Distribution Network Dynamics with Correlated Demands

by

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Abstract

The distribution network designs for two-level supply chains have been analysed using stochastic analytical methods. The market demands faced by multiple retailers are correlated. The correlated demand is modelled as a first order Vector Auto-Regressive process, which is used to represent the progression of and relationships in sets of time series of demand. All participants are assumed to operate an Order-Up-To policy with a Minimum Mean Squared Error forecasting.

Inventory and capacity costs have been considered. Control engineering methods have been exploited to obtain the closed form expressions of the variances of the inventory levels and the order rates. The ratios of costs between the decentralised and centralised systems have been used to evaluate the economic performance of the consolidated distribution network. The variance expressions are the key components for the cost ratios. Insights about the system can also be obtained from the analysis of the variance expressions. The impacts of demand patterns, lead-times and the number of decentralised locations on the consolidation decision have been investigated.

The results show that the auto-correlation and cross-correlation of the market demands highly affect the consolidation decisions. The Square Root Law for Inventory and Bullwhip has been proved to hold with certain demand correlations. Consolidation scenarios that are always attractive under a specific demand pattern and a set of constraints about the lead-times have been presented. The structural transition of the demand into orders placed onto higher echelons has been investigated. The result shows that higher echelons may not need the point-of-sales data as it is already contained in the order they receive from the retailers. Finally, the model has been validated by its application to real world data and has shown to be a useful tool for practitioners to investigate the dynamic behaviour and economic performance of the distribution network design.

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Summary of general notations

DND   Distribution Network Design
VAR   Vector Auto-Regressive
VARMA   Vector Auto-Regressive Moving Average
OUT   Order-Up-To
MMSE   Minimum Mean Square Error
WIP   Work-In-Progress
DC   Distribution Centre

n The number of decentralised locations ($n = 1, 2, 3, \ldots$)

μ_i The mean demand level of retailer i ($i = 1, 2, \ldots, n$)

ϕ_{ii} Auto-correlation coefficient of the demand faced by retailer i

ϕ_{ij} Cross-correlation coefficient between the demand faced by retailer i and retailer j, where $i \neq j$

l Retailer's replenishment lead-time ($l = 0, 1, 2, \ldots$)

L DC's replenishment lead-time ($L = 0, 1, 2, \ldots$)

p the order (lag) of the auto-regressive process

q the order of the moving average process

H Cost per unit for inventory holding

B Cost per unit for inventory backlog

P Cost per unit for over-time working or subcontracting

N Cost per unit for lost capacity or normal-time working

σ_I The standard deviation of the inventory level

σ_O The standard deviation of the order rate

$I_\pounds$ The expected total inventory cost

$C_\pounds$ The expected total capacity cost

$\mathbf{N}$ $(1 \times n)$ Unit vector

$\mathbf{I}_N$ $(n \times n)$ identity matrix

$Ratio[Inv]$ The ratio of the inventory costs between the decentralised and centralised systems

$Ratio[Cap]$ The ratio of the capacity costs between the decentralised and centralised systems

$Ratio[Total]$ The ratio of the total costs between the decentralised and centralised systems

$[.]'$ Matrix transpose function

$[.]^{-1}$ Matrix inverse function

$\text{erf}[.]$ The error function

$\text{erf}^{-1}[.]$ The inverse error function

$\varphi[.]$ The density function of the standard normal distribution

$\Phi^{-1}[.]$ The inverse standard normal cumulative distribution function

$(x)^+$ $\max(0, x)$
det(.)   The matrix determinant
$x(z)$ z-domain version of variable x
z^{-x} z-domain function for the x period time delay
$Z^{-1}[.]$ The inverse z-transform
$E[.]$ Expected value function

Note:

A complete summary of notations used in the mathematical model is in Appendix B.

Table of Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Background

Distribution Network Design (DND) is among the most important strategic decisions in supply chain management as it affects the long term company competitiveness. Excellent DND has led many companies to success. Then again, inadequate DND has also put many companies out of businesses. To adopt a centralised or decentralised system design for a distribution network is one of the major issues to be decided in a DND problem. Although a large number of studies have been concerned with this issue, the impact that the correlation in multiple time series representing market demand have on the centralisation and decentralisation decisions has not been completely understood. Given my experience of mathematical modelling of different network designs and having been awarded a scholarship to pursue a doctoral study in Logistics and Supply Chain Management, the author has developed a strong interest to fulfil this research gap. The research output may be interesting to strategic supply chain decision makers as the impacts of the demand patterns, lead-times and numbers of distribution centres on the benefits of different DNDs will be quantified in this study.

In this chapter, the main concepts related to the study will be first discussed. This includes supply chain management, distribution network consolidation and inventory management. Then, the key issues to be investigated will be identified. These key issues will lead to the research questions that will be addressed in this study. Finally, the structure of this thesis will be explained.

1.2   Supply chain management

"A supply chain consists of all parties involved, directly or indirectly, in fulfilling a customer request. The supply chain includes not only the manufacturer and suppliers, but also transporters, warehouses, retailers, and even customers themselves" (Chopra and Meindl 2007, p. 3). Today's companies have focused their attention on how to achieve efficient supply chains in order to compete in global market where products have a shorter life cycle and customers have higher expectations (Simchi-Levi et al. 2005). The businesses that can manage their supply chain well will succeed in satisfying the customers' requirements while the cost of doing so is under control.

There are many definitions of supply chain management. Simchi-Levi et al. (2000) define it as follows:

> Supply chain management is a set of approaches utilised to efficiently integrate suppliers, manufacturers, warehouses, and stores, so that merchandise is produced and distributed at the right quantities, to the right locations, and at the right time, in order to minimise system-wide costs while satisfying service level requirements.

This thesis is concerned with one of the keys to successful supply chain management, Distribution Network Design (DND). The DND aims to determine the placement of an arbitrary number of stock holding facilities to enable the efficient flow of materials through a supply chain. A distribution network supports the moving and storing of products from the manufacturers to the end consumer. In this study, attention will be paid specifically to the flow of finished products and the number of Distribution Centres (DCs). "Two of the world's most profitable companies, Wal-Mart and Seven-Eleven Japan, have built the success of their entire business around outstanding distribution [network] design and operation" (Chopra and Meindl 2007, p. 75). The DND depends on supply chain objectives ranging from low cost to high responsiveness (Chopra and Meindl 2007). This makes different businesses (even from the same industry) design their distribution networks differently.

The distribution network considered in this study is of the type that products move through DCs when going to retailers in a similar way as that applied by P&G and Texas

Instrument to a proportion of their product distributions (Chopra and Meindl 2007). Figure 1-1 gives an example of the flows of finished products and order information in such distribution networks.
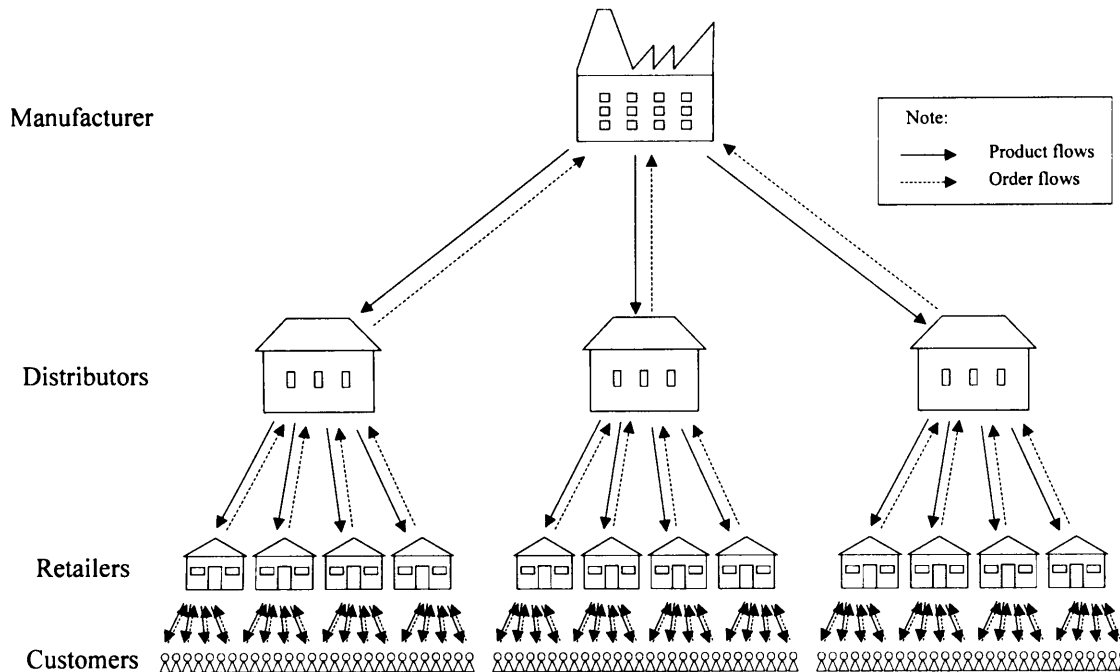
Figure 1-1 The distribution network

1.3 Distribution network consolidation

In this study "distribution network consolidation" is defined as centralising of inventory held at the distribution centre level from multiple locations to a single location. Throughout many years, the trend for companies has been to consolidate distribution network to achieve supply chain and logistics management objectives. A great deal of evidence can be found in the literature, for example:

- In two years National Semiconductor, a US semiconductor manufacturer, reduced its standard delivery time by 47%, reduced distribution costs by 2.5%, and increased sales by 34% by shutting its six warehouses located in different countries and air-freighting its products from a new distribution centre in Singapore (Henkoff 1994).

- "Consolidation and centralisation are proving to be an excellent move for companies that know how to manage logistics effectively" (Rheem 1997).

- Benetton, an Italian global fashion brand, uses one centralised DC in Italy to serve more than 6,000 stores in 83 countries (Dapiran 1992, cited in Teo et al. 2001).
- "Zara is a chain of fashion stores owned by Inditex, Spain's largest apparel manufacturer and retailer. Zara centralised all its European distribution and some of its global distribution through a single distribution centre in Spain" (Chopra and Meindl 2007, p. 17).

Generally, the main benefit that companies can expect from centralising their DCs is to reduce their inventory related costs (Teo et al. 2001). The company will carry fewer inventories as the aggregation reduces replicated inventories held by different decentralised locations. The concept of 'risk pooling' suggests that demand variability is reduced when we aggregate demand between different locations. This is because high demand from one location can compensate for low demand from another (Chopra and Meindl 2007, p. 59). Both safety stock and average inventory will be reduced by the reduction in demand variability.

Besides the inventory benefit, the company can speed up service and gain operational efficiencies by such consolidation (Barnard 2008). A company can achieve higher throughput with a larger DC. This means the company keeps a smaller amount of inventory in stock and does not have to keep it as long as before. A centralised DC has more capacity than a decentralised one and this allows investment in technology such as conveyors and an automated storage and retrieval system to become economical. The increasing automation results in more efficient operations that can be run by a smaller workforce. The operational efficiencies also aid centralised management which benefits from fewer procedures and fewer people involved in decision making.

In terms of other costs, overhead costs are much lower in a centralised system. This is due to economies of scale. Transportation costs are also very important in decisions about network consolidation. This involves inbound transportation costs, which are incurred when products are shipped from manufacturers to DCs, and outbound transportation costs, which are incurred when products are delivered from the DCs to their customers. In decentralised systems where more DCs are placed close to the customers, outbound transportation costs decrease while inbound transportation costs may increase. This is opposite to the centralised systems where the outbound transportation costs typically

increase while the inbound transportation costs can sometimes decrease. In addition, a centralised system can enjoy economies of scale by having full truckloads brought in and shipped out. The shipping rates for full truckloads can be one-half to one-third per kg-km of what they are for less than full truckloads (Barnard 2008). However, it is not immediately clear what the net impact of network consolidation on the total transportation cost might be.

1.4   Inventory management

A mismatch between supply and demand creates inventory in the supply chain (Chopra and Meindl 2007, pp. 50-53). Some inventories are held purposefully in anticipation of future demand whereas some are resulted from exploiting economies of scale in production and distribution operations. Inventory has an important role in supply chain strategy whether it aims for high responsiveness or low cost. High responsiveness can be achieved by carrying more inventories close to the customers. In contrast, a company achieves a lower cost strategy by holding fewer inventories at a centralised location. This demonstrates that inventory management is highly relevant to DND problems.

Inventory management generally involves decisions on cycle inventory. That is, the average amount of inventory held to satisfy demand during the replenishment lead-time. The inventory is managed by the decision on how much to order and how often the order should be placed. Another decision is to determine the safety inventory, which is held to protect the company from uncertainty in market demand. The amount of inventory carried by a company affects the service level and the inventory costs. The service level is a fraction that reflects the proportion of demand that is satisfied directly by stock. Usually the service level is high when more inventory is held by a company. The increased inventory, however, leads to inventory related costs such as the cost of holding the inventory, the cost of obsolescence and the opportunity cost from investing in an inventory which is not used. These costs decrease when less inventory is held but the service level will be lower. Although the service level has been used as a strategic decision that a company takes to compete with its competitors, it should be noted that in this study the strategy will be set to compete on inventory and capacity costs alone.

1.5 Key issues to be investigated in this thesis

The preceding sections have given some overview concepts that this study belongs to, has specified the type of distribution network that will be investigated and has explained the impact distribution network consolidation has on a business. These sections have also described how inventory management relates to DND. Before we approach the research questions, some key issues that will be investigated in this study will now be highlighted.

This thesis will examine how and under what circumstances network consolidation should be adopted. Although many researchers have investigated this matter, the correlation of demands between the retailers has been disregarded by most researchers as it complicates distribution network problems (Chen et al. 2002). This complexity occurs in real supply chains. Neglecting to consider the correlation between retailers' demands may cause significant deviation from the optimal inventory policy (Erkip et al. 1990). Thus, in this thesis, the correlations of market demands both in time and across retailers are investigated to determine its impact on the dynamic behaviour of the system and on the consolidation decision.

Even though most of the literature ignores the impact of demand correlation in supply chain models, a number of papers have paid close attention to the matter in a variety of model settings (Eppen 1979; Evers and Beier 1993; Wanke 2009; Wanke and Saliby 2009; Zinn et al. 1989 for example). All of these papers used statistical models to study the consolidation of inventories and described the correlation of demands between locations by the Pearson product-moment correlation coefficient. Unlike the previous works, the demand correlations in this thesis represent the correlations of the current value of the demand with both previous values of itself and of the demand from other retailers. This definition allows for the cross-correlation between a pair of retailers to be different in both directions, which will allow the model to be more general when compare to previous studies. This contrasts to the correlation represented by Pearson product-moment correlation coefficient that has generally been used by other researchers.

Another distinguishing factor of this thesis is the consideration of the capacity cost when the economic performance of a consolidated distribution network is evaluated. The capacity cost is a function of the variation of the order rates. Thus, it is closely related to

the Bullwhip Effect, a phenomenon where the variability in the orders increases as the order proceeds up the supply chain (Lee et al. 1997). The Bullwhip Effect creates an inefficient supply chain and thus leads to unnecessary supply chain costs. Therefore, as well as inventory costs, this thesis also considers capacity costs in order to capture the variable labour and equipment costs associated with the Bullwhip Effect.

This thesis intends to quantify the benefit of distribution network consolidation, which would be useful not only to demonstrate the magnitude of the benefit, but also to highlight the relationship between the demand correlations, lead-times and benefits from consolidation. The model developed in this study will be restricted to the specific inventory replenishment policy, the forecasting method and demand model. Although this will result in less generality, it will allow us to investigate the structural transition of the end customer demand as it is processed by retailers and DCs before being placed onto the factory. This investigation will provide valuable insights into the value of information sharing between the players in a supply chain.

1.6   Research questions

The research questions addressed in this thesis are:

1) How can we model the distribution network of a supply chain in which the market demand is a correlated multiple time series?

The first research question addresses the technical aspects that are needed to deal with the correlations in multiple time series within the DND context. It can be separated into sub-questions according to distinct parts of the main model as follows:

1.1)   How can the correlated demand be modelled?

1.2)   How can this correlated demand be forecast?

These questions are highlighted because they are expected to be very complex. A new demand representation and forecasting procedure will need to be proposed and this could be a major contribution of this thesis. In order to achieve this, some fundamental techniques in mathematics and statistics will need to be incorporated to the modelling. This leads to the next sub-question:

1.3) Which fundamental techniques are useful for modelling the supply chain's distribution network?

If the system is successfully modelled, the next sub-question that should be addressed to prove that the model is useful is:

1.4) What can we learn from this model compared to previous modelling studies?

2) Under what circumstances should the consolidated distribution network be established?

The second research question centres on what can be learned from the model created in Research Question 1. Again, it can be broken into sub-questions according to factors affecting the consolidation decision.

2.1) What is the impact of the demand correlation on the consolidation decision?

2.2) What is the impact of the lead-times of each player in the distribution network on the consolidation decision?

2.3) What is the impact of the number of decentralised locations on the consolidation decision?

2.4) What is the impact of the cost function on the consolidation decision?

3) How can the stylised analytical results be related to real world demand data?

The final research question is equally important as it links the analytical model to practical aspects. The model is validated to prove that it is applicable to a real supply chain. Some practical issues may be realised through this validation process and this will be a good feedback for model improvement.

1.7 Structure of thesis

This thesis consists of nine chapters in which research questions will be addressed accordingly. Figure 1-2 illustrates the flow and connection of the content in each chapter.

- Chapter 1 introduces the reader to the main area of study addressed in this thesis. The definitions of the key concepts are given for the common understanding of the subjects being considered. After a broad overview of the thesis has been given, the focus of this thesis and the research questions are identified.
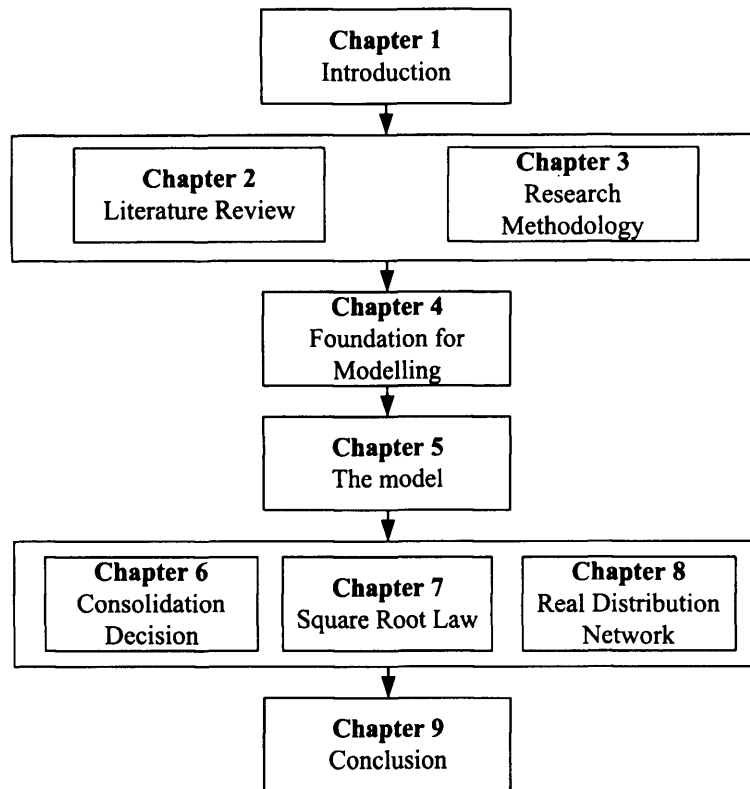
Figure 1-2 Thesis structure

- Chapter 2 provides more extensive reviews of the key concepts specified in Chapter 1. It examines the methodology and results from the related literature in order to identify research gaps. The main subjects to be reviewed include distribution network design, distribution network consolidation, cost consideration in distribution network design and correlated demand modelling.

- Chapter 3 is devoted to the research methodology employed in this thesis. The research perspective, which defines the author's point of view about the knowledge and the nature of the topic, is revealed. The research perspective reflects the appropriate research approaches and methods for this study. The research design is presented together with how this research will be evaluated.

- Chapter 4 presents the theoretical foundations for this research. Some basic knowledge that needs to be understood before proceeding to the analytical modelling is explained. This includes Vector Auto-Regressive demand processes, the Order-Up-To inventory replenishment policy and the Minimum Mean Square Error forecasting. The cost functions employed throughout this thesis are

specified. The ratios of costs, which are used as economic performance measures for consolidated distribution network, are defined.

- Chapter 5 is the core of this thesis. It presents the mathematical model of the distribution network problem with correlated demands. A proposed procedure for forecasting that can deal with the arbitrary lead-times at different locations is presented. An approach for obtaining the variance expressions is presented step-by-step. The first research question (including all of its sub-questions) is answered in this chapter. The model is applied with different situations in the following chapters.

- Chapter 6 investigates the impacts of demand correlation, lead-times and the number of decentralised locations on consolidation decisions. A 'simple' model is studied to gain generalised insights. The inventory variance, order variance, cost ratio and dynamic behaviour of the systems are investigated. These investigations answer the second research question (including all of its sub-questions). One of the interesting findings is the characteristic of the cost ratios under certain circumstances, the "Square Root Law for Inventory" and the "Square Root Law for Bullwhip". This is, therefore, investigated in more detail in Chapter 7.

- Chapter 7 observes the Square Root Law for Inventory and Bullwhip in advanced cases where the demands are correlated both in time and between retailers. The square root law can give a quick and convenient approximation of the benefit to be gained from distribution network consolidation.

- Chapter 8 applies the model created in Chapter 5 to the real market demand data from industry for validation purposes. The model is also applied to the real data of the shipment received by the distribution centres to show the use of the "demand transition" insights derived in Chapter 5. The third research question is answered in this chapter.

- Chapter 9 concludes the thesis by referring back to the research questions. Research contributions are highlighted. Limitations and future research are identified.

1.8 Summary

Considering the fact that most researchers disregard the correlations of market demands when designing a distribution network, this thesis will take this fact into account. This thesis will produce an analytical model that quantifies the benefit of network consolidation. It could be a useful tool for the practitioner when making decisions concerning the decentralisation and centralisation of the distribution network of a supply chain.

A project to redesign the distribution network may be initiated when the current configuration of the distribution network is considered to be unsuitable as shown by inefficient product flows and high costs. Also, a project may be initiated due to the change of the patterns of market demand, the end of lease of the distribution network facilities or an economic crisis. In a survey carried out by Saddle Creek Corporation (2010), the economic recession has shown to have forced two thirds of the 235 companies in their study to make changes to their distribution network design.

The result from this study is most likely to be applied in a grocery business for a fast-moving consumer goods (FMCG) such as pre-packaged foods, toiletries, soft drinks and cleaning products. The FMCG is characterised by high volume, high stock turnover and extensive distribution networks. Thus, it may match well with the model setting for the demand pattern and inventory replenishment policy that will be adopted in this study.

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Chapter 2

Literature review

2.1   Introduction

In this chapter, literature related to Distribution Network Design (DND) will be reviewed. This will begin with an overview of the DND problem where some examples from real businesses that highlight the important of DND are given. Next, performance measures and important decisions related to the DND problem will be described. As researches in this area are generally based their studies on these performance measures and decisions, the research works that are reviewed in the following section will be identified according to their involvement in each of the performance measures and decisions. Finally, more specific issues such as distribution network consolidation, the modelling of demand correlation in supply chain studies and Bullwhip costs are reviewed in order to identify research gaps and the potential contributions of this thesis.

2.2   Distribution network design

A supply chain is dynamic and is concerned with the continuous flows of materials, information, and funds between different players. "There is a close connection between the design and management of supply chain flows (product, information, and funds) and the success of a supply chain" (Chopra and Meindl 2007, p. 6). Chopra and Meindl (2007) provided some examples of companies where the designs of their distribution network had led to the success or failure of their businesses. "Wal-Mart designed its

supply chain with clusters of stores around distribution centres to facilitate frequent replenishment at its retail stores in a cost-effective manner" (Chopra and Meindl 2007, p. 7). This distribution network design was a crucial part of the company's success where the net income was more than $9 billion on revenues of about $250 billion in 2004. Dell, one of the world's largest computer manufacturers, designed its supply chain differently. Dell did not use DCs or retailers. Its finished products were shipped directly to the users. Dell kept very low levels of inventory by centralising its manufacturing and inventories in a few locations and used postponement strategy for the final assembly. This allowed Dell to market its new computer models faster than its competitors. Also, Dell could reduce the risk of holding too much inventory which typically results in price reductions and obsolescence, common symptoms in the computer market. In contrast, "the failure of many e-businesses such as Webvan and Kozmo can be attributed to their inability to design appropriate supply chains or manage supply chain flow effectively" (Chopra and Meindl 2007, p. 8). Webvan and Kozmo were both online grocery businesses. Webvan placed its warehouses in several major cities in the United States delivering groceries to customers' homes. Although Webvan turned over its inventory slightly faster than local traditional supermarkets, its transportation cost was much higher compared to that of the traditional supermarkets that enjoy full-truckload inbound shipments. This led to the end of Webvan's business after just two years. It was a similar story for Kozmo. Thus, it is critical for a company to have a suitable DND to serve its business purposes.

The DND problems are challenging due to their system-wide cost minimisation and uncertain nature. Supply chain dynamics also make these types of problems difficult to solve (Simchi-Levi et al. 2005). There are many important issues associated with DND including network configuration problems, inventory controls, transportation decisions, vehicle fleet management and truck routing, for example.

2.2.1 Conceptual models and issues in distribution network design

Drawing on how real companies manage their supply chain, Chopra (2003) described a framework for designing the distribution network. It was noted that the distribution network is generally designed under two main factors, which are customers' satisfaction and the cost of achieving customers' needs. Table 2-1 summarises the key components for both customer and cost dimensions.

Table 2-1 Performance measures for DND (adapted from Chopra 2003)

Customer service dimension	Cost dimension
• *Response time*: the time interval between when the customer places an order until the order is received.	• *Inventory costs*: cost of having and not having inventory in stock.
• *Product variety*: the number of products that are offered by the distribution network.	• *Transportation costs*: include inbound and outbound transportation costs
• *Product availability*: the probability of having product in stock when a customer order arrives.	• *Facilities and handling costs*: cost of having facilities such as stores, distribution centres, warehouses and factories.
• *Customer experience*: how convenient it is and comfortable customers are when placing and receiving orders.	• *Information costs*: cost of having information flows between facilities.
• *Time to market*: the time before a new product can be launched.	
• *Order visibility*: the ability of customers to track the status of their orders.	
• *Returnability*: how convenient it is for customers to return products.	

Based on the performance measures in Table 2-1, Chopra (2003) proposed six conceptual designs for supply chains' distribution networks. They were classified as follows:

- Manufacturer storage with direct shipping.
- Manufacturer storage with direct shipping and cross-dock DC.
- Distributor storage with package carrier delivery.
- Distributor storage with last mile delivery.
- Manufacturer/distributor storage with costumer pickup.
- Retail storage with customer pickup.

