson 2013). It is possible to extend to monomial punches other models of adhesive contact (DMT, Maugis and various extensions of the Maugis models). Some results in this direction has been already published (Zheng and Yu 2007, see also a discussion by Barthel 2008). Goryacheva and Makhovskaya (2001) derived an extended Maugis model and solved the problems of adhesion and capillary adhesion for monomial punches when d is an even integer. The further extension of the Maugis model for monomial punches was presented by Zheng and Yu (2007) and Zhou *et al.* (2011). Espinasse *et al.* (2010) extended the JKR and DMT models to transversely isotropic materials. However, these problems are out the scope of the paper.

The presented extension of the classic JKR model to the monomial punches for both frictionless and non-slipping boundary conditions is quite important for practical applications. The expressions for the values of the pull-off force and for the corresponding critical contact radius are derived explicitly. It has been shown that for both frictionless and non-slipping boundary conditions, the solutions to the adhesive contact problems can be reduced to the same dimensionless relations between the actual force, displacements and contact radius. Hence, these relations can be considered as the universal adhesive JKR-type dimensionless relations for power-law shaped bodies. The results obtained are applied to problems of nanoindentation when the indenter shape near the tip has some deviation from its nominal shape and the shape function can be approximated by a monomial function of radius.

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