Figure 2-1 illustrates the six designs. The distribution network being investigated in this study is a mixture of these designs, which includes manufacturers, distributors, retailers and customers. All players store inventory and ship products to downstream players upon

request. There will be no direct product and information flows from the manufacturer or distributor to the customer. All product and information flows only occur between a pair of immediate successors in the supply chain.

(a) Manufacturer storage with direct shipping

(b) In-transit merge network

(c) Distrbutor storage with carrier delivery

(d) Distributor storage with last-mile delivery

(e) Manufacturer or distributor storage with customer pickup

(f) Retail storage with customer pickup

Note:
⟶ Product flow
·····▶ Information flow
----▶ Customer flow
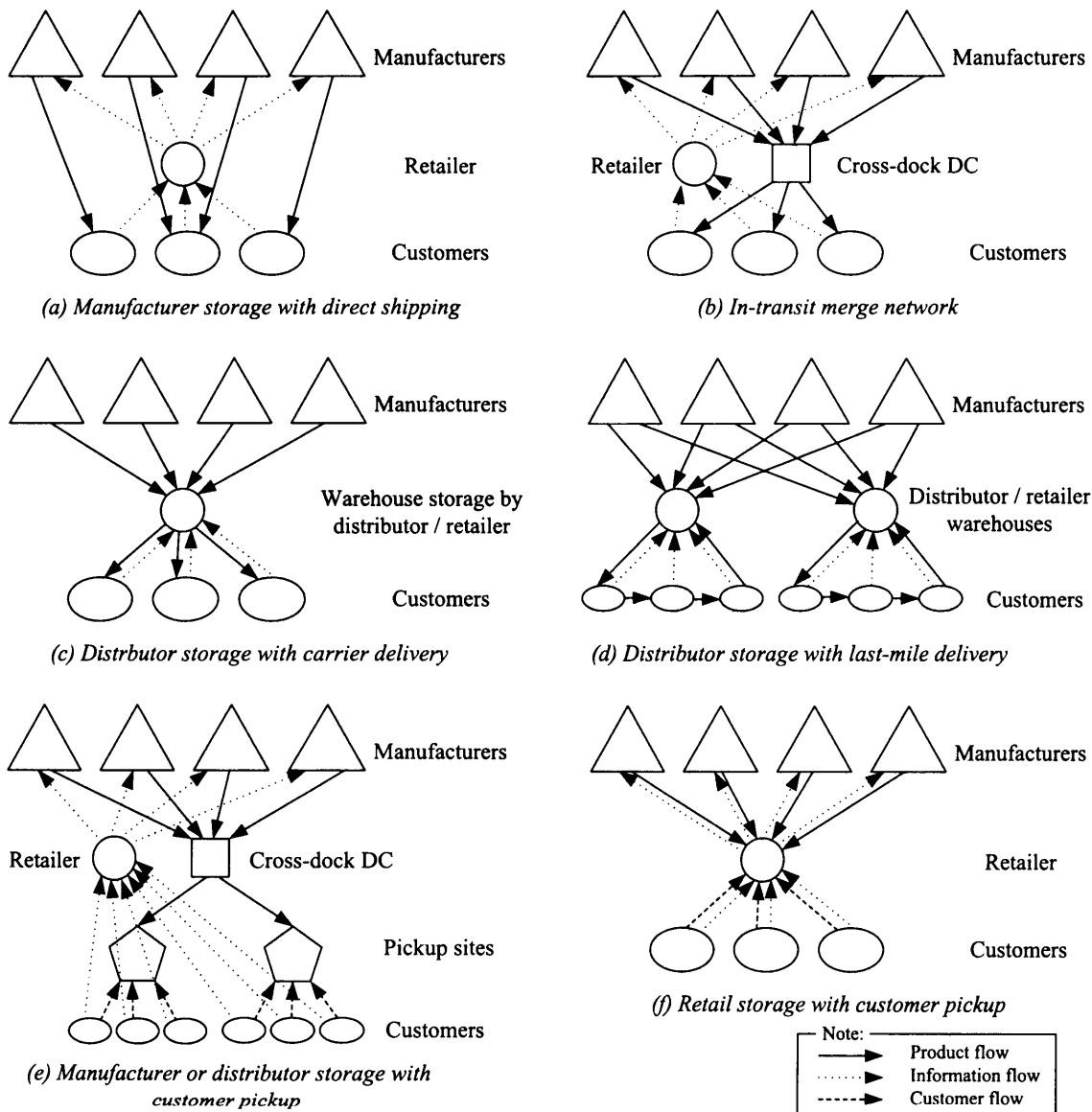
Figure 2-1 Design options for a distribution network (adapted from Chopra 2003)

Although the proposed network designs by Chopra (2003) provided a very good overview of the conceptual options for a distribution network, the magnitude of the economic performance for each option had not been considered. In contrast to Chopra (2003), one of the major aims of this thesis is to provide a quantified benefit of different distribution network so that the practitioners can have a solid figure to support their decision making.

Beamon (1998) categorised supply chain performance measures into qualitative and quantitative measures. The qualitative measures are generally described by more than one numerical measure. Examples of qualitative measures are customer satisfaction, flexibility, information and material flow integration, effective risk management and supplier performance. On the other hand, the quantitative performance measures can be directly described numerically. The quantitative measures that are generally used in supply chain modelling can be based on cost (such as minimisations of costs on inventory management and maximisations of sales, profit and return on investment) or based on customer responsiveness (such as minimisations of product lateness, lead-times, and customer response time and maximisation of the fill-rate). The fill-rate is defined as a measure of the inventory's ability to meet demand. It is generally shown as a percentage of the customers' satisfied from the stock at hand. Beamon (1999) reorganised and extended the supply chain performance measures into three types: resource, output and flexibility. This new framework was claimed to be more appropriate than the past single performance measure, which was not inclusive, ignored interactions within supply chains, and paid no attention to the organisations' strategic goal (Beamon 1999).

Observing the distribution networks given in Figure 2-1, it is understandable that some strategic decisions need to be made during the design process. Simchi-Levi et al. (2000) discussed some important decisions that affect the efficiency of a distribution network. These issues were summarised as follows (Simchi-Levi et al. 2000, pp. 111-120):

- *Centralised versus Decentralised control*: If a decision for a whole supply network is made at a central unit, it is generally a centralised control. The centralised control allows global optimization unlike decentralised control, where each player optimises its own operation. Instantaneous information of the entire network must be available for the central decision maker in order to make the centralised control works.

- *Distribution strategies*: This issue is concerned with the outbound flows of the product. Three strategies have been mentioned. The first strategy is the classical strategy in which inventories are held at the DC and distributed to the retailer when they are required. The second strategy is to ship products directly from manufacturers to the retailers without going through the DC. This strategy can reduce lead-time and eliminate the costs of having a DC. However, transportation

costs will be higher as the manufacturer needs to send more trucks to more locations. This strategy is, therefore, suitable for delivering perishable products to large stores that allow full-truckload deliveries. Another disadvantage of not having a DC is that the benefit from 'risk pooling' cannot be exploited. The third strategy is cross-docking, which was initiated in the retail sector by Wal-Mart in the late 1980s. In a cross-docking system, items received at the DCs are not put into stock but are directly prepared for shipment for retail stores. The role of the DC in this strategy is changed from a stock holding facility (as in the classic strategy) into a "flow through" distributor. As storage time decreases, lead-time and inventory costs decrease too. However, a large initial investment in information technology and the ability to manage the transportation system and efficient forecasting are needed in order to benefit from the cross-docking system. See Bartholdi and Gue (2004) for further details about cross-docking.

- *Transhipment*: Transhipment is defined as "the shipment of items between different facilities at the same level in the supply chain to meet some immediate need" (Simchi-Levi et al. 2000, p. 116). Coordination between facilities, advance information systems and speedy transportation are required to benefit from this strategy. Companies that can implement this transhipment strategy can also take advantage of the "risk pooling" concept without a physical central DC.

- *Push versus Pull systems*: The pull-based system is executed in response to a customer order while the push-based system is executed in anticipation of customer orders (Chopra and Meindl 2007, p. 12). The pull-based system is much quicker to respond to a change in market demands. This leads to reductions in lead-time, inventory at retailers and Bullwhip respectively. However, the pull-based system is not suitable when lead-times are relatively long and when economies of scale cannot be achieved. Companies generally apply both Push- and Pull-based strategies to different processes in their supply chains.

- *Centralised versus Decentralised facilities*: This decision is the main theme of this thesis. For research with quantitative approaches, the decision about whether to use a centralised distribution network is based on safety stock levels, overhead costs, economies of scale, lead-times, services and transportation costs. This topic will be discussed in detail later in this chapter.

In summary, the decision related to each issue above is often not one of "either-or". A mixture of strategies is possible depending on the particular type and objective of businesses. Thus "when" and "how" decisions are probably just as relevant.

2.2.2 Supply chains' distribution network modelling

Beamon (1998) classified models for supply chain design and analysis into four categories as follows:

"1. Deterministic analytical model, in which the variables are known and specified,

2. Stochastic analytical model, where at least one of the variables is unknown and is assumed to follow a particular probability distribution,

3. Economic models,

4. Simulation models" (Beamon 1998).

The combination of the model classification by Beamon (1998) presented above, the performance measures by Chopra (2003) as summarised in Table 2-1 and the DND decisions by Simchi-Levi et al. (2000) as discussed in pages 16 and 17 will be a useful tool to identify the uniqueness of a supply chain model when a literature review is conducted. The research contributions on the views of performance measures employed, DND decision considered, and techniques applied can then be identified. Appendix A summarises selected literature related to DND based on the given criteria.

From the summary table in Appendix A, it can be seen that the majority of the DND research applied Operational Research (OR) tools such as Linear Programming (Abdinnour-Helm 1999; Croxton and Zinn 2005; Ferretti et al. 2008), Integer Linear Programming (Ambrosino and Grazia Scutellà 2005; Hinojosa et al. 2008), Non-linear Programming (Ferretti et al. 2008; Park et al. 2010) and Mixed Integer Programming (Amiri 2006; Easwaran and Uster 2010; Hadi et al. 2009; Jayaraman and Ross 2003; Lee et al. 2010; Longinidis and Georgiadis 2011; Meepetchdee and Shah 2007). These OR programming approaches differ by the linear or non-linear nature of their equations and by the integer or non-integer values of their decision variables.

OR commonly attempts to find an optimal solution for the problem under consideration

by adopting an organisation point of view (Hiller and Lieberman 1995, p. 3). Thus, it resolves the disagreement between the members' objectives in a system in a way that is best for the overall system. The conflict in the objectives frequently found in the context of DND problems typically manifests itself in the trade-off between service level and costs. The OR programming represents a system using a set of equations. An objective equation represents a mutual goal for the system to minimise or maximise under a set of constraints. For example, Ambrosino and Grazia Scutellà (2005) described the DND problem, which involved facility location, warehousing, transportation and inventory decisions, as an Integer Linear Programming. Binary (0,1) variables were used to represent "either-or" decisions involved with facility location, sequence in routing, customer assignment and vehicle assignment. The objective function was the sum of facility, transportation and inventory costs. The model was tested by 12 instances from real case study and data generation. The computational results showed that an optimal solution can be achieved but only for small-sized problems. When the size of the problem was large the computation can take several days.

The long computational time is a major problem faced by the traditional OR programming approach. Alternative approaches that have become attractive are heuristic algorithms such as Genetic Algorithm (Abdinnour-Helm 1999; Costa et al. 2010; Ding et al. 2009), Lagrangian methods (Amiri 2006; Dong et al. 2010; Hinojosa et al. 2008; Park et al. 2010; Sourirajan et al. 2007), Simulated Annealing (Jayaraman and Ross 2003), Artificial Immune System (Tiwari et al. 2010) and other heuristic procedures (Du and Evans 2008; Hadi et al. 2009; Lapierre et al. 2004; Lee et al. 2010). The heuristic approaches aim to achieve optimal or near-optimal solutions with a reasonable computational time. Some heuristic algorithms exploit mathematical techniques to relax or decompose the original OR problems to allow ease of computation. Alternatively, some heuristic algorithms imitate natural mechanisms in searching for a set of better solutions. For example, Costa et al. (2010) used a Genetic Algorithm (GA) to find the best number and location of manufacturing plants and DCs that minimise the total logistics cost resulting from the transportation, location and opening of the facilities in a three-stage supply chain network. The GA employed the mechanics of natural selection and natural genetics to obtain the best solution. The result showed that the proposed GA algorithm quickly converges to the optimal solution in reasonable computational times.

A simulation model was another interesting method used in DND. Bottani and Montanari (2010) used a discrete-event simulation model to observe the effects of different supply chain configurations on the resulting total costs and Bullwhip Effect. They investigated 30 different supply chain designs, with various numbers of echelons, re-order points and inventory management policies, different information sharing strategies, the presence of demand 'peaks' and responsiveness of supply chain members. They concluded that the number of echelons and a specific inventory management policy has a high impact on the total costs and the Bullwhip Effect. Also, demand information sharing could reduce both the Bullwhip Effect and the resulting total costs from decreases in holding costs.

2.3 Consolidation of distribution network

One of the major DND decisions this thesis focuses on is a question of whether to have a centralised or decentralised distribution network. The distribution network centralisation and decentralisation have been studied by many researchers with diverse focuses. For example, Hammant et al. (1999) presented the use of a decision support system (DSS) for an automotive aftermarket supply chain. The service level and costs associated with DND (inventory and transportation costs) were simultaneously considered. The results underlined the benefit of network consolidation of the case study. Melachrinoudisa et al. (2005) proposed a muti-criteria methodology to re-configure a warehouse network through consolidation and elimination. This method could deal with multiple criteria including cost, customer service and intangible benefits related to DND. The model was tested with real data from a company's distribution network and was found to be practical for aiding the management in reconfiguring its distribution network as part of downsizing decisions. Kohn and Brodin (2008) presented a conceptual model for centralised distribution systems in which the environmental impact of logistics can be decreased. They pointed out some positive impacts on the environment as well as costs from a centralised distribution system. This included shipment consolidation, decrease in emergency delivery and the use of intermodal transport (such as rail).

Some scholars evaluated the benefit of network consolidation via inventory reduction. The ratio of inventories between decentralised and centralised systems was used by many scholars to evaluate the benefit of network consolidation. They include Maister (1976),

Zinn et al. (1989), Evers and Beier (1993), Tallon (1993) and Evers and Beier (1998), for example. Disney et al. (2006), Ratanachote and Disney (2008) and Ratanachote and Disney (2009) emplo yed not only the ratio of inventory costs but also the ratio of capacity costs related to the Bullwhip Effect. Under certain circumstances, all of the above studies showed that the ratios was equal to the square root of the number of decentralised locations. This finding was called the Square Root Law for Inventory and Bullwhip. It provides a quick approximation of the benefit from network consolidation. The aforementioned studies, apart from Ratanachote and Disney (2009), did not consider both of the auto- and cross- correlations of the market demand. Although Ratanachote and Disney (2009) did, the result can be only applied to a distribution network that has two retailers, two DCs and the lead-times at all locations are unity. A study that provides more flexibility about the number of locations and the values of lead-times will be more practical for analysing a real distribution network design.

Teo et al. (2001) investigated the impact of demand pattern on the consolidation of DCs based on the total facility investment and inventory costs. The result showed that the differences in the ratios $\left(\text{of the mean demand and its variance}, \dfrac{\mu_i}{\sigma_i^2} \right)$ between locations had a high impact on the effectiveness of the consolidation strategy. The consolidation was more attractive when the differences are small. This was proved for Poisson and i.i.d. demand processes. In contrast, they also showed that the consolidated strategy was very unattractive when the demand was a general stochastic process.

Lu and Van Mieghem (2009) studied a manufacturers' multimarket facility network design based on the network capacity investment perspective. They investigated a situation where a manufacturer produces two products to serve two geographically separated markets (onshore and offshore). Common parts were used in manufacturing the two different products. The main aims were to decide whether the common part should or should not be produced centrally and if it was produced centrally, in which market the plant should be located. The result showed that the optimal location of the centralised plant did not only depend on the relative magnitude of price and manufacturing cost differences but also on the demand size and stability. Similar work was been done by Dong et al. (2010). However, their focus was on the facility network design problem for two markets under demand and exchange rate uncertainty. They found that when the size

of one market increased, the company would not switch from a network of regional to centralised production. Also, the demand and exchange rate uncertainties had opposite effects on optimal centralised output.

2.4   Demand correlation in supply chain modelling

The correlation of demands between the retailers complicates distribution network problems and has been disregarded by most researchers (Chen et al. 2002). This complexity occurs in real consumer products (Erkip et al. 1990). Neglecting to consider the auto-correlation and the cross-correlation of the retailers' demands may cause significant deviation from the optimal inventory policy.

Although most literature ignores the impact of demand correlation in supply chain models, a number of papers have paid close attention to the matter in a variety of model settings. Eppen (1979) demonstrated that the consolidation of demand can reduce the total expected holding and penalty costs in an inventory system. Zinn et al. (1989) showed that the ratio of inventories between the decentralised and the centralised systems depended only on the correlation of demands between decentralised locations and the proportion of the standard deviation of demand of two decentralised locations. However, in their model, they assumed that lead-times were identical at all locations. Evers and Beier (1993) extended the model developed by Zinn et al. (1989) to include variable lead-times. Their model could also be used when centralisation to more than one location (from n to m locations).   More operational aspects about how inventories should be pooled were discussed in Wanke (2009) and Wanke and Saliby (2009).   All of these papers used statistical models to study the consolidation of inventories and described the correlation of demands between locations by the Pearson product-moment correlation coefficient, which only measures the correlation between a pair of historical demands in the same time period. A representation of the correlations that takes into account the influence of the previous value of demand of its own market and other markets could capture the dynamic character of the correlated market demand more accurately due to it general nature.

Other works took the component of time into account when considering the correlated demand. Erkip et al. (1990) developed a depot-warehouse model of a centralised distribution system to show the impact of demand correlations both in time and across warehouses on the optimal safety stock of a periodic review system. The warehouses employed base-stock policies. A first order Autoregressive, AR(1), demand was present and was augmented with a periodic index-variable representing the correlation between demand at different warehouses. The index-variable was assumed to be the same for all locations. This assumption limited the cross-correlation for all locations to be identical. In the real world, the level and nature of the demand correlation between different pair of locations could be different. Thus, this fact should also be taken into account when the mathematical model is built. Güllü (1997) investigated inventory levels and system costs resulting from a proposed forecasting approach by adopting probabilistic demand models. The study allowed correlation through time and among retailers of both demands and demand forecasts. Raghunathan (2003) evaluated the value of and incentives for information sharing in a one-manufacturer and n-retailer setting. The retailers' lead-times were set to zero. The AR(1) demand was assumed at each retailer. A correlation between error terms was used to capture the correlation of demand across retailers. Although Güllü (1997) and Raghunathan (2003) took the time dimension into their correlated demand modelling, they did not investigate DND problems.

There has not yet been a DND research that considers both the auto-correlation and the cross-correlation of demand. According to Erkip et al. (1990), such correlations exist in real world demand of consumer products. Thus, this research gap provides research opportunities which have been addressed in the research questions given in Chapter 1.

2.5   Bullwhip related costs

The Bullwhip Effect is a phenomenon in which the variations of orders grow larger for upstream players in a supply chain. Extensive research has been carried out to understand the causes and cures of the Bullwhip Effect as it has negative impact on supply chain's efficiency and operating costs. Lee et al. (1997a) and Lee et al. (1997b) identified five major causes of the Bullwhip Effect including demand forecast updating, order batching, lead-times, price fluctuation and rationing game. Lambrecht and Dejonckheere (1999) used their simulated experiment based on a spreadsheet application called "Bullwhip

explorer" to investigate the impact that inventory policies have on inventory and order fluctuations in a two-level supply chain. The inventory policies under the investigation included both periodic and continuous review policies such as Order-Up-To, target inventory, demand signalling and multi-echelon strategy. Their experiment confirmed the causes of the Bullwhip Effect as described by Lee et al. (1997a) and Lee et al. (1997b). The total inventory costs resulted from the inventory policies were clearly higher when the Bullwhip Effect occurred.

Wikner et al. (1991) combined an industrial dynamic simulation with transfer function analysis to diagnose the source of demand amplification (or Bullwhip Effect) and indicate improvement strategies for a supply chain. They summarised five approaches to improve the supply chain dynamics;

- "fine tuning" the existing ordering policy parameters,
- reducing system delays,
- removal of the distribution echelon,
- changing the individual echelon decision rules and,
- better use of information flow throughout the supply chain.

The Bullwhip Effect could lead to inefficient inventory, production, transportation and capacity management (Lee et al. 1997a; Lee et al. 1997b). For inventory management, the Bullwhip causes excessive inventory, excessive warehousing expenses and poor customer service due to product being out of stock or long backlogs. For production management, companies may experience uncertain production planning, excessive raw materials, unplanned purchases of supplies. For transportation management, the Bullwhip may cause inefficient scheduling and expedited shipment. For capacity management, it may cause insufficient or excessive inventory. These problems will clearly resulted in high costs from paying for excess inventory holding costs, facility costs, premium transportation costs, over-time and subcontracting costs. From the literature review, previous work in DND generally focused on inventory costs. Thus, this study will also take the capacity cost related to the Bullwhip Effect into consideration. This capacity cost will be related to inefficient utilisation of labour and capital employed and to over-time working, which has not yet been properly tackled in DND research.

2.6 Summary

The literature related to the DND problems has been reviewed in order to identify the research trend in this area and address the research questions. From the literature review, the research gaps have been identified. The major research gaps are the consideration of demand correlation between locations and the consideration of capacity cost related to the Bullwhip Effect in DND problems. The majority of researchers in this field have applied OR programming to tackle the DND problems and usually focus on optimal solution and computational times. To understand the dynamic behaviour of the system's elements involved in DND, a different technique should be employed. This issue will be discussed in Chapter 3 where the research methodology will be explained.

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Chapter 3

Research methodology

3.1 Introduction

This chapter discusses the research methodology applied in this thesis. To start with, some important issues related to the research aspects will be discussed. Research aspects lead to a set of appropriate research methodologies to be applied by researchers. Subsequently, the methods, techniques, theories and tools used in this thesis will be described. Other related issues such as the evaluation of business and management research (which is based on reliability, replication and validity) and the ethical issues will also be discussed.

3.2 Research perspective

There are some research issues that researchers should be aware of when embarking on their research. These include basic issues such as what your topic is (or what specifically you intend to research) and what your stance is (or how you intend to make sense of it); and grounding research issues such as what the nature (or ontology) of the topic is and what might count as knowledge (or epistemology) of the topic being investigated (Willmott 2007). According to Saunders et al. (2007):

> Epistemology concerns what constitutes acceptable knowledge in a field of
> study. Ontology, on the other hand, is concerned with nature of reality. To a
> greater extent than epistemological considerations, this raises questions of

the assumptions researchers have about the way the world operates and the commitment held to particular views. (Saunders et al. 2007, pp. 102, 108).

One of the foundations of good research is that researchers should make their philosophical standpoint very clear. Thus the methods used for conducting research are linked to how researchers picture the nature of social reality and the way it should be explored (Bryman and Bell 2003, p. 4). Therefore, no method is universally suitable for all different viewpoints about the reality. The realisation of the diverse set of philosophical standpoints will enable researchers to select appropriate methodologies and methods for their research and to appreciate other research perspectives.

Saunders et al. (2007) describes research philosophy as an "Onion", see Figure 3-1. Before we can get to the centre of it (where we apply our techniques and procedures), we need to peel away (or to understand) many important layers of the onion (Saunders et al. 2007). Such layers include research philosophies, approaches, strategies, choices of methods, time horizons and finally techniques and procedures at the centre. In each layer, examples of alternatives are given as shown in Figure 3-1. The examples given in each layer are only intended to show how diverse the research aspects can be and thus will not be all discussed in detail.

Positivism is an approach in the philosophy of science that believes only in concrete evidence. Positivists assert that only knowledge gained from scientific method on observable entities is acceptable (Thomas 2004, p. 42). Positivism presumes that there is an outside world that exists independently from our understanding of it and therefore researchers cannot reach or influence it. This also represents the positivism ontological considerations about the social phenomena and their meanings. In terms of the relationship between theory and research, Bryman and Bell (2003, pp. 9-10) conclude that the relationship can be both deductive and inductive. Table 3-1 summarises the differences between the deductive and inductive approaches. The positivist assumes that reality is a set of interacting variables and intends to find a general law or a theory which expresses the relationship amongst them (Thomas 2004, pp. 42-44). This shows that positivists tend to choose extensive rather than intensive research strategies and designs. Therefore, their research strategies are more likely to be quantitative approaches.

Table 3-2 summarises the differences between quantitative and qualitative research strategies. Quantitative tools for social scientists are, for example, self-completed questionnaires, structured interviews, observation, simulation and modelling (Thomas 2004, p. 46). The results will be described and explained by quantitative measures and statistical relationships.
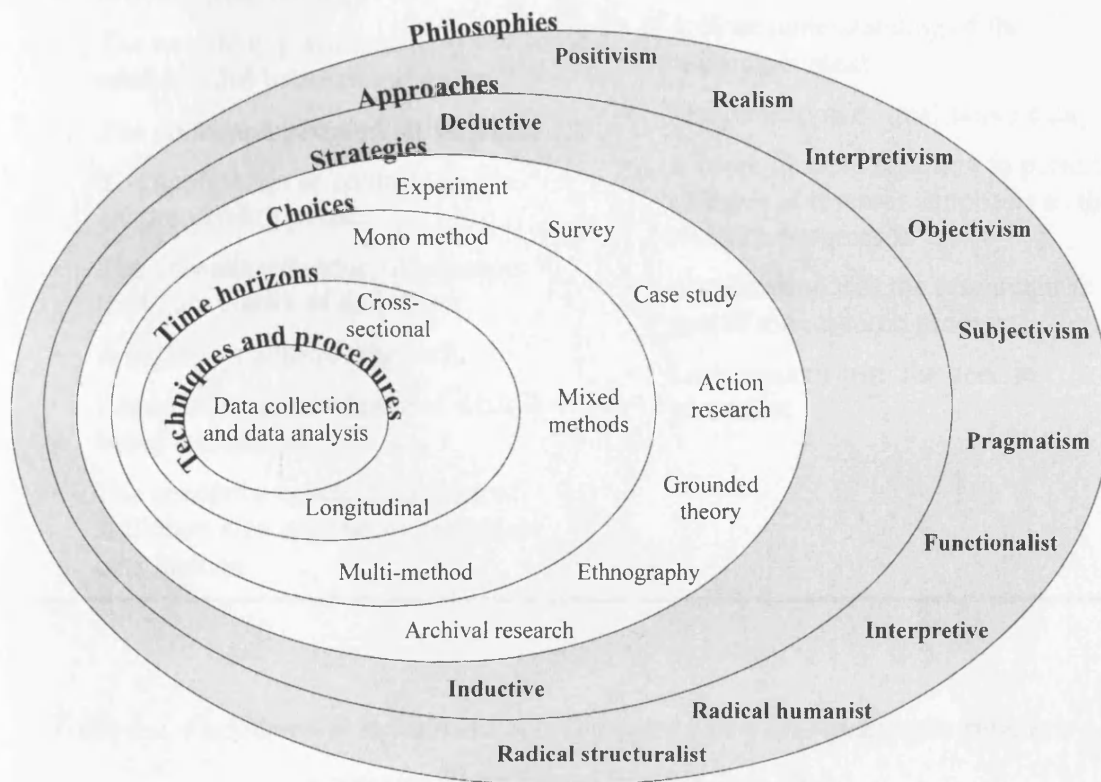
Figure 3-1 The research 'Onion' (Saunders et al. 2007, p. 102)

This study can be characterised as being of a positivism epistemological orientation, where the world is thought of as a set of interacting variables and the purpose of this study is to generalise the relationships among those variables under a set of assumptions about the world. From a positivist's viewpoint, the world is normally called a system, which is given an objective definition and assumptions to scope and elucidate a specific area of interest. The relationship between data and theory is deductive, where theories are tested through observations. Quantitative approaches are applied in this research. Ideas are examined and tested through a clear definition of variables which are observable, tangible and clearly defined.

Table 3-1 Major differences between deductive and inductive approaches to research

(Saunders et al. 2007, p. 120)

Deduction emphasises	Induction emphasises
• Scientific principles • Moving from theory to data • The need to explain casual relationships between variables • The collection of quantitative data • The application of controls to ensure validity of data • The operationalisation of concepts to ensure clarity of definition • A highly structured approach • Researcher independence of what is being researched • The necessity to select samples of sufficient size in order to generalise conclusions	• Gaining an understanding of the meanings humans attach to events • A close understanding of the research context • The collection of qualitative data • A more flexible structure to permit changes of research emphasis as the research progresses • A realisation that the researcher is part of the research process • Less concern with the need to generalise

Table 3-2 Fundamental differences between quantitative and qualitative research strategies (Bryman and Bell 2003, p. 25)

	Quantitative	Qualitative
Principal orientation to the role of theory in relation to research	Deductive; testing of theory	Inductive; generation of theory
Epistemological orientation	Natural science model, in particular positivism	Intepretivism
Ontological orientation	Objectivism	Constructionism

3.3 Research design

This study was designed according to the research perspective stated in the previous section. As a result, quantitative approaches, techniques and tools are employed. Figure 3-2 presents the outline of this study. The literature review is the first task to be conducted. It provides an overview about the previous work in the field of supply chain management (specifically distribution network design), shows the research trend, specifies the methods employed by scholars and identifies research gaps. The information from the literature review helps to design and scope the analysis. The main technique used in this thesis is analytical modelling. It is used to represent the inventory and ordering systems of the distribution network of a supply chain. The result from the analytical model is cross-checked by a simulation model. Both analytical and simulation models are validated by data from the real supply chain. Finally, the research results, implications and conclusions are drawn.
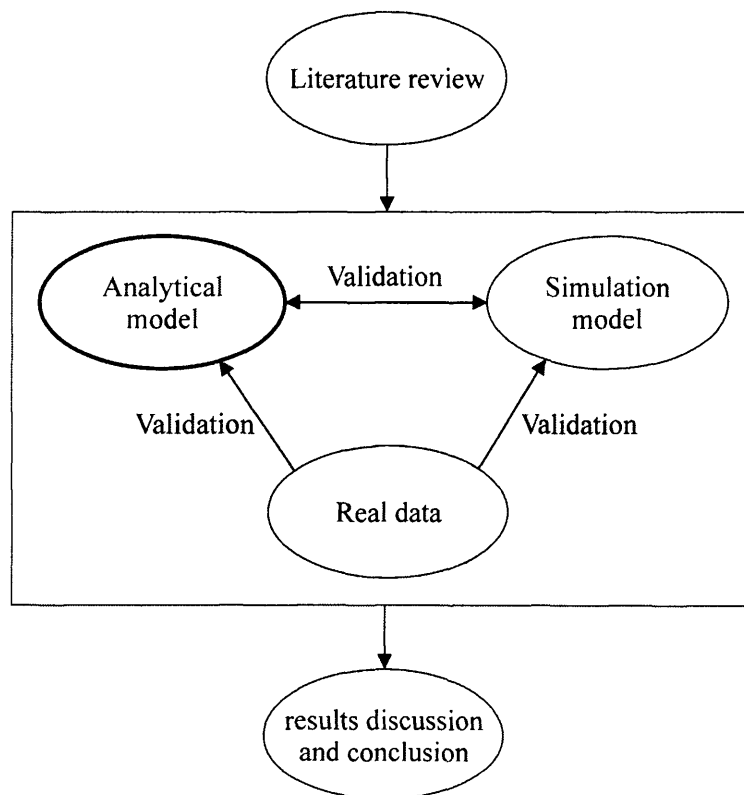
Figure 3-2 Research design

3.3.1 *Literature review*

The literature is searched by using keywords and author names for books, journal papers and conference papers on the major databases such as Cardiff University Library Catalogue, ISI Web of knowledge, EBSCOhost, Google Scholar, Scopus, Science Direct and Emerald. Another efficient method is to trace related work from the references of some key papers. Although from Figure 3-2 the literature review is shown to be done at the beginning of the study, it should also be conducted throughout the study to keep the knowledge up to date.

3.3.2 *Models*

Models are used to describe a system and the relationships between elements within it. The behaviour of the system can then be tested and investigated without disturbing the real system. In this thesis the relationship is represented in mathematical language. In general, the mathematical models can be classified according to their nature in the following ways (Kapur 1998, pp. 7-9):

- Linear vs. non-linear: If all operators in a mathematical model exhibit linearity, such as all equations are polynomials of degree 1, the model will be defined as linear. Otherwise, it will be defined as non-linear.

- Deterministic vs. stochastic (probabilistic): This is concerned with whether or not randomness is taking into account. If all variables are described by unique values and the model always produces the same result for a given input, the model will be defined as deterministic. If the randomness is present and variables are described by a probability distribution, the model will be classified as stochastic.

- Static vs. dynamic: The static model is not concerned with the constituent of time, while the dynamic model is.

- Discrete vs. continuous: A discrete model observes the values of the system state variables at separate points in time such as hourly, daily or weekly, for example. Conversely, a continuous model observes the system continuously.

- Time invariant vs. time-varying: The output of a time-invariant model does not depend explicitly on time; otherwise it is a time-varying model.

In this study, the inventory and ordering system of a distribution network is modelled mathematically and it is assumed to be linear. The demand that activates the inventory and ordering system is uncertain. The demand's error terms will be described by a normally distributed stochastic process. Attention is paid to the inventory of consumer products, which makes it reasonable to assume that the inventory system is monitored in discrete time unlike petrochemical products where production is probably monitored continuously. The parameters such as the lead-times and the correlations of demand are assumed to be stable over time. Thus, the model employed in this study can be defined as being linear, stochastic, dynamic, discrete and time-invariant.

As mentioned before, both analytical and simulation models will be exploited in this study as they have their own strengths and weaknesses (Bertrand and Fransoo 2006). An analytical model is a mathematical model that has a closed form solution. According to Bertrand and Fransoo (2006), the strengths of the analytical model are that it facilitates an analysis and is proof-oriented. It is also useful in obtaining insights about a system. However, sometimes it is difficult or even impossible to model a system that is highly complex. The simulation model, on the other hand, is able to model the more complex relationships between entities in a system. The simulation model is good for exploration and observation of systems behaviour. The system with an uncertain nature, which is normally concerned as a complex system, can be modelled using simulation. Conversely, the simulation model does not provide proofs and the resulting insights may not be clear (Bertrand and Fransoo 2006). However, the researcher could use statistical skills to model, analyse and conclude the results.

This study employs a spreadsheet simulation based on Microsoft® Office Excel. The spreadsheet model is very useful as it can instantly validate the results obtained from the analytical modelling activities that have been undertaken. The Visual Basic Application (VBA) available in Excel is also useful to automate repetitive procedures to collect statistical results.

3.3.3 Theories, methods, techniques and tools

Towill (1982) has stated that "It is generally recognized that an efficient production control system can only be designed and operated if the dynamic behaviour of the

constituent parts is properly understood." In order to achieve this, Control Theory will be employed in this study. Control Theory allows the researcher to systematically investigate the dynamic behaviour of the inventory and ordering systems of a supply chain. It has been used by a number of researchers in this field. Simon (1952), cited in Towill (1982), was the first to employ Control Theory to study inventory problems. Simon applied the concept of Laplace transform to a continuous time system. For the application of Control Theory in discrete time systems, Vassian (195 5) found that the real production and inventory control system are generally discrete and it was more suitable to use z-transform techniques to solve the problems. Recent works that apply discrete control theory include Disney and Towill (2002), Dejonckheere et al. (2003) and Hosoda and Disney (2006), for example.

Control engineering tools are particularly useful for structuring and developing the analytical model. Block diagrams and difference equations will be used to describe the inventory and ordering system. The z-transform techniques are exploited to convert discrete time-domain signals into complex frequency-domain representations. This is because the calculations in the frequency-domain can be much simpler than those of the time-domain (Bissell 1996). The block diagrams will be manipulated using standard techniques (see Nise 1995 for more detail) to obtain "transfer functions", which relates an error term, which is the system input, to a state variable of interest, which is the system output. Although the control theory approaches are sometimes argued to over simplifies the actual situation, it is the only approach where we can achieve a deep understanding of the dynamic behaviour of a system (Towill 1982). The transfer function can then be exploited to find the expression of the variance of a state variable, which is the key component for the economic performance measures. This procedure has also been exploited by Dejonckheere et al. (2003), Hosoda and Disney (2006), Tsypkin (1964) and Vassian (1955). In this study, Mathematica (® Wolfram Research), a computational software program, is used to facilitate the procedure.

Basic mathematical knowledge is certainly useful for the modelling. This knowledge includes the application of matrices/vectors and Series theory. As the distribution network consists of a set of parallel supply chains, it is simpler to represent the model in vector notation. The obvious benefit of the vector notation is in the forecasting model especially when it is assisted by Series theory. This will be discussed in detail in Chapters 4 and 5.

Statistical knowledge is also important as the real data will be dealt with in Chapter 8. Statistical software will be used to help with statistical analysis. This includes Eviews (Econometric Views), SAS (Statistical Analysis System) and JMulti (www.jmulti.de). All of these programs are used together to validate the results obtained. This is also because each program has its own special features and capacity to deal with different processes.

3.4   Alternative research methods

There are other quantitative and qualitative techniques that can be applied with this study such as action research or surveys. For action research, it is important to manipulate the system to test out a strategy for improvement. One major problem is that the nature of the DND problems does not allow the researchers to manipulate real distribution networks very easily. Even if the researchers had access to a distribution network to change in such a manner, the time scales involved would be prohibitive. On the other hand, the data obtained from surveys may be useful in the exploratory phase to appreciate the customers' points of view. It is, however, not sufficient for the design phase where economic performances need to be evaluated. Therefore, the research framework presented in Section 3.3 is most appropriate for DND because improvements can be identified without disturbing the real system and can generate insightful information about the dynamic behaviour of the system and the economic performance of the proposed design.

Many researchers have applied statistical modelling to distribution network design (Evers and Beier 1993, 1998; Maister 1976; Tallon 1993). The model in this study, however, represents a complex situation where auto- and cross-correlations of the demands between multi-retailers are considered. The variance terms will be very complicated if the statistical approaches are used. Moreover, this research intends to study the closed form of the variances analytically and investigate the structural transition of the demand, where statistical models do not advocate doing so. Therefore, the analytical model is the most appropriate solution amongst all available techniques for the problem addressed in this thesis.

3.5 Research evaluation

The evaluation of business and management research as in this thesis is based on the following universal criteria:

- *Reliability and replication:*

"Reliability is concerned with the question of whether the results of a study are repeatable. The idea of reliability is very close to another criterion of research–replication and more especially replicability", Bryman and Bell (2003, pp. 33, 74-77). To achieve this criterion, the modelling procedure and technique will be clearly explained and references will be given when it is appropriate. There will be many variables involved in the model. Thus, the models parameters and variables need to be clearly declared. Their notations will be listed as shown in the 'summary of general notations' and Appendix B. Importantly, the model assumptions will be clearly stated and provided as will be seen in Chapters 4 and 5 and wherever appropriate. This means other researchers can replicate this study to investigate the reliability of the results and future studies may be able to extend the work in this study.

- *Validity:*

"Validity is concerned with the integrity of the conclusions that are generated from a piece of research", Bryman and Bell (2003, pp. 33-34, 77-78). The internal validity will be first discussed. The internal validity ensures that the study actually measures what it is meant to measure. This study employs two different models: analytical and simulation, both of which are employed to model the same situation. Thus, the results can be cross-checked between them. In this way, the internal validation is achieved. Besides the internal validation, the external validity is also considered in order to answer the question of whether the results from this study can be generalised beyond the specific research context (Bryman and Bell 2003, p. 34). Consequently, real data will be analysed to test the assumptions applied in this research. The model will be validated by an application to the real data. This will also allow practical issues to be learned. The analysis of the real data in Chapter 8 leads to consideration of the reliability criterion. The input data needs to be presented to ensure that the result is reliable. This, however, links to research ethic practices concerning the confidentiality of data and anonymity of the participant. Therefore, the data will be presented in graphical terms and the name of the company will be omitted.

3.6   Summary

A stochastic analytical model will be employed to represent the distribution network of a supply chain. This model will allow the researcher to observe the system dynamically and to evaluate the economic performance analytically. A discrete time simulation model will also be employed in order to allow a quick validation of the result from the analytical model. Both analytical and simulation models will, again, be validated against real data. In this way, both internal and external validities that are essential requirements for quantitative research design can be achieved. By following the research framework presented in this chapter, high quality research can be achieved and the integrity of the results will contribute to knowledge in this area.

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Chapter 4

Fundamentals of the distribution network modelling

4.1   Introduction

This chapter provides the foundations of the distribution network modelling presented in this thesis. First, the Vector Auto-Regressive processes, which will be used to represent the demand process faced by the multi-retailer, are introduced. The Order-Up-To inventory replenishment policy and the Minimum Mean Square Error forecasting technique are described. Meanwhile, the benefit of vector notation for the forecasting method is identified. Finally, the two types of costs that are used as measures of performance for the network consolidation decision are specified.

4.2   Vector Auto-Regressive and Vector Auto-Regressive Moving Average processes

The Vector Auto-Regressive (VAR) and the Vector Auto-Regressive Moving Average (VARMA) processes have been used to describe the evolution of and the inter-relationships between multiple time series. The VAR process generalises the Auto-Regressive (AR) process that considers only a single time series. The VARMA generalises the Auto-Regressive Moving Average (ARMA) process that considers only a single time series. The VAR and VARMA models have largely been applied in Economics for forecasting and structural analysis (see Bikker 1998; Groenewold and Hagger 2003; Kurmann 2007; Lütkepohl 1993; Micola and Bunn 2007; Note 2003; Tang et al. 2010 for

example). It has also been applied in weather forecasting, agriculture, tourism and transportation (see Adeyemi et al. 1979; Akinboade and Braimoh 2010; Alfaro and Cid 1999; Andersson 1995; Chandra and Al-Deek 2009; and Kulshreshtha et al. 2001 for examples of the application of VAR process). This thesis introduces the use of the VAR model to the Distribution Network Design problem for the first time.

4.2.1 VAR(p) model

An n-dimensional vector auto-regressive model of order p, VAR(p), is given by

$$\mathbf{D}_t = \mathbf{C} + \mathbf{A}_1 \cdot \mathbf{D}_{t-1} + \mathbf{A}_2 \cdot \mathbf{D}_{t-2} + ... + \mathbf{A}_p \cdot \mathbf{D}_{t-p} + \mathbf{U}_t , \qquad (4.1)$$

where $\mathbf{D}_t$ is an ($n \times 1$) random vector, $\mathbf{C}$ is an ($n \times 1$) fixed vector of intercept terms, $\mathbf{A}_j$ are ($n \times n$) coefficient matrices in which their elements represent the correlations between a pair of the n time series, $\mathbf{U}_t$ is an ($n \times 1$) vector of error terms and " $\cdot$ " symbolises a multiplication of matrices/vectors. In this thesis, it is assumed that the demand process follows the vector auto-regressive of the first order, VAR(1). This assumption will be validated in Chapter 8. This demand process is, thus, given by

$$\mathbf{D}_t = \mathbf{C} + \mathbf{A} \cdot \mathbf{D}_{t-1} + \mathbf{U}_t , \qquad (4.2)$$

where $\mathbf{A}$ is an ($n \times n$) coefficient matrix for the one-period lagged. For convenience, the notation $\mathbf{A}$ is used instead of $\mathbf{A}_1$ as only the order $p = 1$ will be considered in this thesis. The VAR(1) process will be explained through an example in which there are two random variables under consideration. The VAR(1) model presented in Equation (4.2) can be equivalently written by the following system of two equations.

$$\left. \begin{array}{l} d_{1,t} = \mu_1 + \phi_{11}d_{1,t-1} + \phi_{12}d_{2,t-1} + \varepsilon_{1,t} \\ d_{2,t} = \mu_2 + \phi_{21}d_{1,t-1} + \phi_{22}d_{2,t-1} + \varepsilon_{2,t} \end{array} \right\} \qquad (4.3)$$

$d_{1,t}$ and $d_{2,t}$ are random variables, μ_1 and μ_2 are fixed intercept terms, ϕ_{ij} are correlation coefficients and $\varepsilon_{1,t}$ and $\varepsilon_{2,t}$ are independently and identically distributed (white noise)

random processes. In this thesis, it is assumed that $\varepsilon_{i,t}$ is normally distributed with zero mean and unit variance. From Equation (4.3), the value of the variable $d_{1,t}$ at time t is explained by one-period lagged values of itself and one-period lagged of the variable $d_{2,t}$ and by the random error $\varepsilon_{1,t}$ in the current period. In the same manner, the value of the variable $d_{2,t}$ is explained by one-period lagged values of itself and one-period lagged of the variable $d_{1,t}$ and by the random error $\varepsilon_{2,t}$.

The estimated correlation coefficient (ϕ_{ij}) can be obtained from the Pearson product-moment correlation coefficient (Rodgers and Nicewander 1988), which has been used to measured the linear dependence between two variables. In this study, it is a linear dependence for the one-period lagged values. For historical data of finite N time periods, when $i = j$, the estimated correlation coefficients at lag p are actually the auto-correlation and are defined by the following equations (Box and Jenkins 1976, pp. 200-201)

$$\phi_{ii} = \frac{\sum_{t=1}^{N-p}(d_{i,t} - \overline{d}_i)(d_{i,t+p} - \overline{d}_i)}{\sum_{t=1}^{N}(d_{i,t} - \overline{d}_i)^2}, \tag{4.4}$$

where $d_{i,t}$ is the value of the variable d_i at time t and $\overline{d}_i = \frac{\sum_{t=1}^{N} d_{i,t}}{N}$ is the average value of the variable d_i.

When $i \neq j$, the correlation coefficients represent the cross-correlations between a pair of random variables. The cross-correlation coefficient at lag p is estimated by

$$\phi_{ij} = \frac{\sum_{t=1}^{N-p}(d_{i,t+p} - \overline{d}_i)(d_{j,t} - \overline{d}_j)}{\sqrt{\sum_{t=1}^{N}(d_{i,t} - \overline{d}_i)^2}\sqrt{\sum_{t=1}^{N}(d_{j,t} - \overline{d}_j)^2}}, \tag{4.5}$$

where $\overline{d}_i = \frac{\sum_{t=1}^{N} d_{i,t}}{N}$ and $\overline{d}_j = \frac{\sum_{t=1}^{N} d_{j,t}}{N}$.

It must be noted that Equations (4.4) and (4.5) are meant to illustrate the meaning of the correlations. They can only be applied to the case where there are two decentralised locations under consideration. In practice, a computer program for Econometric or multiple time series analysis has a function involving matrix operations to perform this calculation.

4.2.2   VARMA(1,1) model

The VARMA with first order VAR and first order moving average, VARMA(1,1), can be expressed as

$$\mathbf{D}_t = \mathbf{C} + \mathbf{A} \cdot \mathbf{D}_{t-1} + \mathbf{U}_t + \mathbf{M} \cdot \mathbf{U}_{t-1}, \tag{4.6}$$

where $\mathbf{D}_t$ is an $(n \times 1)$ random vector, $\mathbf{C}$ is an $(n \times 1)$ fixed vector of intercept terms, $\mathbf{A}$ is an $(n \times n)$ coefficient matrix, $\mathbf{U}_t$ is an $(n \times 1)$ vector of error terms and $\mathbf{M}$ is an $(n \times n)$ matrix of moving average parameters. It is introduced in this section as a foundation for the investigation of the structural transformation of the VAR(1) process when it is applied in the multi-echelon supply chain context discussed in Chapter 5.

4.3   Order-Up-To (OUT) inventory replenishment policy with MMSE forecasting

This section begins with some assumptions about the inventory replenishment process. It is assumed that all participants including retailers and distribution centres (DCs) review their inventory periodically. Although the reviewing period can be of any length (for example daily, weekly or monthly), it is assumed to be of the same length and to begin at the same point of time for all participants. A normal cycle for inventory replenishment at a particular site starts with the receiving of orders that have been placed previously. Then the demand is fulfilled. The inventory level and the work-in-progress (WIP) are updated. Finally, the quantity of orders to be placed with its supplier is decided. Figure 4-1 illustrates the sequence of these events. The lead-time at each location is assumed to be known and constant. Also, the lead-time is a non-negative integer.
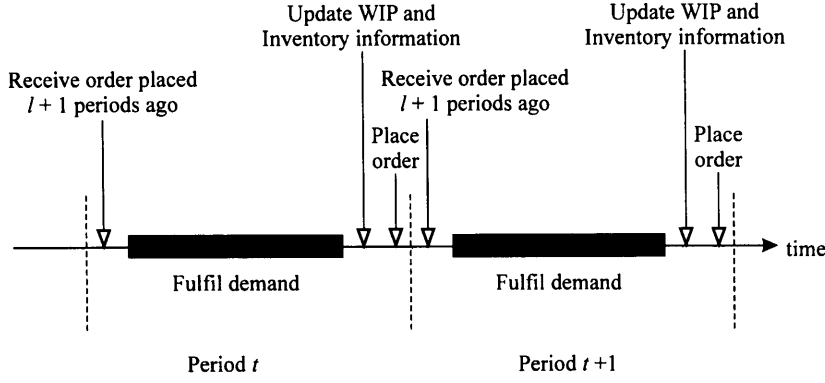
Figure 4-1 Sequence of events in the inventory replenishment system

4.3.1 OUT replenishment policy

All members in the distribution network are assumed to employ the Order-Up-To (OUT) inventory replenishment rule. The OUT policy is a simple but effective policy for controlling high volume products. It has been used in many studies for example Chen et al. (2000), Dejonckheere et al. (2003), Hosoda and Disney (2006), Lee et al. (1997) and Lee et al. (2000). Considering a particular location in the supply chain, the replenishment order at time t, O_t, is given by

$$O_t = \underbrace{\overline{T} + \hat{D}_t}_{\text{OUT level}} - I_t - W_t.$$ (4.7)

Note that the notations used in this section represent the system's variables in general not for a specific level or location in a supply chain. The term $\overline{T} + \hat{D}_t$ is the OUT level, where $\overline{T}$ is the target inventory level and $\hat{D}_t$ is the expected demand during the lead-time and review period calculated at time t. The optimal target inventory level $(\overline{T}^*)$ is defined in Section 4.4.1. I_t is the inventory level at time t and W_t is the work-in-progress (or orders that have been placed but not yet received) at time t. Let l be the inventory replenishment lead-time from the upstream supplier to this particular site. This participant will receive the shipment of the order (that it has placed at time t) at the beginning of time period $t + l + 1$. The relationship expressed in Equation (4.7) allows the order to be negative where the surplus inventory is returned to the supplier without penalty. This assumption is also adopted by Chen et al. (2000), Hosoda and Disney (2006) and Lee et al. (1997), for

example. However, the mean demand level can be set to be high enough when compared to the demand variance to ensure that probability of negative orders is negligible (Johnson and Thompson 1975).

The inventory level at time t, I_t, is described by the inventory balance equation,

$$I_t = I_{t-1} + O_{t-(l+1)} - D_t,$$

(4.8)

where D_t is the demand for period t. Again, from this relationship, the inventory level can be negative in which the excess demand can be either backlogged or satisfied by an alternative source at extra charge (shortage cost). A unit backlog (or shortage) cost per time period is applied. The backlog (or shortage) cost is explained in more detail in Section 4.4.1.

The work-in-progress (WIP) at time t, W_t, is given by

$$W_t = \begin{cases} 0 & \text{if the lead-time } l = 0, \\ \sum_{j=1}^{l} O_{t-j} & \text{otherwise.} \end{cases}$$

(4.9)

A number of scholars (Chen et al. 2000; Hosoda and Disney 2006; Lee et al. 2000) express the OUT policy in the following equations;

$$O_t = D_t + (S_t - S_{t-1}),$$

(4.10)

$$\text{where } S_t = \hat{D}_t + k\sigma_l,$$

(4.11)

in which S_t is the OUT level for time period t, k is a constant used to set up the desired service level and σ_l is the standard deviation of the demand over the lead-time l. Hosoda and Disney (2006) have shown that the expressions of the OUT policy in Equation (4.7) and in Equations (4.10) are identical under certain circumstances. This is also true for this study where optimum $k\sigma_l$ (or T^* which is defined in Section 4.4.1) is employed. Both presentations of the OUT policy will be used in this thesis. Mainly, the presentation in

Equation (4.7) is used as the order rate and the inventory level are explicitly presented. The other presentation in Equation (4.10) is used in the proof of the demand transition discussed in Section 5.6 owing to the simplicity of its presentation.

4.3.2 MMSE forecasting

It is assumed that all participants in the supply chain apply the Minimum Mean Squared Error (MMSE) forecasting. The MMSE forecast for this study can be described by the conditional expectation of the future demand that minimises the forecast error as the demand's error term is normally distributed i.i.d white noise with a zero mean (Box and Jenkins 1976; Pindyck and Rubinfeld 1976). This forecasting method exploits knowledge about the structure of the demand to produce a forecast. For example, if the demand faced by two locations from the same level of a supply chain follows the VAR(1) process presented in Equation (4.3) and both locations have lead-time l, the conditional expected demand over the lead-time and review period for location i calculated at time t can be expressed as

$$
\begin{aligned}
\hat{d}_{i,t} &= E[d_{i,t+1} + d_{i,t+2} + d_{i,t+3} + \ldots + d_{i,t+(l+1)} \mid \tau_t] \\
&= E[d_{i,t+1} \mid \tau_t] + E[d_{i,t+2} \mid \tau_t] + E[d_{i,t+3} \mid \tau_t] + \ldots + E[d_{i,t+(l+1)} \mid \tau_t], \ \forall\, i \in \{1,2\},
\end{aligned}
\tag{4.12}
$$

where $E[.]$ is a function for an expected value, and $\tau_t = \{d_{i,t}, d_{i,t-1}, d_{i,t-2}, \ldots\}$ is the historical demands of the two locations, $i = \{1, 2\}$ Note that given $d_{i,t}$ is known, it will follow that $E[d_{i,t} \mid d_{i,t}] = d_{i,t}$. As the error, $\varepsilon_{i,t}$, is i.i.d. white noise with a mean of zero, thus $E[\varepsilon_{i,t}] = 0$.

For VAR(1) process in Equation (4.3):

If $l = 0$, it follows that

$$
\begin{aligned}
\hat{d}_{i,t} &= E[d_{i,t+1} \mid \tau_t] \\
&= \mu_i + \phi_{ii} d_{i,t} + \phi_{ij} d_{j,t}, \ \forall\, i, j = \{1, 2\} \text{ and } i \neq j.
\end{aligned}
\tag{4.13}
$$

If $l = 1$, it follows that

$$\hat{d}_{i,t} = E[d_{i,t+1} \mid \tau_t] + E[d_{i,t+2} \mid \tau_t]$$
$$= 2\mu_i + (\phi_{ii} + \phi_{ii}^2 + \phi_{ij}\phi_{ji})d_{i,t} + (\phi_{ij} + \phi_{ii}\phi_{ij} + \phi_{ij}\phi_{jj})d_{j,t}, \forall\, i,j = \{1,2\} \text{ and } i \neq j.$$

(4.14)

If $l = 2$, it follows that

$$\hat{d}_{i,t} = E[d_{i,t+1} \mid \tau_t] + E[d_{i,t+2} \mid \tau_t] + E[d_{i,t+3} \mid \tau_t]$$
$$= 3\mu_i + (\phi_{ii} + \phi_{ii}^2 + \phi_{ii}^3 + \phi_{ij}\phi_{ji} + 2\phi_{ii}\phi_{ij}\phi_{ji} + \phi_{ij}\phi_{ji}\phi_{jj})d_{i,t} +$$

(4.15)

$$(\phi_{ij} + \phi_{ii}\phi_{ij} + \phi_{ij}\phi_{jj} + \phi_{ii}^2\phi_{ij} + \phi_{ii}\phi_{ij}\phi_{jj} + \phi_{ij}^2\phi_{ji} + \phi_{ij}\phi_{jj}^2)d_{j,t}, \forall\, i,j = \{1,2\} \text{ and } i \neq j.$$

Equations (4.13) to (4.15) demonstrate how the formulas of the conditional expected demand become increasingly complicated when lead-time l increases. Alternatively, the MMSE forecast of the VAR(1) demands could be obtained while the VAR(1) demand is maintained in its vector notation, as in Equation (4.2). The vector notation allows convenient derivation of the general expression of the MMSE forecast for the VAR(1) demand that involves n locations. The conditional expected demand over arbitrary lead-time l and review period for n locations can be presented by an $(n \times 1)$ vector, $\hat{\mathbf{D}}_t$, as follows

$$\hat{\mathbf{D}}_t = E[\mathbf{D}_{t+1} + \mathbf{D}_{t+2} + \mathbf{D}_{t+3} + .. + \mathbf{D}_{t+(l+1)} \mid \tau_t]$$
$$= (l+1)\mathbf{C} + (\mathbf{A} + \mathbf{A}^2 + \mathbf{A}^3 + ... + \mathbf{A}^{l+1}) \cdot \mathbf{D}_t$$
$$= (l+1)\mathbf{C} + \left(\sum_{j=1}^{l+1} \mathbf{A}^j\right) \cdot \mathbf{D}_t$$

(4.16)

$$= (l+1)\mathbf{C} + \mathbf{A} \cdot (\mathbf{I}_N - \mathbf{A}^{l+1}) \cdot [\mathbf{I}_N - \mathbf{A}]^{-1} \cdot \mathbf{D}_t,$$

where the $\mathbf{I}_N$ is an $(n \times n)$ identity matrix and $[\mathbf{X}]^{-1}$ denotes the inverse of matrix $\mathbf{X}$. The τ_t is the demand history of all n locations which is given by $\tau_t = \{\mathbf{D}_{i,t}, \mathbf{D}_{i,t-1}, \mathbf{D}_{i,t-2}, ...\}$ where $i = \{1, 2, ..., n\}$. The use of vector notation has clearly simplified the general expression of the MMSE forecast for the VAR(1) demand. It also has a tremendous effect on simplifying the block diagram of the multi-location and multi-echelon distribution network.

50

Equation (4.16) can only be applied at the retailer level. Thus the MMSE forecasting is now generalised so that it is applicable for any level of the supply chain. This will be useful for the MMSE forecasting for the DCs (as will be discussed in Sections 5.3.2 and 5.3.3) and other players in the supply chain including factories and suppliers (which will be discussed in Section 5.6). However, the forecast in this section will be limited to cases where all members at the same echelon have a 'common lead-time'. It is assumed that the upstream players know that the demand process follows a VAR(1) model with known parameters μ_i, ϕ_{ij} and variances of error terms. This assumption is reasonable by means of periodic meetings between the retailers and the upstream players or by providing the upstream player with demand history (Lee et al. 2000).

Let L_C be the common lead-time of the local echelon, L_D be the sum of the common lead-times and review periods of all downstream echelons and L_T be the sum of the common lead-times and review periods of the local and all downstream echelons. For example if the retailer's common lead-time is l, the DC's common lead-time is L and the DC echelon is considered, it follows that $L_T = (l+1) + (L+1) = l + L + 2$ and $L_D = (l+1)$. It is assumed that all players in the supply chain employ the OUT inventory replenishment policy with the MMSE forecasting scheme. As a result, the conditional expected demand over the lead-time and review period calculated at time t for a certain echelon of a supply chain, $\hat{\mathbf{D}}_t^{local}$, is given by

$$\hat{\mathbf{D}}_t^{local} = (L_C + 1)\mathbf{C} + \left(\sum_{j=L_D+1}^{L_T} \mathbf{A}^j \right) \cdot \mathbf{D}_t, \qquad (4.17)$$

where $\mathbf{D}_t$ is an $(n \times 1)$ VAR(1) demand vector for time period t.

As has been shown, the formulas for the MMSE forecasting presented in this section are only applicable to cases where all players in a particular echelon have the same lead-time. The case where each player has an arbitrary lead-time is more complicated and will be presented in detail in Chapter 5.

51

4.4 Cost model and performance measures for the network consolidation

Two main types of costs are considered in this study; inventory and capacity. The cost model is based on Disney et al. (2006). Note that the notations used in this section represent the system's variables in general not for a specific level or location in a supply chain.

4.4.1 Inventory costs

The inventory cost includes the inventory holding and backlog (or shortage) costs. This study considers the case where a fixed ordering cost is either absent, negligible or a constant incurred in every period (regardless of how much is ordered) and the unit inventory holding and backlog costs are constant over time. The inventory holding and backlog costs are assumed to be piece-wise linear and convex. From Equation (4.8), the inventory level can fall below zero and this means demand is not completely satisfied. The main model in Chapter 5 will consider a two-level supply chain in which the players include the retailers and the DCs. For the retailers, the demand that cannot be satisfied in each period is accumulated in a backlog until the inventory becomes available. A backlog cost per time period is applied to each unit backlogged. For the DCs, the excess demand is met by acquiring some units from an alternative source. Each unit obtained from the alternative source is paid for an extra rate which can be considered as a penalty for shortage. The DC is responsible for resupplying the alternative source when its inventory becomes available. In this manner, the DC promises shipment to the retailer. In Section 5.6 where higher supply chain level (factories and suppliers) is considered, the above assumption used for the DCs will also be applied. This assumption on inventory shortage at the retailers and at the DCs is also applied by Gavirneni (2006) and Lee et al. (2000).

The inventory cost incurred in each time period, I_t, can be expressed as follows.

$$\text{Inventory cost for period } t = \begin{cases} H(I_t), & \text{when } I_t \geq 0, \\ B(-I_t), & \text{when } I_t < 0, \end{cases} \text{ or } I_t = H(I_t)^+ + B(-I_t)^+, \qquad (4.18)$$

where I_t is the inventory level at time t, H and B are the unit costs per period for holding inventory and inventory backlog (or shortage) respectively and the notation $(x)^+$ is used to describe $\max(0, x)$. Figure 4-2 illustrates the inventory cost.
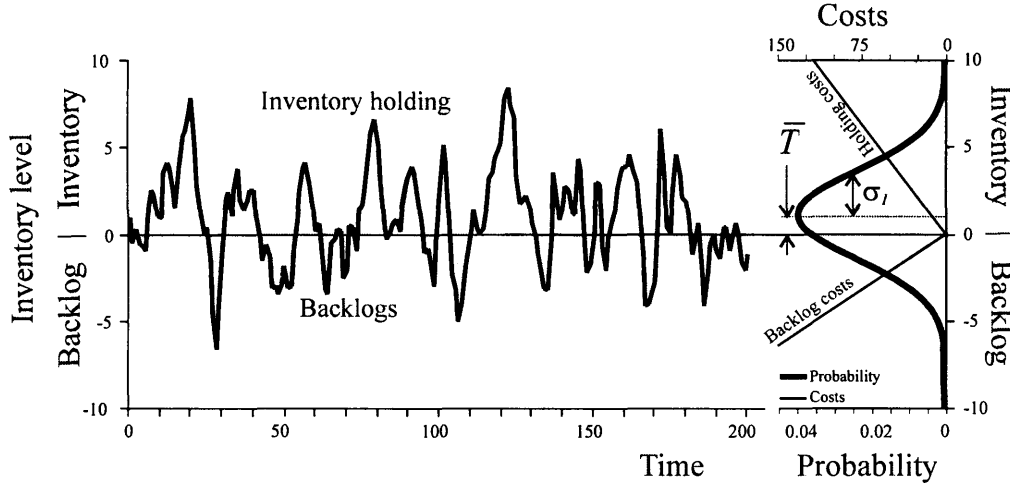
Figure 4-2 How the inventory costs are generated over time (Disney et al. 2006)

As the error terms $\varepsilon_{i,t}$ are assumed to be normally distributed and it is assumed that a linear system exists then the distribution of the inventory levels (and the order rates which will be considered later on) will also be normally distributed. Therefore, the expected inventory holding cost can be derived via the probability density function of the normal distribution and is, thus, given by

$$
\begin{aligned}
\text{Holding cost} &= \frac{H}{\sigma_I \sqrt{2\pi}} \int_0^\infty \exp\left[-\frac{(x - \bar{T})^2}{2\sigma_I^2} \right] x\, dx \\
&= \frac{H}{2} \left(e^{-\frac{\bar{T}^2}{2\sigma_I^2}} \sqrt{\frac{2}{\pi}} \sigma_I + \bar{T} + \bar{T}\, \text{erf}\left[\frac{\bar{T}}{\sqrt{2}\sigma_I} \right] \right),
\end{aligned}
\tag{4.19}
$$

where $\bar{T}$ is the target inventory level, σ_I is the standard deviation of the inventory level and erf[.] is the error function.

53

Similarly, the expected backlog cost is given by

$$\text{Backlog cost} = \frac{B}{\sigma_I\sqrt{2\pi}}\int_{-\infty}^{0}\exp\left[-\frac{(x-\overline{T})^2}{2\sigma_I^2}\right](-x)dx$$

$$= \frac{B}{2}\left(e^{-\frac{\overline{T}^2}{2\sigma_I^2}}\sqrt{\frac{2}{\pi}}\sigma_I - \overline{T} + \overline{T}\operatorname{erf}\left[\frac{\overline{T}}{\sqrt{2}\sigma_I}\right]\right). \qquad (4.20)$$

The expected total inventory cost, $I_{\pounds}$, is the sum of the holding and backlog costs from Equations (4.19) and (4.20) as follows.

$$I_{\pounds} = \frac{H}{2}\left(e^{-\frac{\overline{T}^2}{2\sigma_I^2}}\sqrt{\frac{2}{\pi}}\sigma_I + \overline{T} + \overline{T}\operatorname{erf}\left[\frac{\overline{T}}{\sqrt{2}\sigma_I}\right]\right) + \frac{B}{2}\left(e^{-\frac{\overline{T}^2}{2\sigma_I^2}}\sqrt{\frac{2}{\pi}}\sigma_I - \overline{T} + \overline{T}\operatorname{erf}\left[\frac{\overline{T}}{\sqrt{2}\sigma_I}\right]\right). \qquad (4.21)$$

In order to choose the best $\overline{T}$ that minimised the total inventory related costs, the inventory cost in Equation (4.21) is partially differentiated with respect to $\overline{T}$. Then it is solved for zero gradient to obtain the optimum target inventory level ($\overline{T}^*$). The $\overline{T}^*$ is given by

$$\overline{T}^* = \sigma_I\sqrt{2}\operatorname{erf}^{-1}\left[\frac{B-H}{B+H}\right] = \sigma_I\Phi^{-1}\left[\frac{B}{B+H}\right], \qquad (4.22)$$

where $\operatorname{erf}^{-1}[.]$ is the inverse error function and $\Phi^{-1}[.]$ is the inverse standard normal cumulative distribution function. Note that $\Phi^{-1}[x] = \sqrt{2}\operatorname{erf}^{-1}[2x-1]$. The inventory cost will be minimised when the optimum target inventory level is applied. The optimal inventory cost per period, $I_{\pounds}^*$, is given by

$$I_{\pounds}^* = \sigma_I(B+H)\varphi\left[\Phi^{-1}\left[\frac{B}{B+H}\right]\right], \qquad (4.23)$$

where $\varphi[.]$ is the density function of the standard normal distribution. Note that $\varphi[x] = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}$, $x \in \Re$. Note also that $\overline{T}^*$ and $I_{\pounds}^*$ in Equations (4.22) and (4.23) are

linear functions of the standard deviation of the inventory levels as $\Phi^{-1}\left[\dfrac{B}{B+H}\right]$ and

$(B+H)\varphi\left[\Phi^{-1}\left[\dfrac{B}{B+H}\right]\right]$ are simple numbers related to the unit inventory backlog and holding costs.

4.4.2 Capacity costs

The capacity costs have been added to capture the opportunity costs associated with the Bullwhip Effect. The normal capacity level is set to $(\mu_d + S)$ where μ_d is the mean demand and S is spare capacity above (or below) the mean demand. If the order quantity is smaller than the normal capacity, this is considered as a lost capacity situation which has opportunity costs. Then again, if the order is larger than the normal capacity, it will be paid at premium; either for overtime capacity or subcontractors. It is also assumed that piece-wise linear and convex lost capacity and overtime costs exist. The capacity cost incurred in each time period, $C_£$, can be described as

$$\text{Capacity cost for period } t = \begin{cases} N((\mu_d + S) - O_t), & \text{when } O_t \le (\mu_d + S), \\ P(O_t - (\mu_d + S)), & \text{when } O_t > (\mu_d + S), \end{cases} \tag{4.24}$$

$$\text{or } C_£ = N(-(O_t - (\mu_d + S)))^+ + P((O_t - (\mu_d + S)))^+,$$

where O_t is the order rate at time t, N and P are the unit costs of lost capacity and of over-time working respectively. The lost capacity cost is mainly described by inefficient usage of labour, space and material handling equipment. Figure 4-3 illustrates the capacity cost.

The expected over-time cost per period is given by

$$\text{Over - time cost} = \frac{P}{\sigma_O \sqrt{2\pi}} \int_0^\infty \exp\left[-\frac{(x+S)^2}{2\sigma_O^2}\right] x\, dx$$

$$= \frac{P}{2}\left(e^{-\frac{S^2}{2\sigma_O^2}} \sqrt{\frac{2}{\pi}}\sigma_O - S + S\, \text{erf}\left[\frac{S}{\sqrt{2}\sigma_O}\right]\right), \tag{4.25}$$

where σ_O is the standard deviation of the order rate.
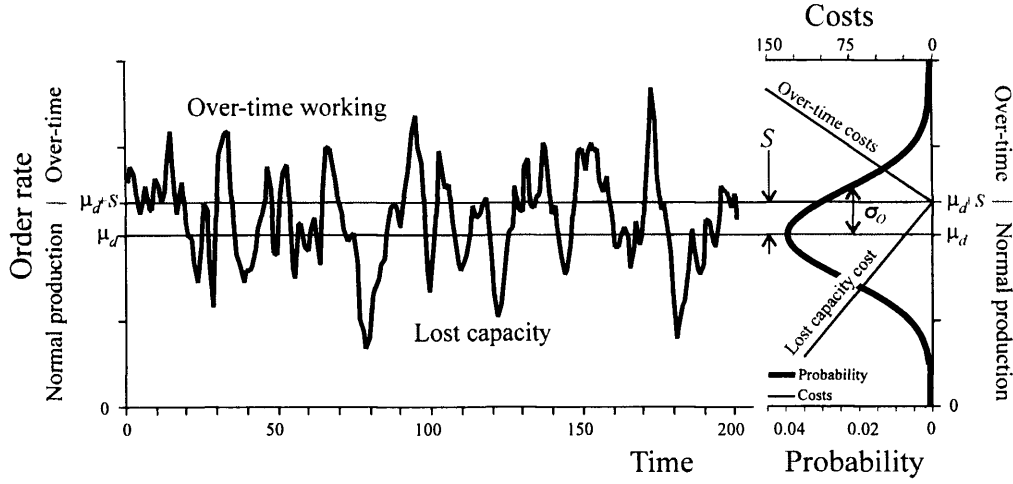
Figure 4-3 How the capacity costs are generated over time (Disney et al. 2006)

The expected lost capacity cost per period is given by

$$\text{Lost capacity cost} = \frac{-N}{\sigma_O \sqrt{2\pi}} \int_{-\infty}^{0} \exp\left[-\frac{(x+S)^2}{2\sigma_O^2}\right] x\,dx$$

$$= \frac{N}{2}\left(e^{-\frac{S^2}{2\sigma_O^2}}\sqrt{\frac{2}{\pi}}\sigma_O + S + S\,\text{erf}\left[\frac{S}{\sqrt{2}\sigma_O}\right]\right). \tag{4.26}$$

The expected total capacity cost, $C_£$, is the sum of the costs of lost capacity and of over-time working from Equations (4.25) and (4.26), which is given by

$$C_£ = \frac{N}{2}\left(e^{-\frac{S^2}{2\sigma_O^2}}\sqrt{\frac{2}{\pi}}\sigma_O + S + S\,\text{erf}\left[\frac{S}{\sqrt{2}\sigma_O}\right]\right) + \frac{P}{2}\left(e^{-\frac{S^2}{2\sigma_O^2}}\sqrt{\frac{2}{\pi}}\sigma_O - S + S\,\text{erf}\left[\frac{S}{\sqrt{2}\sigma_O}\right]\right). \tag{4.27}$$

In a similar treatment to the inventory cost, the capacity cost in Equation (4.27) is partially differentiated with respect to S. Then it is solved for zero gradient to obtain the optimum spare capacity (S^*) which minimises the capacity cost. The optimum spare capacity is given by

$$S^* = \sigma_O \sqrt{2}\,\text{erf}^{-1}\left[\frac{P-N}{N+P}\right] = \sigma_O \Phi^{-1}\left[\frac{P}{N+P}\right]. \tag{4.28}$$

56

The capacity cost is minimised when the optimum spare capacity is applied. The optimal capacity cost per period, $C_\pounds^*$, is given by

$$C_\pounds^* = \sigma_O(N+P)\varphi\left[\Phi^{-1}\left[\frac{P}{N+P}\right]\right].$$

(4.29)

Also note that S^* and $C_\pounds^*$ in Equations (4.28) and (4.29) are linear functions of the standard deviation of the order rates as $\Phi^{-1}\left[\frac{P}{N+P}\right]$ and $(N+P)\varphi\left[\Phi^{-1}\left[\frac{P}{N+P}\right]\right]$ are simple numbers related to the unit lost capacity and over-time working costs.

4.4.3 The ratio of costs

One of the main objectives of this thesis is to evaluate the benefit of the consolidation of a distribution network. In order to investigate the impact of network consolidation, the ratio of costs between the decentralised and centralised systems is used as an economic measure. This includes the ratios of the inventory costs, the capacity costs and the total costs.

The ratio of the inventory costs is denoted by *Ratio*[*Inv*]. The optimal inventory cost for each location is obtained from Equation (4.23). Thus, the *Ratio*[*Inv*] can be expressed as

$$Ratio[Inv] = \frac{\text{Inventory costs in the decentralised system}}{\text{Inventory costs in the centralied system}}.$$

(4.30)

The numerator is the sum of the inventory costs of all players in the decentralised system.

The ratio of the capacity costs is denoted by *Ratio*[*Cap*]. The optimal capacity cost for each location is obtained from Equation (4.29). Thus, the *Ratio*[*Cap*] can be expressed as

$$Ratio[Cap] = \frac{\text{Capacity costs in the decentralised system}}{\text{Capacity costs in the centralied system}}.$$

(4.31)

Similarly, the numerator is the sum of the order costs of all players in the decentralised system.

The total cost is the sum of inventory and capacity costs. The ratio of the total costs is denoted by *Ratio[Total]*. The optimal total cost for each location is obtained from Equations (4.23) and (4.29). The *Ratio[Total]* is given by

$$Ratio[Total] = \frac{\text{Total cost of the decentralised system}}{\text{Total cost of the centralied system}}, \qquad (4.32)$$

where the numerator is the sum of the order costs and the inventory costs of all players in the decentralised system.

It is found that, when the cost structure provided in Section 4.4.1 and 4.4.2 is applied with fixed unit costs at all locations, the only factor that affects the value of *Ratio[Inv]* is the inventory variance and of the *Ratio[Cap]* is the order variance. The procedure for finding the closed form of the variances of the inventory levels and of the order rates is presented in Chapter 5.

4.5   Summary

The fundamentals of the distribution network modelling that will be applied throughout this thesis have been introduced in this chapter. The VAR models and the formulas for obtaining its correlation coefficients have been expressed. The OUT inventory replenishment policy has been described by two different presentations as the two presentations will be used for different purposes. The MMSE forecasting for a simple lead-time case has been explained to provide a basis for more complex situations described in the following chapters. These include the situation when lead-times at all locations can be different in both decentralised and centralised systems. The optimal inventory and capacity costs have been presented. The ratios of costs including *Ratio[Inv]* and *Ratio[Cap]*, which will be used as the economic performance measures for the consolidation decision of a distribution network, have also been described.

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Chapter 5

The analytical model

5.1 Introduction

In this chapter, a mathematical model for DND will be presented. A proposed procedure for MMSE forecasting, when lead-times at each location in a supply chain echelon can be different, is developed for both centralised and decentralised systems. Note that the MMSE forecast introduced in Section 4.3.2 only deals with decentralised cases where the lead-times at each level are identical. A z-transform approach is used to obtain variances of system states. Finally, the evolution of orders placed over the supply chain is investigated.

To evaluate the impact of the DND on its dynamics and economic performance, two different distribution networks are considered; a decentralised system and a centralised system both consisting of two echelons. In the lower echelon of each system, there are n retailers operating OUT replenishment policies with MMSE forecasting. For the upper level, there are n distribution centres (DCs) in the decentralised system and a single distribution centre in the centralised system. All DCs operate an OUT policy with MMSE forecasting. Arbitrary lead-times are assumed to be known and constant at each location in both distribution systems. Figure 5-1 depicts the two systems. Note that the n customers and n retailers shown in Figure 5-1 can also represent n groups of customers and n groups of retailers.
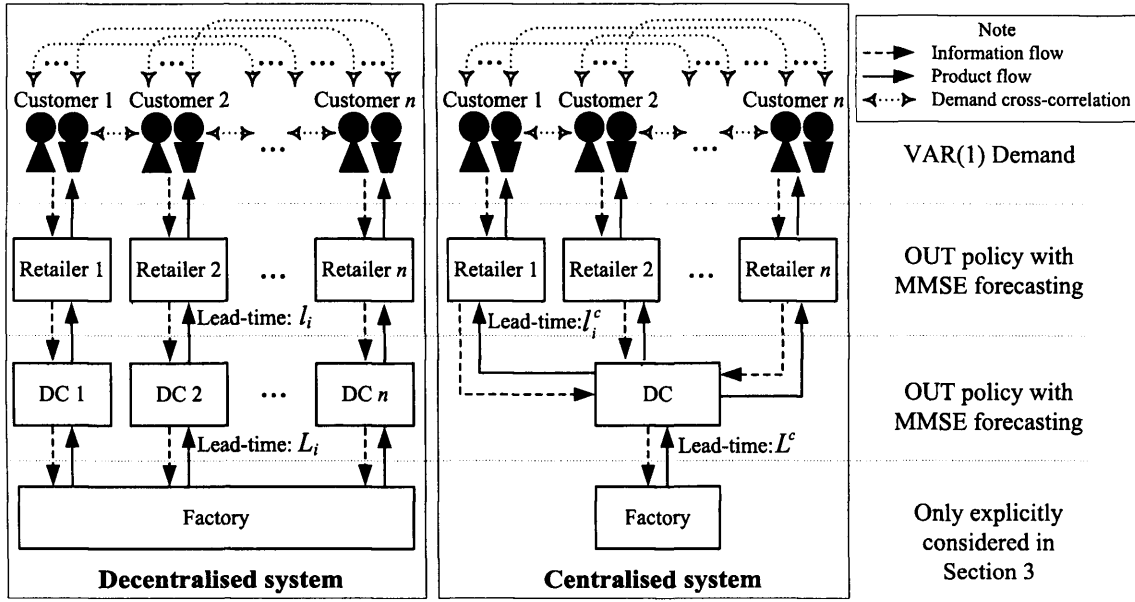
Figure 5-1 The decentralised and the centralised distribution networks

5.2　Demand model

This study assumes that the customer demands follow a first order vector autoregressive, VAR(1), demand process. Specifically, the mean-centred VAR(1) demand process is employed. Comparing to the standard VAR(1) process in Equation (4.2), the mean-centred version of the VAR(1) process removes the dependency on magnitude and focuses on the variation of the variables by subtracting each previous values of the demand by the mean demand. Thus, the mean-centred VAR(1) is given by

$$\mathbf{D}_t = \mathbf{C} + \mathbf{A} \cdot (\mathbf{D}_{t-1} - \mathbf{C}) + \mathbf{U}_t, \tag{5.1}$$

where $\mathbf{D}_t$ is an $(n \times 1)$ demand column vector, $\mathbf{C}$ is an $(n \times 1)$ mean demand vector, $\mathbf{A}$ is an $(n \times n)$ square coefficient matrix and $\mathbf{U}_t$ is an $(n \times 1)$ error column vector. Note that the expected value of $\mathbf{U}_t$, $E[\mathbf{U}_t] = 0$, $E[\mathbf{U}_t \mathbf{U}_t'] = $ covariance matrix and $E[\mathbf{U}_t \mathbf{U}_s'] = 0$ for $s \neq t$. This means there is no correlation across time for error terms.

The concise VAR(1) model in Equation (5.1) can be written in large matrix notation as

$$\underbrace{\begin{bmatrix} d_{1,t} \\ d_{2,t} \\ \vdots \\ d_{n,t} \end{bmatrix}}_{\mathbf{D}_t} = \underbrace{\begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix}}_{\mathbf{C}} + \underbrace{\begin{bmatrix} \phi_{11} & \phi_{12} & \cdots & \phi_{1n} \\ \phi_{21} & \phi_{22} & \cdots & \phi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n1} & \phi_{n2} & \cdots & \phi_{nn} \end{bmatrix}}_{\mathbf{A}} \cdot \underbrace{\begin{bmatrix} d_{1,t-1} - \mu_1 \\ d_{2,t-1} - \mu_2 \\ \vdots \\ d_{n,t-1} - \mu_n \end{bmatrix}}_{\mathbf{D}_{t-1} - \mathbf{C}} + \underbrace{\begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \vdots \\ \varepsilon_{n,t} \end{bmatrix}}_{\mathbf{U}_t} , \qquad (5.2)$$

where $d_{i,t}$ is the mean centred demand for retailer i at time t, μ_i is the mean demand level of retailer i and ϕ_{ij} is a cross-correlation coefficient of the demand at retailer i with the previous realisation of demand at retailer j, where $i, j = \{1, 2, ..., n\}$. For cases where the coefficient $i = j$, ϕ_{ii}, it is an autoregressive term of one period with itself. Note again that $\varepsilon_{i,t}$ is an independently and identically distributed (white noise) random process and is normally distributed with zero mean and unit variance. It is assumed that these error terms are uncorrelated as this simplifies the mathematics considerably.

This study considers only a stable VAR(1) demand process. The VAR(1) process is stable if all eigenvalues of matrix $\mathbf{A}$ have modulus less than 1. The condition is equivalent to

$$\det(\mathbf{I}_N - \mathbf{A}z) \neq 0 \qquad \text{for } |z| \leq 1, \qquad (5.3)$$

where $\mathbf{I}_N$ is an $(n \times n)$ identity matrix.

This stability condition places limits on the variance ratio analysis that is exploited later at each of the different DCs to the circumstances which are within the polygons shown in Figure 5-2. Note that the examples in Figure 5-2 describe only the special cases where all $\phi_{ii} = \phi$ and all $\phi_{ij} = \theta$ to allow plotting of a 2-dimensional graph. An example of an explicit set of the stationary criteria in Equation (5.3) will be presented in Chapter 6.

5.3   MMSE forecasting of VAR(1) with arbitrary lead-times

Lee et al. (2000), Gilbert (2005) and Hosoda and Disney (2006), for example, have successfully applied the MMSE forecasting in their supply chain models. However, their studies considered only supply chains with a single player at each level. In this section, a proposed procedure to achieve the MMSE forecasting, for the case where there are multiple players at each supply chain level and cross-correlations between their demands exist, is presented. Importantly, the lead-times of each player are allowed to be different. In addition, the MMSE forecasting for the demand faced by a centralised DC where the demand is an aggregation of orders from multiple retailers is introduced. Vector notation can enhance facilitation of this procedure.
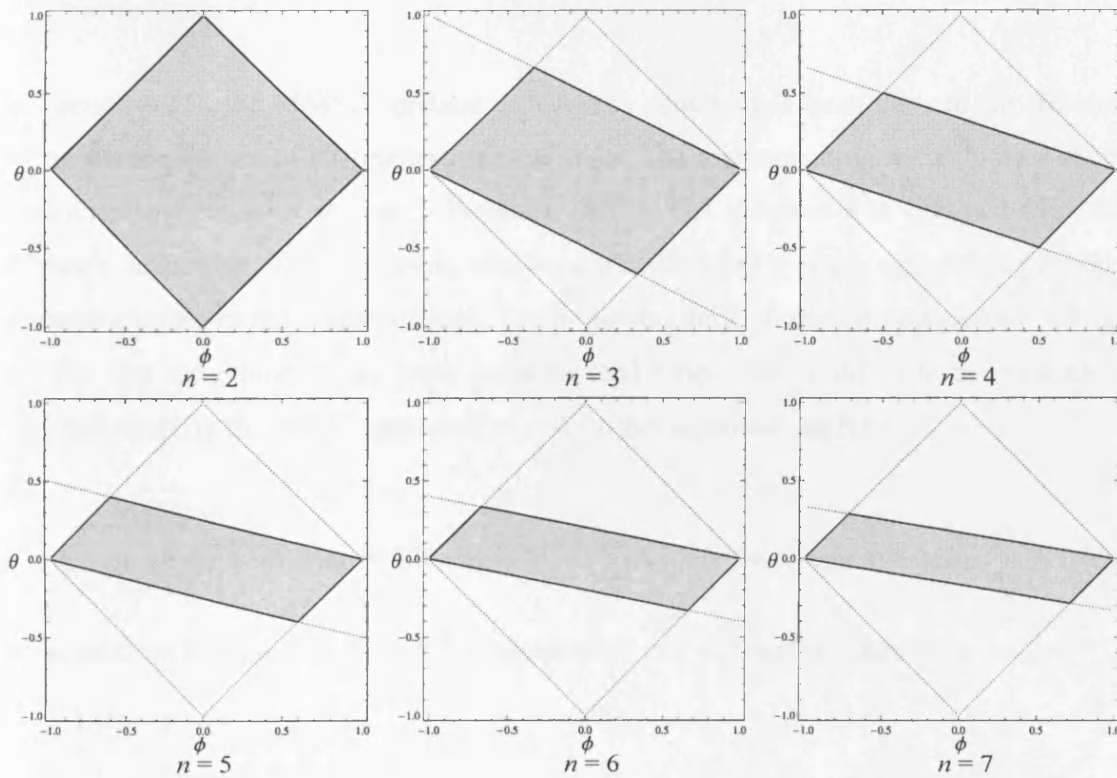
Figure 5-2 Admissible regions for a stable VAR(1) demand process at different n shown by the shaded areas (when all $\phi_{ii} = \phi$ and all $\phi_{ij} = \theta$)

5.3.1 MMSE forecasting for the retailers

In the decentralised system, $\hat{\mathbf{D}}_t$, an $(n \times 1)$ vector, denotes the sum of the expected demand over the lead-times and review period of the retailers calculated at time t, where $\hat{\mathbf{D}}_t = [\hat{d}_{1,t}, \hat{d}_{2,t}, ..., \hat{d}_{n,t}]'$. The notation $[.]'$ is for the vector (or matrix) transpose. $\hat{d}_{i,t}$ is the sum of the expected demand over the lead-times and review period for retailer i calculated at time t. The lead-time of retailer i is denoted by l_i. Let $\mathbf{L}$ be an $(n \times n)$ diagonal matrix, which the element on the main diagonal is equal to l_i,

$$\mathbf{L} = \begin{bmatrix} l_1 & 0 & \cdots & 0 \\ 0 & l_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & l_n \end{bmatrix}.$$

In Section 4.3.2, the MMSE forecast of VAR(1) demand has been derived for the case where the lead-times of all retailers are the same. The forecast needs the xth power of the coefficient matrix $\mathbf{A}$ as shown in Equation (4.17). The value of x is obtained from the retailer's lead-time. To this point, the equation does not permit different lead-times amongst players in the same echelon. The following procedure is introduced to manage the fact that the retailers may have different lead-times. The result from this procedure will also simplify the model presentation and further algebraic analysis.

Let $\mathbf{Y}_i^R$ be an $(n \times n)$ matrix in which $\mathbf{Y}_i^R = \sum_{j=1}^{l_i+1} \mathbf{A}^j, \forall i$. Note that the term $\sum_{j=1}^{l_i+1} \mathbf{A}^j$ was obtained from Equation (4.17). $\mathbf{Y}_i^R$ is represented in a full matrix notation as

$$\mathbf{Y}_i^R = \begin{bmatrix} y_{11}^i & y_{12}^i & \cdots & y_{1n}^i \\ y_{21}^i & y_{22}^i & \cdots & y_{2n}^i \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}^i & y_{n2}^i & \cdots & y_{nn}^i \end{bmatrix}.$$

Let $\mathbf{N}^i$ be a $(1 \times n)$ vector where the element in the ith column is equal to unity and all other elements are zeros. Then, let $\mathbf{Y}^R$ be an $(n \times n)$ matrix in which its elements in the ith row are $\mathbf{N}^i \cdot \mathbf{Y}_i^R$. In this manner, the 1st row of matrix $\mathbf{Y}^R$ is the 1st row of matrix $\mathbf{Y}_1^R$, the 2nd row of matrix $\mathbf{Y}^R$ is from the 2nd row of matrix $\mathbf{Y}_2^R$ and so on. That is

$$\mathbf{Y}^R = \begin{bmatrix} y_{11}^1 & y_{12}^1 & \cdots & y_{1n}^1 \\ y_{21}^2 & y_{22}^2 & \cdots & y_{2n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}^n & y_{n2}^n & \cdots & y_{nn}^n \end{bmatrix}.$$

Matrix $\mathbf{Y}^R$ can now be used to obtain the conditional expectation of the retailer's demand when the lead-time at each retailer can be different. The sum of the expected demand over the lead-time and review period of the retailers calculated at time t, $\hat{\mathbf{D}}_t$, is given by

$$\hat{\mathbf{D}}_t = (\mathbf{L} + \mathbf{I}_N) \cdot \mathbf{C} + \mathbf{Y}^R \cdot (\mathbf{D}_t - \mathbf{C}). \tag{5.4}$$

Note that the MMSE forecast presented in Section 5.3 is for a mean-centred VAR(1) process as presented in Equation (5.1). Thus, when compared to Equation (4.17), the demand at time t ($\mathbf{D}_t$) is subtracted by the mean demand and becomes ($\mathbf{D}_t - \mathbf{C}$).

The model for the retailers in the centralised system can be derived in exactly the same way as in the decentralised system. All the variables involved with the retailers in the centralised system are differentiated by the super script 'c', such as the lead-time, l_i^c, the lead-time matrix, $\mathbf{L}^c$, the coefficient matrix for forecasting, $\mathbf{Y}^{c:R}$, and the expected demand, $\hat{\mathbf{D}}_t^c$. A summary of the notation used in this model can be found in Appendix B.

5.3.2 MMSE forecasting for the decentralised DCs

Let $\hat{\mathbf{D}}_t^{DC}$, an $(n \times 1)$ vector, denote the sum of the expected demand over the lead-times and review period for the decentralised DC calculated at time t, where $\hat{\mathbf{D}}_t^{DC} = [\hat{D}_{1,t}, \hat{D}_{2,t}, ..., \hat{D}_{n,t}]'$. $\hat{D}_{i,t}$ is the sum of the expected demand over the lead-times and review period for the decentralised DC i calculated at time t. The lead-time for

decentralised DC i is denoted by L_i. Let $\mathbf{L}^{DC}$ be an $(n \times n)$ diagonal matrix, in which the element on the main diagonal is equal to L_i. That is

$$\mathbf{L}^{DC} = \begin{bmatrix} L_1 & 0 & \cdots & 0 \\ 0 & L_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L_n \end{bmatrix}.$$

The same method from the previous section is adopted to deal with the cases where lead-times at each decentralised DC can be different. From Equation (4.17) and based on the assumption that OUT policy is employed, the term $\sum_{j=l_i+2}^{l_i+L_i+2} \mathbf{A}^j$ is used to obtain $\hat{\mathbf{D}}_t^{DC}$. Let

$\mathbf{Y}_i^{DC}$ be an $(n \times n)$ matrix in which $\mathbf{Y}_i^{DC} = \sum_{j=l_i+2}^{l_i+L_i+2} \mathbf{A}^j, \forall i$. In a similar manner, let $\mathbf{Y}^{DC}$ be an

$(n \times n)$ matrix in which the elements on the ith row of matrix $\mathbf{Y}^{DC}$ are $\mathbf{N}^i \cdot \mathbf{Y}_i^{DC}$. Thus, the sum of the expected demand over the lead-times and review period of the decentralised DCs calculated at time t, $\hat{\mathbf{D}}_t^{DC}$, can be expressed as

$$\hat{\mathbf{D}}_t^{DC} = (\mathbf{L}^{DC} + \mathbf{I}_N) \cdot \mathbf{C} + \mathbf{Y}^{DC} \cdot (\mathbf{D}_t - \mathbf{C}). \tag{5.5}$$

5.3.3 MMSE forecasting for the centralised DC

Let $\hat{D}_t^c$ denote the sum of the expected demand over the lead-times and review period calculated at time t for the centralised DC. Please note that $\hat{D}_t^c$ is a scalar, a (1×1) vector, as there is a single DC in the centralised system. For OUT replenishment policy, the $\hat{D}_t^c$ can be derived from a column-wise sum of the expected demand matrix of decentralised DCs ($\hat{\mathbf{D}}_t^{DC}$); That is $\hat{D}_t^c = \hat{D}_{1,t} + \hat{D}_{2,t} + \ldots + \hat{D}_{n,t}$. The column-wise sum of matrix $\mathbf{X}$ of order $(n \times n)$ can be obtained by a multiplication of a $(1 \times n)$ unit vector with matrix $\mathbf{X}$. It follows that $[1 \quad 1 \quad \cdots \quad 1] \cdot \hat{\mathbf{D}}_t^{DC} = \sum_{i=1}^{n} \hat{D}_{i,t} = \hat{D}_t^c$.

Matrix $\mathbf{Y}^{DC}$ from Equation (5.5) is adapted to account for the fact that the centralised system has a single DC with lead-time L^c and centralised retailer i has lead-time l_i^c. Let $\mathbf{Y}_i^{c:DC}$ be an ($n \times n$) matrix in which $\mathbf{Y}_i^{c:DC} = \sum_{j=l_i^c+2}^{l_i^c+L^c+2} \mathbf{A}^j$. Then, let $\mathbf{Y}^{c:DC}$ be an ($n \times n$) matrix in which elements in i^{th} row of matrix $\mathbf{Y}^{c:DC}$ are $\mathbf{N}^i \cdot \mathbf{Y}_i^{c:DC}$. Let $\mathbf{N}$ be a ($1 \times n$) unit vector utilised in order to obtain the column-wise sum of a matrix. The expected demand over the lead-times and review period of the centralised DC calculated at time t, $\hat{D}_t^c$, is expressed as

$$\hat{D}_t^c = (L^c + 1)(\mathbf{N} \cdot \mathbf{C}) + (\mathbf{N} \cdot \mathbf{Y}^{c:DC}) \cdot (\mathbf{D}_t - \mathbf{C}). \tag{5.6}$$

5.4 Inventory replenishment policy

All locations in our supply chain are assumed to adopt the Order-Up-To (OUT) inventory replenishment policy. The OUT policy has already been described in Section 4.3.1. This section, however, intends to use a vector notation for the OUT policy for multiple locations in a distribution network. This vector notation will allow a coherent connection between the MMSE forecasting and the inventory models.

5.4.1 Inventory replenishment policy for the retailers

Let $\mathbf{I}_t$ be an ($n \times 1$) vector in which its elements are the inventory level of retailer i at time t ($i_{i,t}$), $\mathbf{W}_t$ be an ($n \times 1$) vector in which its elements are the WIP of retailer i at time t ($w_{i,t}$), $\mathbf{O}_t$ be an ($n \times 1$) vector in which its elements are the order for retailer i at time t ($o_{i,t}$), $\mathbf{T}$ be an ($n \times 1$) vector where its elements are the target inventory level for retailer i ($\bar{t}_i$).

That is, $\mathbf{I}_t = [i_{1,t}, i_{2,t}, ..., i_{n,t}]'$, $\mathbf{W}_t = [w_{1,t}, w_{2,t}, ..., w_{n,t}]'$, $\mathbf{T} = [\bar{t}_1, \bar{t}_2, ..., \bar{t}_n]'$ and $\mathbf{O}_t = [o_{1,t}, o_{2,t}, ..., o_{n,t}]'$.

The replenishment order of the retailer at time t is given in a vector notation by

$$\mathbf{O}_t = \overline{\mathbf{T}} + \hat{\mathbf{D}}_t - \mathbf{I}_t - \mathbf{W}_t. \tag{5.7}$$

The inventory level of the retailer at time t for the decentralised system is given by

$$\mathbf{I}_t = \mathbf{I}_{t-1} + \mathbf{O}_{t-(l_i+1)} - \mathbf{D}_t, \tag{5.8}$$

where vector $\mathbf{O}_{t-(l_i+1)} = [o_{t-(l_1+1)}, o_{t-(l_2+1)}, ..., o_{t-(l_n+1)}]'$.

The work in progress of the retailer at time t for the decentralised system is given by

$$\mathbf{W}_t = \begin{cases} 0 & \text{if the lead-time is 0,} \\ \sum_{j=1}^{l_i} \mathbf{O}_{t-j} & \text{otherwise,} \end{cases} \tag{5.9}$$

where vector $\sum_{j=1}^{l_i} \mathbf{O}_{t-j} = [\sum_{j=1}^{l_1} o_{t-j}, \sum_{j=1}^{l_2} o_{t-j}, ..., \sum_{j=1}^{l_n} o_{t-j}]'$. For the retailers in the centralised

system, l_i^c replaces l_i in Equations (5.8) and (5.9) and all other variables are described by adding the superscript 'c'.

5.4.2 Inventory replenishment policy for the decentralised DCs

Let $\mathbf{I}_t^{DC}$ be an ($n \times 1$) vector in which its elements are the inventory level of DC i at time t ($I_{i,t}$), $\mathbf{W}_t^{DC}$ be an ($n \times 1$) vector in which its elements are the WIP of DC i at time t ($W_{i,t}$), $\mathbf{O}_t^{DC}$ be an ($n \times 1$) vector in which its elements are the order for DC i at time t ($O_{i,t}$), $\overline{\mathbf{T}}^{DC}$ be an ($n \times 1$) vector where its elements are the target inventory level for DC i ($\overline{T}_i$). Thus, $\mathbf{I}_t^{DC} = [I_{1,t}, I_{2,t}, ..., I_{n,t}]'$, $\mathbf{W}_t^{DC} = [W_{1,t}, W_{2,t}, ..., W_{n,t}]'$, $\overline{\mathbf{T}}^{DC} = [\overline{T}_1, \overline{T}_2, ..., \overline{T}_n]'$ and $\mathbf{O}_t^{DC} = [O_{1,t}, O_{2,t}, ..., O_{n,t}]'$.

The replenishment order of the decentralised DC at time t is given in a vector notation by

$$\mathbf{O}_t^{DC} = \overline{\mathbf{T}}^{DC} + \hat{\mathbf{D}}_t^{DC} - \mathbf{I}_t^{DC} - \mathbf{W}_t^{DC} .$$ (5.10)

The inventory level of the decentralised DC at time t is given by

$$\mathbf{I}_t^{DC} = \mathbf{I}_{t-1}^{DC} + \mathbf{O}_{t-(L_i+1)}^{DC} - \mathbf{D}_t^{DC} ,$$ (5.11)

where vector $\mathbf{O}_{t-(L_i+1)}^{DC} = [O_{t-(L_1+1)}, O_{t-(L_2+1)}, ..., O_{t-(L_n+1)}]'$. The demand for the decentralised DC at time t is actually an order of the corresponding retailer that is placed at time t and is passed directly to DC i. It is assumed that orders from the retailer are passed to the DC without delay. This means the demand that DC i faced at time t is the order that retailer i placed at time t. This assumption is motivated by today's information technology.

The work in progress of the decentralised DC at time t is given by

$$\mathbf{W}^{DC} = \begin{cases} 0 & \text{if the lead-time is } 0, \\ \sum_{j=1}^{L_i} \mathbf{O}_{t-j}^{DC} & \text{otherwise,} \end{cases}$$ (5.12)

where $\sum_{j=1}^{L_i} \mathbf{O}_{t-j}^{DC} = [\sum_{j=1}^{L_1} O_{t-j}^{DC}, \sum_{j=1}^{L_2} O_{t-j}^{DC}, ..., \sum_{j=1}^{L_n} O_{t-j}^{DC}]'$.

5.4.3 Inventory replenishment policy for the centralised DC

Italic font and the superscript 'c' without the index i (as it is a single centralised DC) are used to indicate that the centralised DC is being considered.

The replenishment order of the centralised DC at time t is given by

$$O_t^c = \overline{T}^c + \hat{D}_t^c - I_t^c - W_t^c .$$ (5.13)

The inventory level of the centralised DC at time t is given by

$$I_t^c = I_{t-1}^c + O_{t-(L^c+1)}^c - D_t^c,$$ (5.14)

where the demand for the centralised DC at time t is assumed to be the aggregation of orders from all retailers that are placed at time t. This aggregated order is assumed to be received by the DC without delay.

The work in progress of the centralised DC at time t is given by

$$W_t^c = \begin{cases} 0 & \text{if the lead-time is 0,} \\ \sum_{j=1}^{L^c} O_{t-j}^c & \text{otherwise.} \end{cases}$$ (5.15)

5.5 The variances of the order rates and the inventory levels

Control Engineering tools are used to obtain expressions for the variances of the inventory level and order rates. The variances are essential inputs to the cost model that is used to evaluate the economic performance of the distribution networks. The block diagrams in Figure 5-3 and Figure 5-4 represent the inventory replenishment decisions in the discrete time z-domain for the decentralised and centralised systems. Firstly, the block diagram is manipulated to get the transfer function which relates the error term and the state variable of interest. Please refer to Nise (1995) for background reading on Control Theory and to Hosoda and Disney (2006) for its application to supply chain problems. The transfer function is then used to obtain the time domain impulse response and, ultimately, the expression of variances.

5.5.1 The VAR(1) demand in the z-domain

The notation $x(z)$ represents a z-domain version of variable x. The VAR(1) demand model in Equation (5.1) is rewritten in z-domain as $\mathbf{D}(z) = \mathbf{A} \cdot (z^{-1} \times \mathbf{D}(z)) + \mathbf{U}(z)$, given that $\mu_i = 0$, $\forall i$. z^{-k} represents a z-domain function for an k period time delay. As linear systems exist and the focus is on the variance, the mean demand is set to zero, $\mu_i = 0$, $\forall i$,

without loss of generality. Equation (5.16) gives the transfer function vector of the demand process, which is derived by a simple matrix application for solving simultaneous equations.

$$\mathbf{D}(z) = \left[\mathbf{I}_N - \mathbf{A} \times z^{-1}\right]^{-1} \cdot \mathbf{U}(z),$$

(5.16)

where $[.]^{-1}$ denotes matrix inverse function.

5.5.2 Transfer functions for the retailers

Arranging the block diagram in Figure 5-3 using standard techniques yields the following transfer function vector which relates the error terms to the order rates.

$$\mathbf{O}(z) = \mathbf{Y}^R \cdot \left(\frac{z-1}{z} \times \mathbf{D}(z)\right) + \mathbf{D}(z).$$

(5.17)

Note that the error terms are not explicitly presented in the transfer function but are kept within $\mathbf{D}(z)$, as described by Equation (5.16), to keep the formula simple.

The transfer function vector that relates the error terms to the inventory levels of the retailer is

$$\mathbf{I}(z) = \mathbf{Y}^R \cdot \left(z^{-l_i-1} \cdot \mathbf{D}(z)\right) + \frac{z^{-l_i} - z}{z-1} \cdot \mathbf{D}(z).$$

(5.18)

For notation convenience, the above z^{-l_i-1} to denote diag[z^{-l_1-1}, z^{-l_2-1}, ..., z^{-l_n-1}] and $\frac{z^{-l_i} - z}{z-1}$ to represent diag[$\frac{z^{-l_1} - z}{z-1}$, $\frac{z^{-l_2} - z}{z-1}$, ..., $\frac{z^{-l_n} - z}{z-1}$], where diag[x_1, x_2, ..,x_n] is an $(n \times n)$ diagonal matrix in which those leading diagonal elements are $x_1, x_2, ..., x_n$. Again, for the retailer in the centralised system, l_i^c substitutes l_i in Equation (5.18).

5.5.3 Transfer functions for the decentralised DCs

The decentralised DC's replenishment decision is depicted in Figure 5-3. The demand for the DC is the order from the relevant retailer.

The transfer function vector that relates the error terms to the order rates of the decentralised DC is

$$\mathbf{O}^{DC}(z) = \mathbf{Y}^{DC} \cdot \left(\frac{z-1}{z} \times \mathbf{D}(z)\right) + \mathbf{Y}^{R} \cdot \left(\frac{z-1}{z} \times \mathbf{D}(z)\right) + \mathbf{D}(z). \tag{5.19}$$

The transfer function vector for the inventory level of the decentralised DC is

$$\mathbf{I}^{DC}(z) = \mathbf{Y}^{DC} \cdot \left(z^{-L_i-1} \cdot \mathbf{D}(z)\right) + \mathbf{Y}^{R} \cdot \left(z^{-L_i-1} \cdot \mathbf{D}(z)\right) - \mathbf{Y}^{R} \cdot \mathbf{D}(z) + \frac{z^{-L_i} - z}{z-1} \cdot \mathbf{D}(z). \tag{5.20}$$

Again, z^{-L_i-1} denotes diag[z^{-L_1-1}, z^{-L_2-1}, ..., z^{-L_n-1}] and $\dfrac{z^{-L_i} - z}{z-1}$ denotes diag[$\dfrac{z^{-L_1} - z}{z-1}$,

$\dfrac{z^{-L_2} - z}{z-1}$, ..., $\dfrac{z^{-L_n} - z}{z-1}$].

Figure 5-3 Block diagram of the replenishment decision for the decentralised system

74

5.5.4 Transfer functions for the centralised DC

The centralised DC's replenishment decision is depicted in Figure 5-4. The demand for the centralised DC is the sum of the orders of all retailers. Again, $\mathbf{N}$, a $(1 \times n)$ unit vector is employed to deal with the column-wise summation.

The transfer function that relates the error term to the order rate of the centralised DC is

$$O^c(z) = \frac{z-1}{z} \times \left(\left(\mathbf{N} \cdot \mathbf{Y}^{c:DC} \right) \cdot \mathbf{D}(z) \right) + \mathbf{N} \cdot \left(\mathbf{Y}^{c:R} \cdot \left(\frac{z-1}{z} \times \mathbf{D}(z) \right) + \mathbf{D}(z) \right). \tag{5.21}$$

The transfer function for the inventory level of the centralised DC is

$$I^c(z) = z^{-L^c-1} \times \left(\left(\mathbf{N} \cdot \mathbf{Y}^{c:DC} \right) \cdot \mathbf{D}(z) \right) + \frac{z^{-L^c} - z}{z-1} \times \left(\mathbf{N} \cdot \left(\mathbf{Y}^{c:R} \cdot \left(\frac{z-1}{z} \times \mathbf{D}(z) \right) + \mathbf{D}(z) \right) \right). \tag{5.22}$$
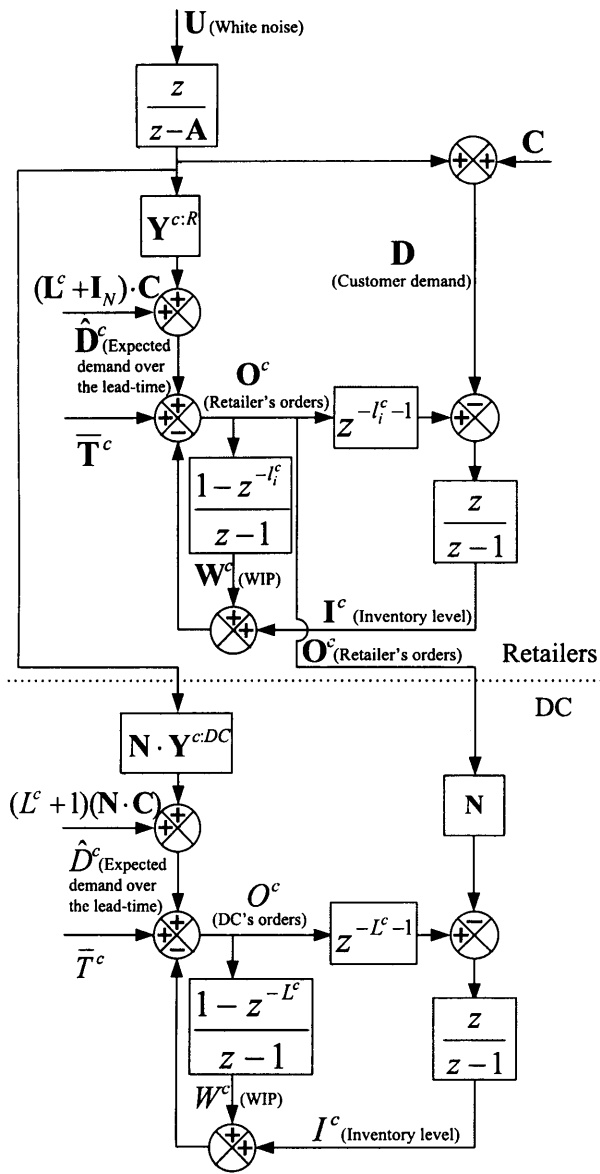
Figure 5-4 Block diagram of the replenishment decision for the centralised system

5.5.5 Approaching variance expressions by Control Theory

The transfer function acquired in Sections 5.5.2 – 5.5.4 will be used to derive variance expressions for system states. The principle of finding the variance[1] is the same for both the retailer and the DC. Therefore, one example will be given for each system state. Note that in this section we intend to give an indication of the procedure for obtaining a closed form of the variance of interest. A complete example, which includes both closed form expressions of the variances and numerical results, will be given in Chapter 6.

5.5.5.1 Variances of the order rate

Equation (5.17) presents the transfer function vector of the retailer's orders, $\mathbf{O}(z)$, where $\mathbf{O}(z) = [o_1(z), o_2(z), ..., o_n(z)]'$. $o_i(z)$ is the transfer function of the order rates of retailer i. In order to find the variance of orders, the inverse z-transform of $\mathbf{O}(z)$ is taken. This will give the time domain impulse response of the order rate,

$$\mathbf{O}_t = Z^{-1}[\mathbf{O}(z)] = \begin{bmatrix} Z^{-1}[o_1(z)] \\ Z^{-1}[o_2(z)] \\ \vdots \\ Z^{-1}[o_n(z)] \end{bmatrix} = \begin{bmatrix} o_{1,t} \\ o_{2,t} \\ \vdots \\ o_{n,t} \end{bmatrix}, \tag{5.23}$$

where $Z^{-1}[x]$ is the inverse z-transform of x. Then, the element-wise[2] squares of the time-domain impulse response is found. Finally, the long run variance of the order rate at the retailer can be found by a summation of its element-wise square from $t = 0$ to infinity. This operation is known as Tsypkin's Relation (Tsypkin 1964).

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[1] A number of error terms are involved when the VAR demand pattern is applied. For example, the model with 3 retailers (or $n = 3$) will involve three independent error terms i.e. $\mathbf{U}_t = [\varepsilon_{1,t}, \varepsilon_{2,t}, \varepsilon_{3,t}]'$. Therefore, the variance is found by a summation of the variances affected by each of the errors one at a time.

[2] To square a matrix element-wise is different from squaring a matrix. For example, let $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Squaring matrix $\mathbf{A} = \mathbf{A} \cdot \mathbf{A}$, which resulted in $\begin{pmatrix} a^2 + cd & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix}$, while squaring element-wise yields $\begin{pmatrix} a^2 & b^2 \\ c^2 & d^2 \end{pmatrix}$.

The retailer order variance is

$$VAR[\mathbf{O}] = [\sum_{t=0}^{\infty}(Z^{-1}[o_1(z)])^2, \quad \sum_{t=0}^{\infty}(Z^{-1}[o_2(z)])^2, \quad \cdots, \quad \sum_{t=0}^{\infty}(Z^{-1}[o_n(z)])^2]'. \tag{5.24}$$

Note that the long run order variance can only be found when the demand is stable; that is Equation (5.3) is satisfied. Otherwise, the order variance is infinite.

5.5.5.2 Variances of the inventory level

Equation (5.18) shows the transfer function of the retailer's inventory level ($\mathbf{I}(z)$), where $\mathbf{I}(z) = [i_1(z), i_2(z), ..., i_n(z)]'$. $i_i(z)$ is the transfer function of the inventory level of retailer i. In the same way, the inverse z-transform of $\mathbf{I}(z)$ can be taken. The time domain impulse response of the inventory level is

$$\mathbf{I}_t = Z^{-1}[\mathbf{I}(z)] = \begin{bmatrix} Z^{-1}[i_1(z)] \\ Z^{-1}[i_2(z)] \\ \vdots \\ Z^{-1}[i_n(z)] \end{bmatrix} = \begin{bmatrix} i_{1,t} \\ i_{2,t} \\ \vdots \\ i_{n,t} \end{bmatrix}. \tag{5.25}$$

However, an extra procedure is needed to find inventory variance as its transfer function shown in Equation (5.18) includes the lead-time term (l_i). For each retailer i, its own lead-time (l_i) is substituted into the transfer function one at a time. Therefore, this requires repetition until all different lead-times have been applied. This calculation can cause some confusion. As in each repetition, a particular row (which represents a particular retailer) of the resulted matrix, as in Equation (5.25), is valid only with its relevant value of lead-time.

To facilitate the calculation, the fact that the inventory level is zero in the case of impulse responses when $t > l_i$ is utilised. The largest value of lead-times amongst the retailers is used in Equation (5.18). After getting the time domain impulse response in Equation (5.25), the Heaviside step function, $h[x]$, is utilised. $h[x]$ is 1, if $x \geq 0$, and 0, otherwise.

The $h[l_i - t]$ is added to the time domain impulse response in Equation (5.25) to achieve the fact that the inventory level is zero when $t > l_i$.

$$\mathbf{I}_t = \left(Z^{-1}[\mathbf{I}(z)]\right) \times \left(h[l_i - t]\right) = \begin{bmatrix} \left(Z^{-1}[i_1(z)]\right)\left(h[l_1 - t]\right) \\ \left(Z^{-1}[i_2(z)]\right)\left(h[l_2 - t]\right) \\ \vdots \\ \left(Z^{-1}[i_n(z)]\right)\left(h[l_n - t]\right) \end{bmatrix} = \begin{bmatrix} i_{1,t} \\ i_{2,t} \\ \vdots \\ i_{n,t} \end{bmatrix}. \tag{5.26}$$

Finally, the long run variance of the inventory level of the retailer is obtained from

$$Var[\mathbf{I}] = \sum_{t=0}^{l_i, \forall i} \begin{bmatrix} (i_{1,t})^2 \\ (i_{2,t})^2 \\ \vdots \\ (i_{n,t})^2 \end{bmatrix} = \begin{bmatrix} \sum_{t=0}^{l_1} (i_{1,t})^2 \\ \sum_{t=0}^{l_2} (i_{2,t})^2 \\ \vdots \\ \sum_{t=0}^{l_n} (i_{n,t})^2 \end{bmatrix}. \tag{5.27}$$

Note that the infinite summation is reduced to a sum of the first l_i time periods as the inventory level is zero when $t > l_i$.

5.6 Transition of the VAR(1) demand process

In this section, the transition of VAR(1) demand into orders for a multi-stage supply chain will be investigated. Gilbert (2005) proved that the Autoregressive Integrated Moving Average, ARIMA(p, d, q) demand process result to the ARIMA(p, d, q) order for OUT policy with MMSE forecasting. The orders will have the same autoregressive and difference operators. However, the moving-average operator is changed. This transition is applicable to all upstream players when it is assumed that an order of a downstream player becomes the demand of the upstream player and the OUT policy with MMSE forecasting is being used by all players. Similar results were found by Graves (1999) for the ARIMA(0,1,1) demand process, Zhang (2004) for the ARMA(p, q) demand process and Hosoda and Disney (2006) for the AR(1) demand process.

5.6.1 Transition of the demand in the decentralised system

A vector version of the OUT policy expressed in Equation 4.10 is exploited in this investigation. Note that in this section $\mu_i, \forall i$ is assumed to be zero without loss of generality. With an MMSE forecasting scheme, the retailers' orders can be expressed by

$$
\begin{aligned}
\mathbf{O}_t &= \mathbf{D}_t + (\hat{\mathbf{D}}_t - \hat{\mathbf{D}}_{t-1}) \\
&= \mathbf{D}_t + \mathbf{Y}^R \cdot (\mathbf{D}_t - \mathbf{D}_{t-1}) \\
&= (\mathbf{A} \cdot \mathbf{D}_{t-1} + \mathbf{U}_t) + \mathbf{Y}^R \cdot ((\mathbf{A} \cdot \mathbf{D}_{t-1} + \mathbf{U}_t) - (\mathbf{A} \cdot \mathbf{D}_{t-2} + \mathbf{U}_{t-1})) \\
&= (\mathbf{A} \cdot \mathbf{D}_{t-1} + \mathbf{Y}^R \cdot (\mathbf{A} \cdot \mathbf{D}_{t-1} - \mathbf{A} \cdot \mathbf{D}_{t-2})) + \mathbf{U}_t + \mathbf{Y}^R \cdot \mathbf{U}_t - \mathbf{Y}^R \cdot \mathbf{U}_{t-1} \\
\mathbf{O}_t &= \mathbf{A} \cdot \mathbf{O}_{t-1} + (\mathbf{I}_N + \mathbf{Y}^R) \cdot \mathbf{U}_t - \mathbf{Y}^R \cdot \mathbf{U}_{t-1},
\end{aligned}
\tag{5.28}
$$

which is a scaled Vector Auto-Regressive Moving Average with first order VAR and first order moving average components, VARMA(1,1). The VARMA(1,1) process has been described in Section 4.2.2.
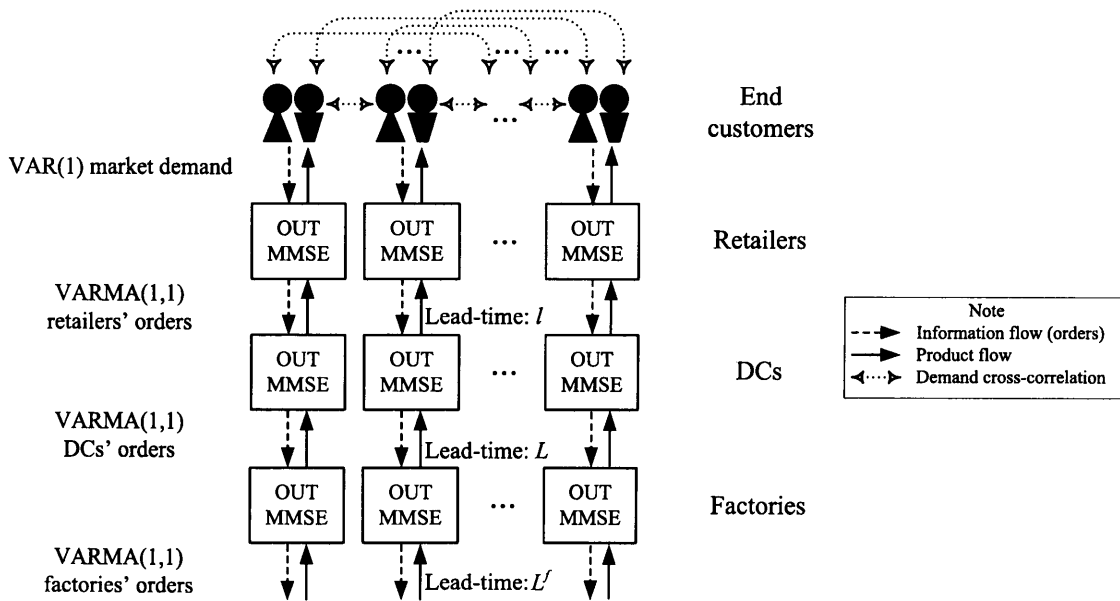
Figure 5-5 Transition of the VAR(1) market demand in the decentralised system

The order expression in Equation (5.28) is now generalised to be applicable for any particular upstream level. It is assumed that all participants in a particular level have the same lead-time and this is called a common lead-time. Let L_T denote the sum of all common lead-times and review periods of the local level and its downstream levels. For example, if the common lead-time for the retailers is l, the common lead-time for the DCs is L and the DC level is being considered as a local level, $L_T = (L+1) + (l+1) = L + l + 2$. A general expression of the VARMA(1,1) orders at a particular level of a supply chain can be expressed as

$$\mathbf{O}_t = \mathbf{A} \cdot \mathbf{O}_{t-1} + \mathbf{U}_t' - \mathbf{A} \cdot (\mathbf{I}_N - \mathbf{A}^{L_T}) \cdot [\mathbf{I}_N - \mathbf{A} + \mathbf{A} \cdot (\mathbf{I}_N - \mathbf{A}^{L_T})]^{-1} \cdot \mathbf{U}_{t-1}', \qquad (5.29)$$

where $[.]^{-1}$ denotes the inverse of a matrix and $\mathbf{U}_t' = \left(\mathbf{I}_N + \mathbf{A} \cdot (\mathbf{I}_N - \mathbf{A}^{L_T}) \cdot [\mathbf{I}_N - \mathbf{A}]^{-1}\right) \cdot \mathbf{U}_t$.

5.6.2 Transition of the demand in the centralised system

Orders that are placed by the retailers onto the centralised DC also have a VARMA(1,1) structure. The proof is the same as for the retailers in the decentralised system. The centralised retailers' orders are given by the following expression

$$\mathbf{O}_t^c = \mathbf{A} \cdot \mathbf{O}_{t-1}^c + (\mathbf{I}_N + \mathbf{Y}^{c:R}) \cdot \mathbf{U}_t - \mathbf{Y}^{c:R} \cdot \mathbf{U}_{t-1}, \qquad (5.30)$$

which is a VARMA(1,1) process. The demand faced by the centralised DC (D_t^c) is an aggregation of all orders from the retailers. Thus, D_t^c is a column-wise sum of a VARMA(1,1) process and is given by

$$D_t^c = \mathbf{N} \cdot \mathbf{O}_t^c, \qquad (5.31)$$

where $\mathbf{N}$ is a $(1 \times n)$ unit vector.

In a similar manner as previously shown, the order placed by the centralised DC to the factory can be given by

$$
\begin{aligned}
O_t^c &= D_t^c + (\hat{D}_t^c - \hat{D}_{t-1}^c) \\
&= \mathbf{N} \cdot \mathbf{O}_t^c + \mathbf{N} \cdot \mathbf{Y}^{c:DC} \cdot (\mathbf{D}_t - \mathbf{D}_{t-1}) \\
&= \mathbf{N} \cdot \left(\mathbf{O}_t^c + \mathbf{Y}^{c:DC} \cdot (\mathbf{D}_t - \mathbf{D}_{t-1}) \right) \\
&= \mathbf{N} \cdot \begin{pmatrix} \left(\mathbf{A} \cdot \mathbf{O}_{t-1}^c + (\mathbf{I}_N + \mathbf{Y}^{c:R}) \cdot \mathbf{U}_t - \mathbf{Y}^{c:R} \cdot \mathbf{U}_{t-1} \right) + \\ \mathbf{Y}^{c:DC} \cdot \left((\mathbf{A} \cdot \mathbf{D}_{t-1} + \mathbf{U}_t) - (\mathbf{A} \cdot \mathbf{D}_{t-2} + \mathbf{U}_{t-1}) \right) \end{pmatrix}
\end{aligned}
$$

$$
O_t^c = \mathbf{N} \cdot \left(\mathbf{A} \cdot \mathbf{O}_{t-1}^{c:DC} + (\mathbf{I}_N + \mathbf{Y}^{c:R} + \mathbf{Y}^{c:DC}) \cdot \mathbf{U}_t - (\mathbf{Y}^{c:R} + \mathbf{Y}^{c:DC}) \cdot \mathbf{U}_{t-1} \right), \tag{5.32}
$$

where $\mathbf{O}_t^{c:DC}$ is the order vector of pseudo-decentralised DCs. The purpose of using these pseudo-decentralised DCs is purely for the proof of the demand transition in the centralised system (the centralised DC's lead-time is applied at each pseudo-decentralised DC in order to obtain the correct expected demand for the centralised DC).

$$
\mathbf{O}_t^{c:DC} = \mathbf{O}_t^c + \mathbf{Y}^{c:DC} \cdot (\mathbf{D}_t - \mathbf{D}_{t-1}) \tag{5.33}
$$

From Equations (5.32) and (5.33), it may be shown that

$$
\mathbf{O}_t^{c:DC} = \mathbf{A} \cdot \mathbf{O}_{t-1}^{c:DC} + (\mathbf{I}_N + \mathbf{Y}^{c:R} + \mathbf{Y}^{c:DC}) \cdot \mathbf{U}_t - (\mathbf{Y}^{c:R} + \mathbf{Y}^{c:DC}) \cdot \mathbf{U}_{t-1}, \tag{5.34}
$$

which is a scaled noise VARMA(1,1) process. Therefore, the centralised DC's order (O_t^c) shown in Equation (5.32) is a column-wise sum of VARMA(1,1).

Thus, the same conclusion as in Graves (1999), Zhang (2004), Gilbert (2005) and Hosoda and Disney (2006) can be made even for cases where the demands are correlated with the previous values of itself and of other retailers. That is the point-of-sale data for information sharing scheme are redundant as upstream players can actually identify such information from the order they received. This is true for both the decentralised and the centralised systems given that all locations apply the OUT policy with MMSE forecasting scheme and the market demand is a stable VAR(1) process. The proof provided is limited to the condition that the lead-times for the players in the same level are identical. I speculate that the arbitrary lead-times will make not alter this conclusion.
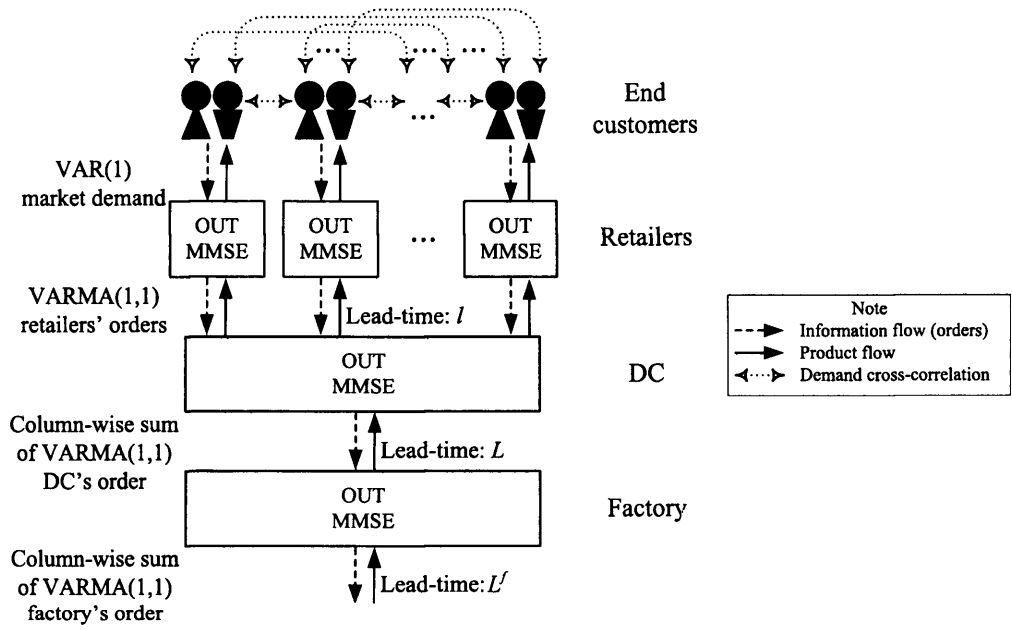
Figure 5-6 Transition of the VAR(1) market demand in the centralised system

5.7 Summary

The core model of this thesis has been presented in this chapter. This analytical model can be used to evaluate the dynamic behaviours of a 2-level supply chain with multi-distribution centres and arbitrary lead-times at each location. Exact formulas of inventory and order variances can be obtained from the model.

The impact of the correlations of demands both in time (auto-correlation) and across retailers (cross-correlation) is the main concern of this research. The VAR model, which is generally used in Econometrics and is new to supply chain modelling, has been introduced in this research to represent such correlations. This allows for both auto- and cross- correlations to be analysed. The major contribution is the development of an approach that combines knowledge and techniques from different fields to deal with such complex situations. A method for presenting the MMSE forecast of the VAR(1) demands with different lead-times at each locations has also been developed and proposed in this chapter. Moreover, the transition of the VAR(1) demand process for multi-level supply chain has been investigated in both the decentralised and the centralised systems.

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Chapter 6

Example models and variance analyses

6.1 Introduction

The model presented in Chapter 5 allows an analysis of a complete distribution network where the number of decentralised DCs, lead-times at each retailer and each DC, level of the demand correlations for both the auto-correlation and the cross-correlation are arbitrary. The cross-correlation coefficients for any pair of retailers can be different in both directions. In the following section, simple examples are given so that readers could appreciate the application of the model. Some insightful investigations about the variances are also presented. This is followed by more cases where the capability of the model is exploited. The impacts of the auto- and cross-correlations, the lead-times and the number of decentralised locations on consolidation decisions can be understood through these examples. Finally, an example of a consolidation scenario is presented and analysed.

6.2 A simple model for the case where $n = 2$, $l_i = l_i^c = L_i = L^c = 1$

The aim of this example is to show how the model works and how a graphical and algebraic analysis can be performed. More complicated examples that fully employ the model capability will be shown later in this chapter and in Chapter 8. In this particular example there are 2 retailers and 2 decentralised DCs; that is $n = 2$. Unit lead-times are assumed at all locations in both decentralised and centralised systems; that is $l_i = l_i^c = L_i$

$= L^c = 1$. The demand model is simplified to allow a thorough evaluation by employing the following assumptions.

1. The auto-correlation coefficients of the demands are the same for all retailers and are equal to ϕ; that is $\phi_{11} = \phi_{22} = \phi$.

2. The cross-correlation coefficients between the two retailers are the same in both directions and are equal to θ; that is $\phi_{12} = \phi_{21} = \theta$.

Throughout this thesis a model that applies the above two assumptions will be referred to as a "simple model".

6.2.1 An example of the simple model

Applying the modelling approach from Chapter 5, Table 6-1 summarises the model for this particular example. Each formula refers to the corresponding equation that has been presented in Chapter 5. For the simple model with all unit lead-times, the values of the variances of interest are the same for all decentralised locations, for example $Var[O_1] = Var[O_2]$, which is denoted by $Var[O_i]$ in Table 6-1.

For stable processes the variance of the demand at retailer i is given by

$$Var[d] = \frac{1 - \theta^2 - \phi^2}{\theta^4 + (\phi^2 - 1)^2 - 2\theta^2(\phi^2 + 1)}.$$ (6.1)

The contour plot in Figure 6-1 presents the demand variance for all possible demand patterns for the simple model. The demand variance is purely influenced by the value of auto- and cross- correlations. $Var[d]$ increases with the magnitude of $|\phi|$ and $|\theta|$ regardless of the signs of ϕ and θ as the value of $Var[d]$ is symmetric along the x-axis as well as the y-axis. The plot clearly demonstrates the demand stability area, which was first mentioned in Section 5.2. The area outside the polygon is where the demand is unstable and the exact demand variance is indeterminate.
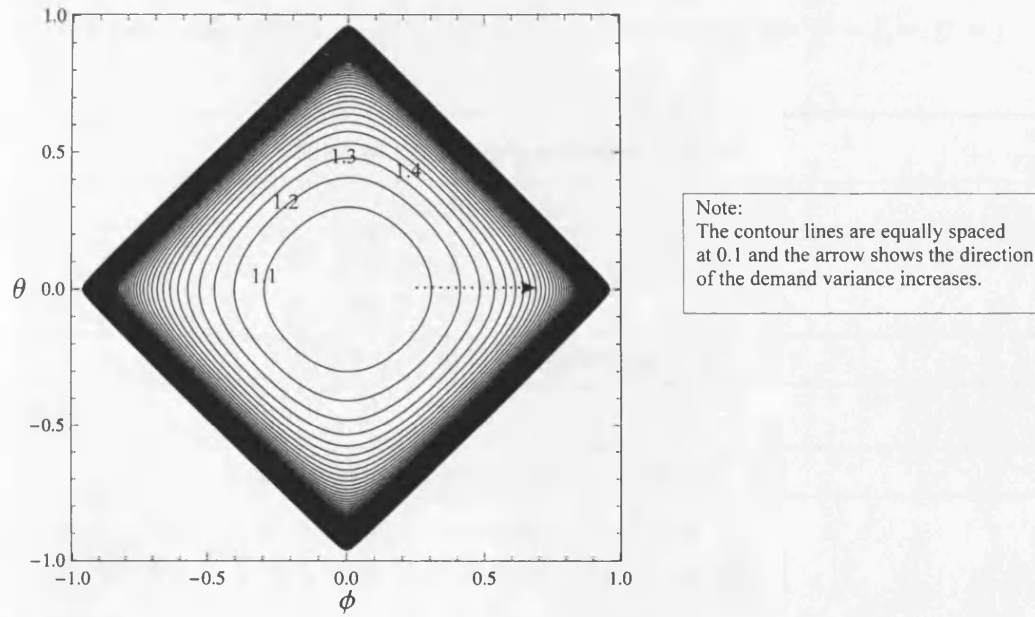
Figure 6-1 The contour plot of the demand variance for the simple model with n = 2

The contour plots in Figure 6-2 show the inventory and order variances at each location for all possible demand patterns. The plots correspond to the variance equations in Table 6-1. The graphs of order variances are only valid within the area where demand is stable. The order variance outside the stability area is infinite. Although the graphs of inventory variances are not limited by the demand stability area, the inventory variance can be extremely high when the demand correlations are close to ±1. Therefore, the upcoming analyses will be limited to the parameter within the demand stability area.

Considering the formula of the demand variance in Equation (6.1), the numerator is $1 - \theta^2 - \phi^2$ and the denominator can be rearranged to be $1 - \theta^2 - \phi^2 - \theta^2 - \phi^2 + \theta^4 - 2\theta^2\phi^2$. Noticeably, all terms of the numerator is contained in the expression of the denominator. Subtracting the expression of the denominator by the expression of the numerator resulted in the following terms $-\theta^2 - \phi^2 + \theta^4 - 2\theta^2\phi^2$, which will always be negative (as we assume $|\theta| < 1$) regardless the values of ϕ and θ and will exponentially decrease (more negative) with the higher values of ϕ and θ. Thus, from this analytical analysis the demand variance will be very high when $|\phi|$ and $|\theta|$ close to stability boundary as shown in Figure 6-1.

87

Table 6-1 The simple model for the case that n = 2 and $l_i = l_i^c = L_i = L^c = 1$

Summary of the simple example model	*Refer to Equations*
Demand model	
$\mathbf{D}_t = \begin{bmatrix} d_{1,t} \\ d_{2,t} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \phi & \theta \\ \theta & \phi \end{bmatrix} \cdot \begin{bmatrix} d_{1,t-1} - \mu_1 \\ d_{2,t-1} - \mu_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$	*(5.2)*
Demand stability conditions	
$\lvert \phi \pm \theta \rvert < 1$	*(5.3)*
Expected demands (MMSE forecasting)	
$\hat{\mathbf{D}}_t = \begin{bmatrix} \hat{d}_{1,t} \\ \hat{d}_{2,t} \end{bmatrix} = \begin{bmatrix} 2\mu_1 \\ 2\mu_2 \end{bmatrix} + \begin{bmatrix} \theta^2 + \phi + \phi^2 & \theta + 2\theta\phi \\ \theta + 2\theta\phi & \theta^2 + \phi + \phi^2 \end{bmatrix} \cdot \begin{bmatrix} d_{1,t} - \mu_1 \\ d_{2,t} - \mu_2 \end{bmatrix}$	*(5.4)*
$\hat{\mathbf{D}}_t^{DC} = \begin{bmatrix} \hat{D}_{1,t} \\ \hat{D}_{2,t} \end{bmatrix} = \begin{bmatrix} 2\mu_1 \\ 2\mu_2 \end{bmatrix} + \begin{bmatrix} \theta^4 + \phi^3(1+\phi) + 3\theta^2\phi(1+2\phi) & \theta(\theta^2(1+4\phi) + \phi^2(3+4\phi)) \\ \theta(\theta^2(1+4\phi) + \phi^2(3+4\phi)) & \theta^4 + \phi^3(1+\phi) + 3\theta^2\phi(1+2\phi) \end{bmatrix} \cdot \begin{bmatrix} d_{1,t} - \mu_1 \\ d_{2,t} - \mu_2 \end{bmatrix}$ (5.5)	
$\hat{D}_t^c = 2(\mu_1 + \mu_2) + ((\theta + \phi)^3(1 + \theta + \phi))((d_{1,t} - \mu_1) + (d_{2,t} - \mu_2))$	*(5.6)*
Variances for retailer i	
$Var[i_i] = 2 + \theta^2 + \phi(2 + \phi) = Var[i_i^c]$	*(5.27)*
$Var[o_i] = \dfrac{1}{4}\left(8\theta^2(1+3\phi) + 8\phi(1+\phi+\phi^2) - \dfrac{4(\theta^2 + \phi^2 - 1)}{\theta^4 + (\phi^2-1)^2 - 2\theta^2(\phi^2+1)} \right) = Var[o_i^c]$	*(5.24)*
Variances for decentralised DC i	
$Var[I_i] = \begin{pmatrix} (\theta + 2\theta\phi)^2 + (1 + \theta^2 + \phi + \phi^2)^2 + (1 + \phi + \phi^2 + \phi^3 + \theta^2(1+3\phi))^2 + \\ \theta^2(1 + \theta^2 + \phi(2+3\phi))^2 \end{pmatrix}$	*(5.27)*
$Var[O_i] = \dfrac{1}{4}\begin{pmatrix} 8\theta^6(1+7\phi) + 8\phi(1+\phi^2)(1+\phi+\phi^2+\phi^3+\phi^4) - \\ \dfrac{4(\theta^2+\phi^2-1)}{\theta^4+(\phi^2-1)^2-2\theta^2(\phi^2+1)} + 8\theta^4(2+5\phi(2+\phi(3+7\phi))) + \\ 8\theta^2(1+\phi(6+\phi(12+\phi(20+3\phi(5+7\phi))))) \end{pmatrix}$	*(5.24)*
Variances for the centralised DC	
$Var[I^c] = 2((1 + \theta + \theta^2 + \phi + 2\theta\phi + \phi^2)^2 + (1 + \theta + \phi)^2(1 + (\theta + \phi)^2)^2)$	*(5.27)*
$Var[O^c] = \begin{pmatrix} 2\left(\begin{array}{l} 1 + \theta(1 + \theta + \theta^2 + \theta^3) + \phi + \theta(2 + \theta(3 + 4\theta))\phi + (1 + 3\theta + 6\theta^2)\phi^2 \\ + (1 + 4\theta)\phi^3 + \phi^4 \end{array}\right)^2 - \\ \dfrac{2(\theta + \phi)^{10}}{(\theta + \phi - 1)(\theta + \phi + 1)} \end{pmatrix}$	*(5.24)*

Variance of inventory levels — Variance of orders

Retailer

$Var[i_i]$ or $Var[i_i^c]$ $Var[o_i]$ or $Var[o_i^c]$

Decentralised DC

$Var[I_i]$ $Var[O_i]$

Centralised DC

$Var[I^c]$ $Var[O^c]$

Note: The contour lines are equally spaced at 0.5 and the arrow shows the direction
in which the variance increases. The dashed line shows the demand stability area.

*Figure 6-2 Contour plots of the variances of the system states of the simple model with
$n = 2$ and $l_i = l_i^c = L_i = L^c = 1$*

89

6.2.2 Analytical analysis of the cost ratios of the example model

In order to decide whether the consolidation of the distribution network is attractive, the ratios of costs between the decentralised and the centralised systems will be investigated. The cost model and the cost ratios have been introduced in Section 4.4. From Equations (4.23), (4.29) and (4.30) – (4.32), when the unit costs of all players are the same, the cost ratios become the ratio of the standard deviations (SDs) of the two systems. For the inventory level, it is given that

$$Ratio[Inv] = \frac{\sum_{i=1}^{n} \sqrt{Var[I_i]}}{\sqrt{Var[I^c]}},$$
(6.2)

where $Ratio[Inv]$ denotes the ratio of the inventory costs between the decentralised and the centralised systems, $\sum_{i=1}^{n} \sqrt{Var[I_i]}$ denotes the summation of the SDs of the inventory level of the decentralised DCs and $\sqrt{Var[I^c]}$ denotes the SD of the inventory level of the centralised DC. It follows that the consolidation is attractive when $Ratio[Inv] > 1$. Since the model presented in Chapter 5 can obtain the closed form of both inventory and order variances, the cost ratio in Equation (6.2) can be investigated without difficulty.

For the example model presented in Section 6.2.1,

$$Ratio[Inv] = \frac{\sqrt{2}\sqrt{\left(\begin{array}{l}(\theta + 2\theta\phi)^2 + (1 + \theta^2 + \phi + \phi^2)^2 + \\ (1 + \phi + \phi^2 + \phi^3 + \theta^2(1 + 3\phi))^2 + \theta^2(1 + \theta^2 + \phi(2 + 3\phi))^2\end{array}\right)}}{\sqrt{(1 + \theta + \theta^2 + \phi + 2\theta\phi + \phi^2)^2 + (1 + \theta + \phi)^2(1 + (\theta + \phi)^2)^2}}.$$
(6.3)

Mathematica has been used to evaluate all possible cases of (ϕ, θ) within the stability region. The result confirms that the $Ratio[Inv]$ is always greater than 1. The contour plot for this case is shown in Figure 6-3 (a). The consolidation is, therefore, attractive for all pairs of (ϕ, θ) when all lead-times are unity.

By assuming that all lead-times are equal to L, this result can be extended to more general cases, where the overall lead-time (L) is greater than one. The formula of the $Ratio[Inv]$ for this case is presented by Equation (C.7) in Appendix C. The test by graphical method (presented in Figure D-1 in Appendix D) shows that $Ratio[Inv] > 1$ for all L. As a result, it can be concluded that the consolidation of the distribution network is always attractive when lead-times of all players are the same and maintained at the same value after consolidation. To achieve these in practice will involve good logistics and operations management performance to control the lead-times of the players in the centralised system. Furthermore, it is assumed favourable geographical conditions exist.

For the order rate, the capacity cost model introduced in Section 4.4.2 is applied. Again with the assumption that the unit costs of all players are the same, it is given that

$$Ratio[Cap] = \frac{\sum_{i=1}^{n} \sqrt{VAR[O_i]}}{\sqrt{VAR[O^c]}}, \tag{6.4}$$

where $Ratio[Cap]$ denotes the ratio of the capacity costs between the decentralised and the centralised systems, $\sum_{i=1}^{n} \sqrt{VAR[O_i]}$ denotes the summation of the SDs of the order rates of the decentralised DCs and $\sqrt{VAR[O^c]}$ denotes the SD of the order rates of the centralised DC. Similar to the case of the $Ratio[Inv]$, the consolidation is attractive when the $Ratio[Cap] > 1$. For the example model presented in Section 6.2.1, it can be shown that

$$Ratio[Cap] = \frac{\sqrt{\frac{1}{2}\left(\begin{array}{l} 8\theta^6(1+7\phi) + 8\phi(1+\phi^2)(1+\phi+\phi^2+\phi^3+\phi^4) - \\ \dfrac{4(\theta^2+\phi^2-1)}{\theta^4+(\phi^2-1)^2-2\theta^2(\phi^2+1)} + 8\theta^4(2+5\phi(2+\phi(3+7\phi))) + \\ 8\theta^2(1+\phi(6+\phi(12+\phi(20+3\phi(5+7\phi))))) \end{array}\right)}}{\sqrt{\left(\begin{array}{l} 2(1+\theta(1+\theta+\theta^2+\theta^3)+\theta(2+\theta(3+4\theta))\phi+(1+3\theta+6\theta^2)\phi^2 + \\ \phi+(1+4\theta)\phi^3+\phi^4)^2 - \dfrac{2(\theta+\phi)^{10}}{(\theta+\phi-1)(\theta+\phi+1)} \end{array}\right)}}. \tag{6.5}$$

91

The result from Mathematica shows that $Ratio[Cap] > 1$ for all possible cases of (ϕ, θ) when al lead-times are units. Again, by assuming that all lead-times are equal to L, the formula for $Ratio[Cap]$ can be expressed by Equation (C.8) in Appendix C. The contour plots of the $Ratio[Cap]$ at different values of lead-time L are shown in Figure D-2 in Appendix D. This result shows that for the simple model where all lead-times are equal, the consolidation is also attractive when the order cost is considered.

The analytical analysis presented in this Section considered the cost ratios as either-or type problems. When the ratio is greater than 1, the centralised distribution network is more preferable. Otherwise, the decentralised distribution network is more preferable. In practice, the magnitude of the ratio is more important than just the ratio is greater than one or not. The magnitude indicates the percentage of savings resulted from the network consolidation. It also allows the decision makers to incorporate other distribution network costs other than inventory and capacity costs into consideration by setting a benchmark. The consideration of the magnitude of the ratios will be presented in the next section.

6.2.3 Numerical analysis on the cost ratios of the example model

Based on the mathematical model in Table 6-1, a numerical example of decision making on distribution network consolidation will now be presented. The $Ratio[Inv]$, $Ratio[Cap]$ and $Ratio[Total]$, which have been presented in Section 4.4.3, will be used as economic performance measures in order to see the impact of the auto-correlation (ϕ) and cross-correlation (θ) on the consolidation decision. The unit costs of $H = 1$, $B = 9$, $N = 4$ and $P = 6$ are assumed to be presented at all locations in the distribution network.

Consolidation decisions can be made by investigating a contour plot of the cost ratio as shown in Figure 6-3. When the ratio is greater than one, it means the centralised system is more cost effective than the decentralised system. This is, however, based on particular costs engaged with a particular ratio. For this example, all cost ratios suggest that the centralised system will be more economical than the decentralised system. Although the analytical analysis presented in the previous section has already confirmed this finding, the magnitude of the ratio can be easily comprehended by the graphical method.

(a) Ratio[Inv]

(b) Ratio[Cap]

(c) Ratio[Total]
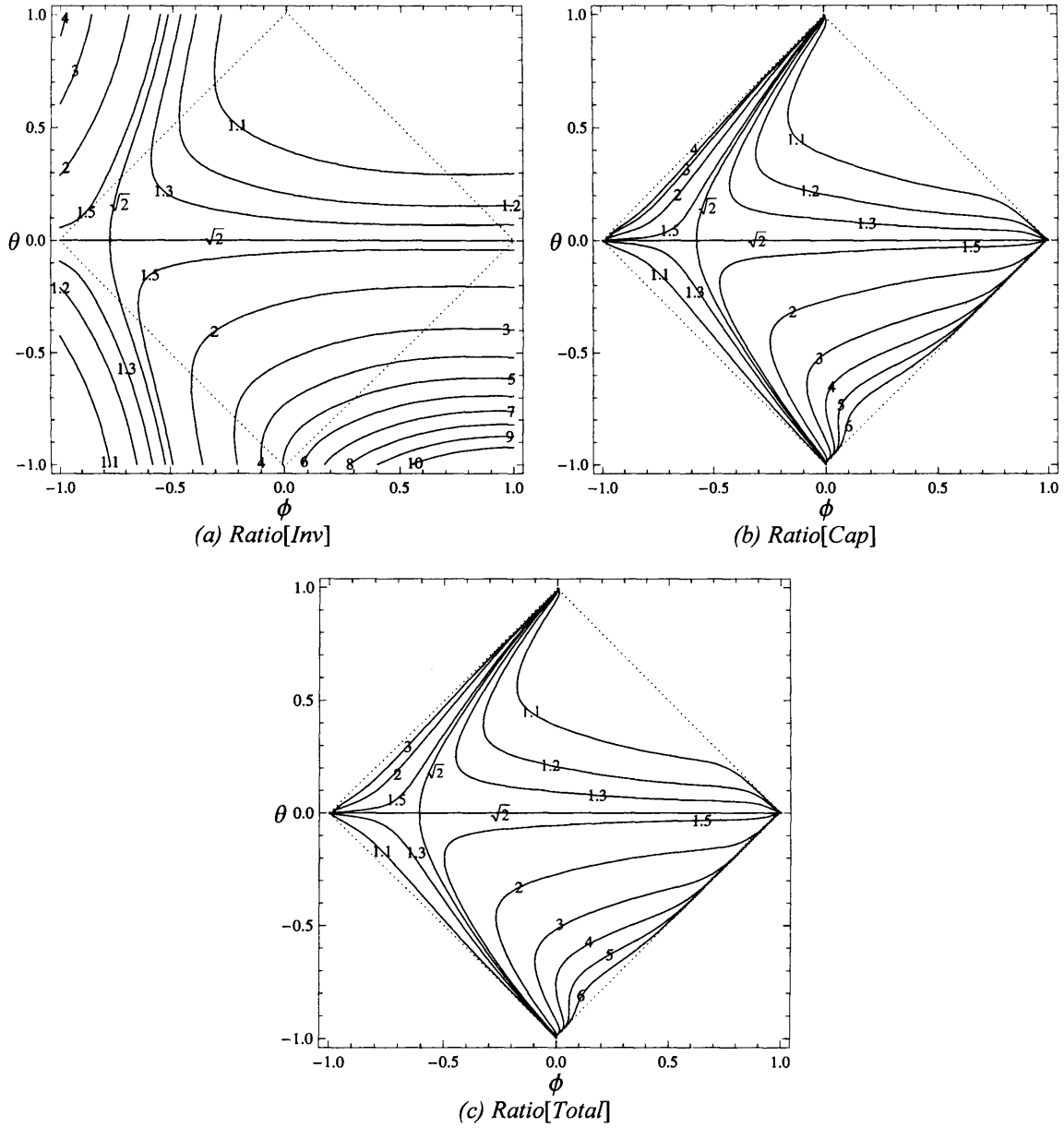
Figure 6-3 Impact of the demand patterns on the ratios of costs for the simple model with

$$n = 2 \quad and \quad l_i = l_i^c = L_i = L^c = 1$$

93

In practice, each company may pick a cost ratio that represents the major cost of its business. Then, it should set the critical value which suggests the consolidation of the DCs when the ratio is greater than or at the critical value. This benchmark could also be considered by the proportion of other extra costs and/or savings involved with distribution network consolidation such as transportation, customer satisfaction and facilities costs. For example, if the total cost ratio is selected and the critical value is set to be 1.25 this will ensure that the network consolidation will provide at least 20 percent total savings in combined inventory and capacity costs.[1]

While Figure 6-3 presents all possible cases of the demand pattern, Figure 6-4 shows the cross-sections of the contour plot at fixed ϕ's so the impact of different cost ratios and the impact of the cross-correlation can be conveniently observed. Obviously, the graphs for different ϕ's are different. This illustrates the impact of the auto-correlation. At a particular ϕ, costs saving from a consolidation can range between 0% and 85% by the impact the cross-correlation. For example, for cases where the auto-correlation is not high $\phi = 0.3$, the range $Ratio[Inv]$ is between 1 (for highly positive θ) and 5 (for highly negative θ) which can be calculated as 0% and 80% savings in inventory costs respectively. $Ratio[Cap]$ can generate even greater savings as the range is between 1 (for highly positive θ) and more than 7 (for highly negative θ) which can be calculated as 0% and more than 85.7% savings in capacity costs respectively. **In summary, the results indicate that a consolidation decision should also be based on both levels of auto-correlation (ϕ) and cross-correlation (θ) to avoid making errors in the design of a distribution network.**

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[1] A cost ratio of 1.25 means the decentralised cost is 1.25 times of the centralised cost.

Thus, the saving by consolidation $= \dfrac{\text{Decentralised costs} - \text{Centralised costs}}{\text{Decentralised costs}}$

$= \dfrac{1.25(\text{Centralised costs}) - \text{Centralised costs}}{1.25(\text{Centralised costs})} \times 100\%$

$= \dfrac{1.25 - 1}{1.25} \times 100\% = 20\%$.
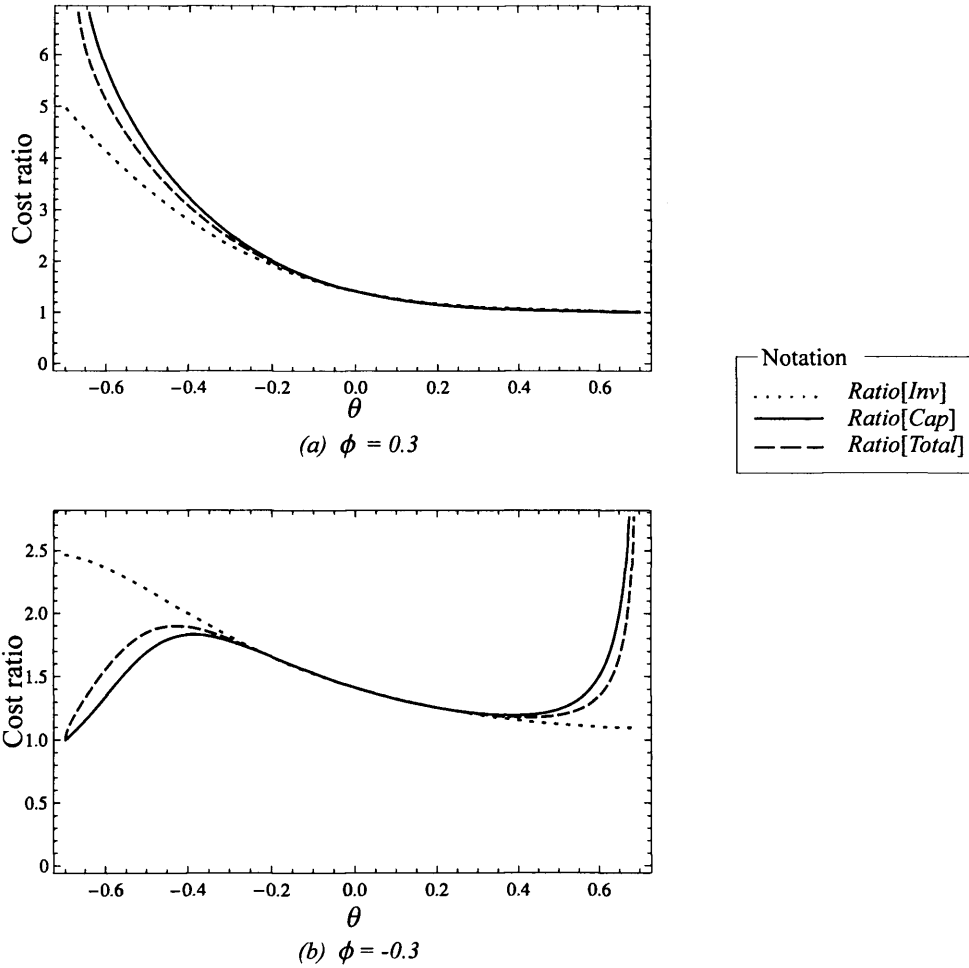
Figure 6-4 Impact of the cross-correlation on the cost ratios and consolidation decisions for the simple model with $n = 2$ and $l_i = l_i^c = L_i = L^c = 1$

In addition, the plots show that different types of cost ratios may support different consolidation decisions. Although all cost ratios are greater than one, the magnitude of each ratio can be different. This means consolidations can be more or less encouraging depending on the demand pattern as well as the type of costs under consideration.

6.3 Analytical analysis of the variances

6.3.1 Symmetric property of the variances at the decentralised locations

For the simple model in which it is assumed that all $\phi_{ii} = \phi$ and all $\phi_{ij} = \theta$, the signs of θ do not affect the value of the inventory variance and of the order variance. This is true for all locations in the decentralised system. Figure 6-2 shows that the contour plots of variances of the retailers and the decentralised DCs are symmetric along the x-axis. Therefore, at a fixed ϕ the variances for the demand pattern with (ϕ, θ) and $(\phi, -\theta)$ are of the same value. This can be easily proved by observing the formulas of $Var[i_i]$, $Var[o_i]$, $Var[I_i]$ and $Var[O_i]$ in Table 6-1. All the terms in the formulas that contain θ are to the power of an even number. Therefore, the values of the variances are not affected by the signs of θ.

This symmetry property also holds for cases where lead-times are arbitrary. The general formulas of $Var[i_i]$, $Var[o_i]$, $Var[I_i]$ and $Var[O_i]$ for the simple model with $n = 2$ are presented in Appendix C in Equations (C.1) to (C.4) respectively. This can be proved by substituting θ by $-\theta$. The result shows that the formulas are exactly the same for both θ and $-\theta$.

6.3.2 Constancy property of the variances at the centralised locations

From Figure 6-2 the contour plot for the centralised DC is apparently linear. This resulted from the specific assumptions applied to the simple model where all $\phi_{ii} = \phi$ and all $\phi_{ij} = \theta$. This presumption allows any pair of (ϕ, θ) that satisfy an equation $\phi + \theta = c$, where c is a constant, to have the same value of variances. Note that the given equation is a parallel line corresponding to the stability condition. This property can be proved by substituting ϕ by $(c - \theta)$ into the variance formulas of $Var[I^c]$ and $Var[O^c]$ from Table 6-1.

Let $\phi = c - \theta$, the formula for the inventory variance for the centralised DC becomes

$$Var[I^c] = 2((1 + \theta + \theta^2 + \phi + 2\theta\phi + \phi^2)^2 + (1 + \theta + \phi)^2(1 + (\theta + \phi)^2)^2)$$

$$= 2((1 + \theta + \theta^2 + (c - \theta) + 2\theta(c - \theta) + (c - \theta)^2)^2$$

$$+ (1 + \theta + (c - \theta))^2(1 + (\theta + (c - \theta))^2)^2)$$

$$= 2((1 + c(1 + c))^2 + (1 + c)^2(1 + c^2)^2).$$

In similar manner, the formula for the order variance for the centralised DC becomes

$$Var[O^c] = 2(1 + \theta(1 + \theta + \theta^2 + \theta^3) + \phi + \theta(2 + \theta(3 + 4\theta))\phi + (1 + 3\theta + 6\theta^2)\phi^2$$

$$+ (1 + 4\theta)\phi^3 + \phi^4)^2 - \frac{2(\theta + \phi)^{10}}{(\theta + \phi - 1)(\theta + \phi + 1)}$$

$$= \frac{4c(c - 1)(c + 1)(1 + c^2)(1 + c + c^2 + c^3 + c^4) - 2}{(c - 1)(c + 1)}.$$

The above formulas show that both $Var[I^c]$ and $Var[O^c]$ depend only on the value of c. Therefore, for the simple model, the values of $Var[I^c]$ and $Var[O^c]$ will be constant for the demand pattern whose (ϕ, θ) satisfies the condition $\phi + \theta = c$.

This property also holds for cases where lead-times are arbitrary. The general formulas of $Var[I^c]$ and $Var[O^c]$ for the simple model with $n = 2$ are presented in Appendix C in Equations (C.5) and (C.6) respectively. This assertion can be proved in the same manner as before. Furthermore, this constancy property also holds for cases where n is greater than 2 where the condition that allows constant variances becomes $\phi + (n - 1)\theta = c$.

6.3.3 Influence of the downstream lead-times on the variance of order rate

In this section, an insight made by Hosoda and Disney (2006) is further investigated. Their study considered AR(1) demand in a 3-level supply chain with one player in each level. They showed that when the auto-correlation is positive the order variance of a higher echelon is not affected by the values of the lead-time of the lower levels or the local lead-time. The order variance will keep the same value under the constraint that the accumulation of the local and downstream lead-times is constant. This finding will be tested further in this section on cases where the cross-correlation exists. Table 6-2 shows

the order variances for different demand patterns and lead-time settings. The results confirm that for a particular demand pattern the variance of the decentralised DC i keeps the same value when the $l_i + L_i$ is constant. Therefore, when all participants in the decentralised system use the OUT policy with the MMSE forecasting scheme, the variance of order rates of the decentralised DC will keep the same value under the constraint that the summation of the downstream lead-time and the local lead-time is constant even for the case where the customer demands are cross-correlated. This insight could be useful for assessing the impact of re-allocating lead-times between echelons. This, for example, is when a company plans to reduce the lead-time at the retailer level but with limited budget some resources such as labour, equipment and technology from the DC level is re-allocated at the retailer. This makes the lead-time at the retailer shorter and the lead-time at the DC longer.

Table 6-2 Var[O_i] for different demand pattern and lead-times settings

Case	Demand pattern				Lead-time settings				Order variances	
Number	ϕ_{11}	ϕ_{12}	ϕ_{21}	ϕ_{22}	l_1	L_1	l_2	L_2	$Var[O_1]$	$Var[O_2]$
1	0.7	-0.1	-0.1	0.7	2	5	2	5	13.063	13.063
2	0.7	-0.1	-0.1	0.7	5	2	5	2	13.063	13.063
3	0.6	0.2	-0.3	0.5	3	4	3	4	4.319	3.729
4	0.6	0.2	-0.3	0.5	4	3	4	3	4.319	3.729
5	0.7	-0.1	0.25	0.4	1	4	2	3	8.457	3.550
6	0.7	-0.1	0.25	0.4	4	1	3	2	8.457	3.550
7	-0.8	0.1	-0.2	0.7	1	5	3	4	0.425	9.980
8	-0.8	0.1	-0.2	0.7	5	1	4	3	0.425	9.980

Note: The auto-correlation of the demand at retailer 1 (ϕ_{11}); The auto-correlation of the demand at retailer 2 (ϕ_{22}); The cross-correlation of the demand at retailer i to the one period lag of the demand at retailer j (ϕ_{ij}) for $i \neq j$; Lead-time at retailer i (l_i); Lead-time at DC i (L_i); The variance of order rate for decentralised DC i ($Var[O_i]$).

To prove this insight for the case of a two-level supply chain, let l_e represent $l_i + L_i$, which is the accumulation of the lead-times of the downstream and the local locations i. Substituting l_e into the expression of the order variance, $Var[O_i]$, shown in Equation C.4 in Appendix C becomes

$$VAR[O_i] = \frac{1}{4}\left(\frac{1}{\left(\theta^2 - (\phi-1)^2\right)^2}\left(\begin{array}{l}\left(\begin{array}{l}2 - (\theta-1)\theta^3(\phi-\theta)^{l_e} + (\phi-\theta-1)(\phi+\theta)^{3+l_e} + \\ \phi\left((\phi-\theta)^{l_e}(2\theta^3 + (\phi-1)\phi^2) - 3\theta^2 + \theta(3-2\phi)\phi - 2\right)\end{array}\right)^2 + \\ \left(\begin{array}{l}(\phi-\theta)^{l_e}\left(\theta^4 + 3\theta^2\phi - (\phi-1)\phi^3 - \theta^3(1+2\phi)\right) - \\ (\phi+\theta)^{l_e}\left(\theta^3(1+\theta) + \phi^3 - \phi^4 + \theta^2(3+2\theta)\phi\right) + \\ \theta\left(2 + \phi^2(2\phi-3)\right)\left((\phi-\theta)^{l_e} + (\phi+\theta)^{l_e}\right)\end{array}\right)^2\end{array}\right) \\ - \frac{2(\phi-\theta)^{6+2l_e}}{(\theta-\phi)^2 - 1} - \frac{2(\phi+\theta)^{6+2l_e}}{(\phi+\theta-1)(\phi+\theta+1)}\right).$$

(6.6)

Interestingly Equation (6.6) will always be constant as long as l_e is constant.

6.3.4 Impact of the lead-times on the inventory and order variances for the different demand patterns

Lead-times have different impacts on the inventory and order variances when the demand patterns are different. The effects of the increases of the lead-time on the inventory variances at the retailer level for different demand patterns are shown in Figure 6-5. The increasing of the lead-time generally increases the inventory variance. However, the speed of the rises is influenced by the demand patterns. The inventory variances of the demands with positive ϕ (presented by a regular line) increase more dramatically when compared to that of the demands with negative ϕ (presented by a dotted line). The values of θ enhance the influence of the demand patterns on the inventory variance. For positive ϕ, the term $\phi + |\theta|$ determines the magnitude of the effect of the lead-time on the value of variances.

Variance of inventory levels at retailer i

($Var[i_i]$)

140

120 — (0.7, -0.2) and (0.7, 0.2)

100 — (0.2, -0.7) and (0.2, 0.7)

(-0.7, -0.2), (-0.7, 0.2),
(-0.5, -0.2), (-0.5, 0.2),
(-0.7, 0.0), (-0.5, 0.0)

80 — (0.7, 0.0)

60

(0.5, -0.2) and (0.5, 0.2)
40 — (0.2, -0.5) and (0.2, 0.5)
(0.5, 0.0)

20 — (-0.2, -0.7) and (-0.2, 0.7)
(-0.2, -0.5) and (-0.2, 0.5)

0

Lead-time of retailer i (l_i)

1   2   3   4   5   6   7   8   9   10

Notation

.... lines for negative auto-correlation
———— lines for positive auto-correlation
(x, y) a pair of (ϕ, θ)
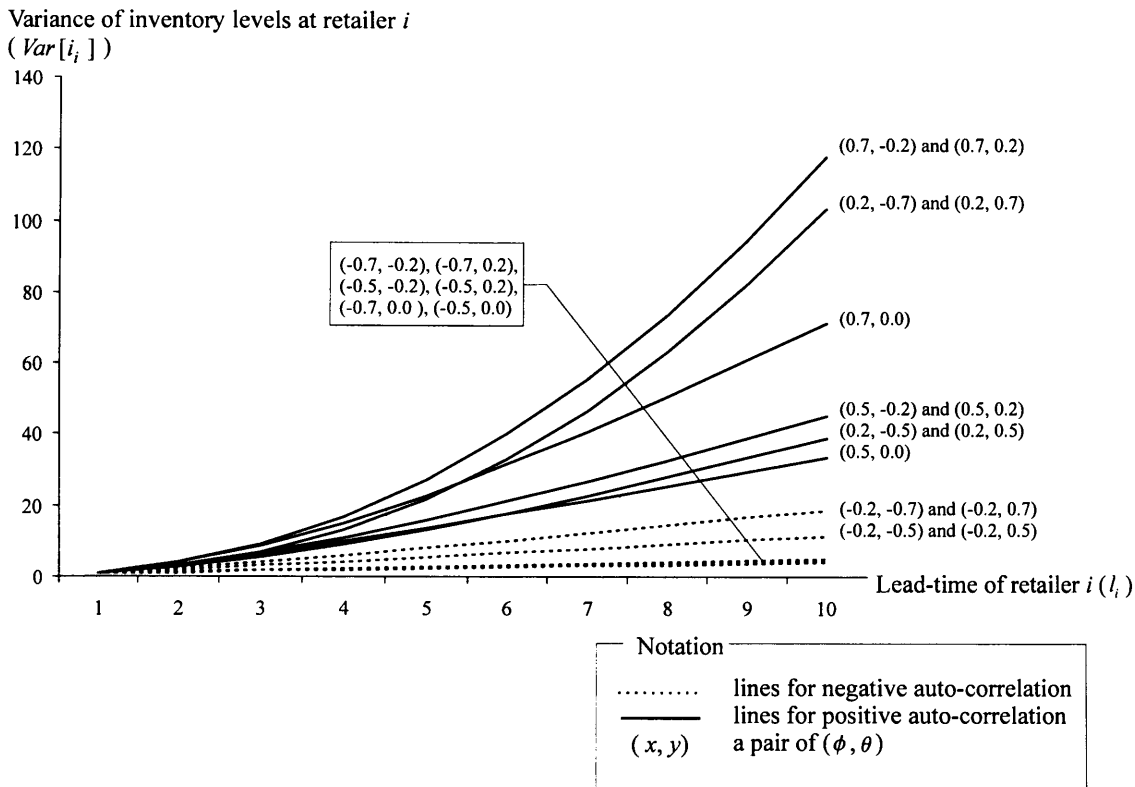
Figure 6-5   The impact of lead-times on the inventory variance for different demand patterns

A similar plot of the order variances is presented in Figure 6-6. Once again, the demand patterns highly affect the changes of the order variances when the lead-time increases. The order variances of the demands with positive ϕ (presented by a regular line) increase with the lead-times. On the other hand, the order variances of the demands with negative ϕ (presented by a dotted line) fluctuate up and down by the increment of the lead-times. These figures have illustrated how the lead-times and the demand patterns influence the variances and how the influence of the demand patterns can overshadow the influence of the lead-times in some situations. For instance this could happen when the demand pattern has negative ϕ. The values of the order variance for this case are modestly changed by the increase of the lead-time.

Variance of order rates at retailer i

($Var[o_i]$)
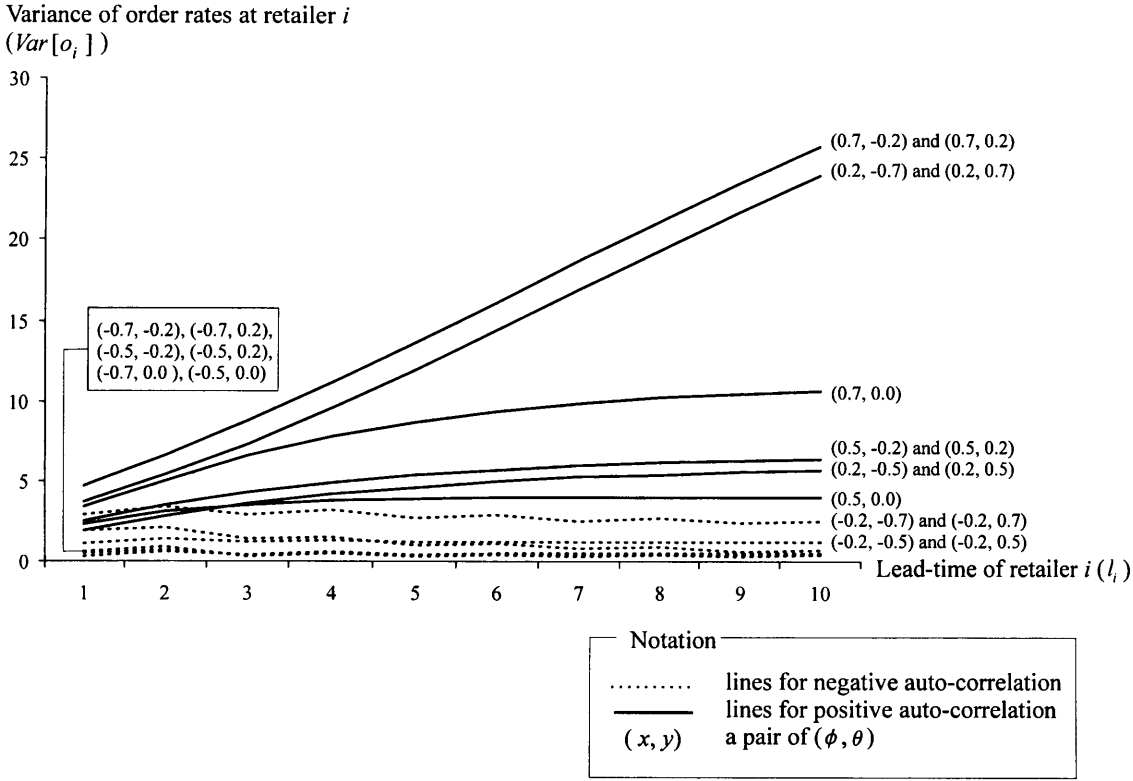
Figure 6-6 The impact of lead-times on the order variance for different demand patterns

6.4 Impact of the model parameters on the cost ratios

This section presents the impact of the model's parameters on consolidation decisions. The simple model, where all $\phi_{ii} = \phi$ and all $\phi_{ij} = \theta$, is again exploited throughout this section. Impacts of the lead-times of each player in both decentralised and centralised systems and the number of decentralised locations will be considered. The auto-correlation coefficient (ϕ) is assumed to be 0.7 (Lee et al. 2000; Raghunathan 2003, for example, advocated this setting) and the cross-correlation coefficient (θ) is between (−0.3, 0.3), which is within the stability region.

6.4.1 Impact of the overall lead-time

To see the impact of the lead-time in general, in this section all locations are assumed to have the same lead-time (called here the lead-time L). The lead-time L has a similar effect on all cost ratios. That is when lead-time L increases, for negative cross-correlations, the ratios will also increase but for positive cross-correlations the cost ratios will slightly

decrease. This is shown in Figure 6-7. The cross-correlation, therefore, plays a major role in the consolidation decision as it can intensify (as in this example when θ is highly negative) or weaken (as in this example when θ is positive) the benefit of consolidation. **Consolidations of a distribution network should therefore be emphasised more when the cross-correlation is more negative and the lead-time L is higher.**
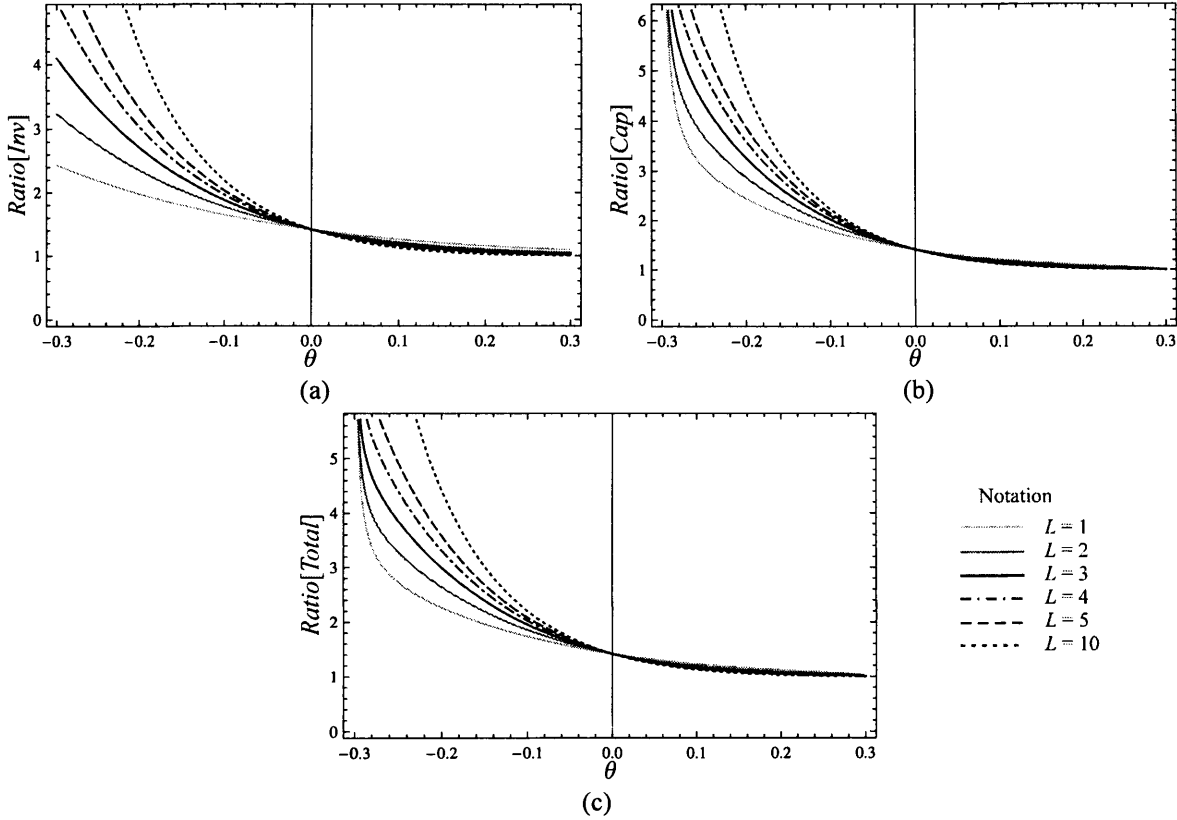
Figure 6-7 Impact of the lead-time L on the cost ratios when $\phi = 0.7$

(for the simple model with $n = 2$ and $l_i = l_i^c = L_i = L^c = L$)

Interestingly, from Figure 6-7, it can be seen that all cost ratios of all different lead-time L are at the same point when $\theta = 0$. This characteristic of the inventory cost ratios is called 'The Square Root Law for inventory' (Maister 1976). When inventories from n identical and independent decentralised locations are consolidated into a single centralised location, Maister (1976) proved that the approximation of the ratio between decentralised and centralised inventories is equal to the square root of the number of n. In the cases in Figure 6-7 it is assumed that all locations have the same lead-times and auto-correlation coefficients. This means all decentralised locations are identical. Therefore, when $\theta = 0$,

the cost ratios reduce to $\sqrt{n} = 1.414214$ (as in this case $n = 2$). Note that in these cases the Square Root Law holds for all of the cost ratios: $Ratio[Inv]$, $Ratio[Cap]$ and $Ratio[Total]$. This means, when all decentralised locations are identical and the cross-correlation is equal to zero, the benefit of network consolidation will be the same (and equal to $\sqrt{n}$) regardless of the values of the auto-correlation and of the lead-time L. Section 6.4.4 will numerically demonstrate the Square Root Law when n is greater than two. Chapter 7 will provide more details and prove the Square Root Law for inventory and for bullwhip in a formal fashion for general n and general lead-times.

6.4.2 Impact of the lead-times of the players in the decentralised system

In this section the impact of individual lead-times of each player in the decentralised system on the cost ratios is investigated. Figure 6-8 and Figure 6 -9 show that the individual lead-times of the players in a decentralised system do not have much effect on the cost ratios as long as they can maintain the total lead-time of all players in each supply chain level. Note that the examples given in Figure 6-8 and Figure 6-9 are set at relatively low lead-times. If the lead-times are higher, the effect will be clearer. On the other hand, the impact of the total lead-time in each echelon can be clearly seen in Figure 6-10 and Figure 6-11. When the total lead-time increases, the ratio of inventory cost shifts equally over the possible range of θ while the ratio of capacity cost changes depending more on the value of θ.

Notation

—— — $Ratio[Inv]$-Case I
— — — — $Ratio[Inv]$-Case II
— —·— $Ratio[Cap]$-Case I
—·—·— $Ratio[Cap]$-Case II
———— $Ratio[Total]$-Case I
——— $Ratio[Total]$-Case II

Case I : $l_1 = 1, l_2 = 3$

Case II : $l_1 = 2, l_2 = 2$
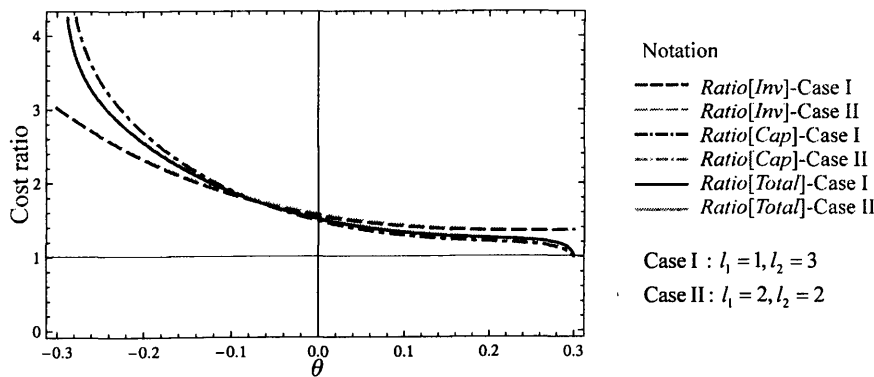
Figure 6-8 Impact of the individual retailers' lead-times (l_i) when $\phi = 0.7$

(for the simple model with $n = 2$, $\sum_i l_i = 4$ and $l_i^c = L_i = L^c = 1$)

Notation

---- $Ratio[Inv]$-Case I
----- $Ratio[Inv]$-Case II
--·-- $Ratio[Cap]$-Case I
--··-- $Ratio[Cap]$-Case II
—— $Ratio[Total]$-Case I
······· $Ratio[Total]$-Case II

Case I : $L_1 = 1, L_2 = 3$
Case II : $L_1 = 2, L_2 = 2$
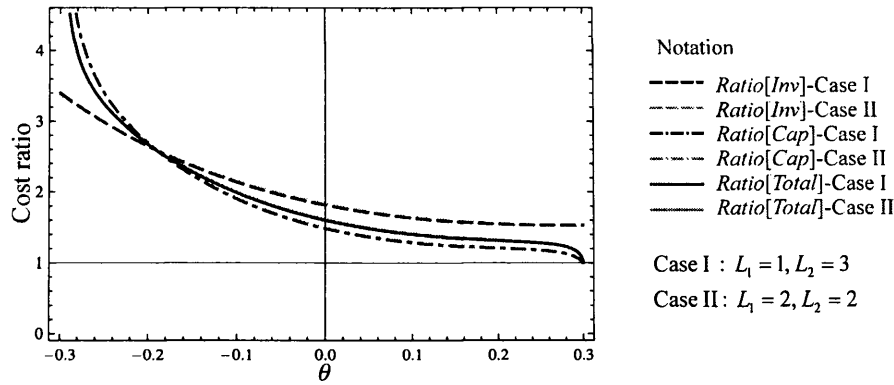
Figure 6-9 Impact of the individual decentralised DC's lead-times (L_i) on the cost ratios when $\phi = 0.7$ (for the simple model with $n = 2$, $\sum_i L_i = 4$ and $l_i = l_i^c = L^c = 1$)

Notation

---- $Ratio[Inv]$-Case I
----- $Ratio[Inv]$-Case II
--·-- $Ratio[Cap]$-Case I
--··-- $Ratio[Cap]$-Case II
—— $Ratio[Total]$-Case I
······· $Ratio[Total]$-Case II

Case I : $\sum_i l_i = 2$
Case II : $\sum_i l_i = 4$
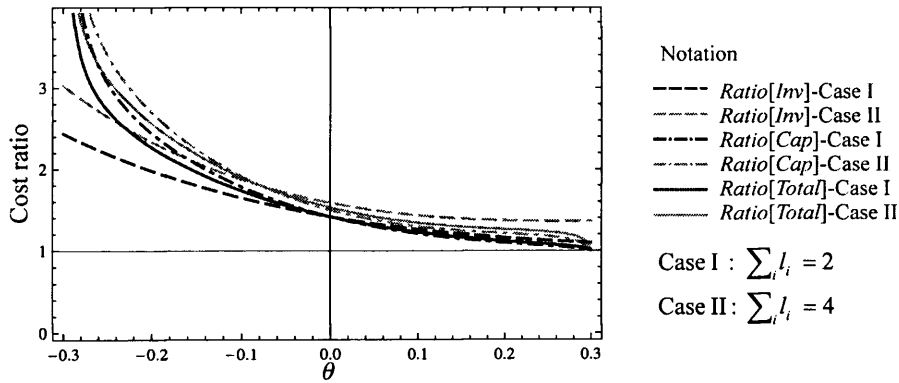
Figure 6-10 Impact of the retailers' total lead-time $(\sum_i l_i)$ on the cost ratios when $\phi = 0.7$ (for the simple model with $n = 2$ and $l_i^c = L_i = L^c = 1$)

Notation

---- $Ratio[Inv]$-Case I
----- $Ratio[Inv]$-Case II
--·-- $Ratio[Cap]$-Case I
--··-- $Ratio[Cap]$-Case II
—— $Ratio[Total]$-Case I
······· $Ratio[Total]$-Case II

Case I : $\sum_i L_i = 2$
Case II : $\sum_i L_i = 4$
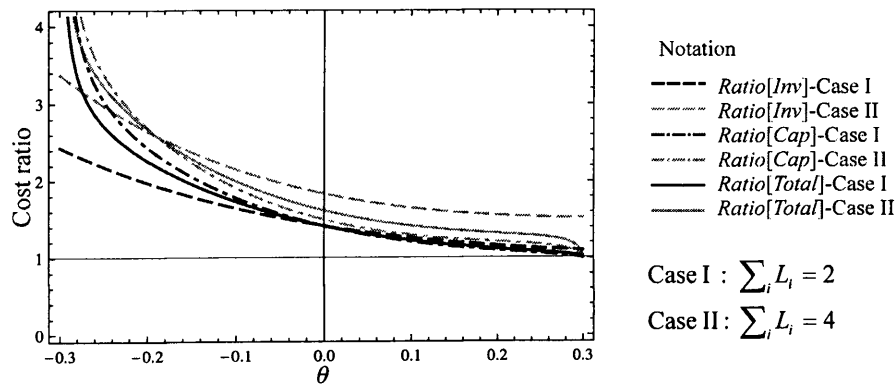
Figure 6-11 Impact of the decentralised DC's total lead-time $(\sum_i L_i)$ on the cost ratios when $\phi = 0.7$ (for the simple model with $n = 2$ and $l_i = l_i^c = L^c = 1$)

6.4.3 Impact of the lead-times of the players in the centralised system

In the centralised system, the individual lead-times show a visible impact on the cost ratio as shown in Figures 6-12 to 6-14. This is especially true for the centralised DC's lead-time. Investigating the relative change of the expressions of the variances for the decentralised and centralised systems, a higher positive θ generates greater variations to inventory levels and order rates in the centralised system. The impact is great as it can bring the cost ratios below 1 for positive θ; this means that the consolidation is cost-ineffective especially when θ is near 0.3. **Thus, the centralised DC's lead-time is a critical factor affecting the benefit from the consolidation of a distribution network.**
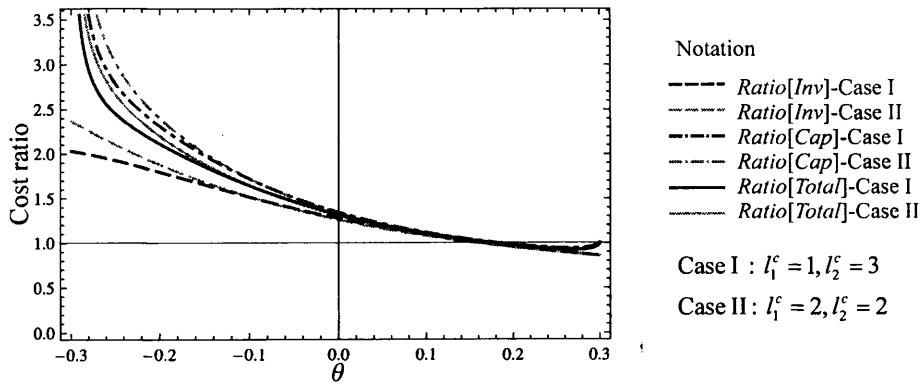
Figure 6-12 Impact of the individual centralised retailers' lead-times (l_i^c) on the cost ratios when $\phi = 0.7$ (for the simple model with $n = 2$, $\sum_i l_i^c = 4$ and $l_i = L_i = L^c = 1$)
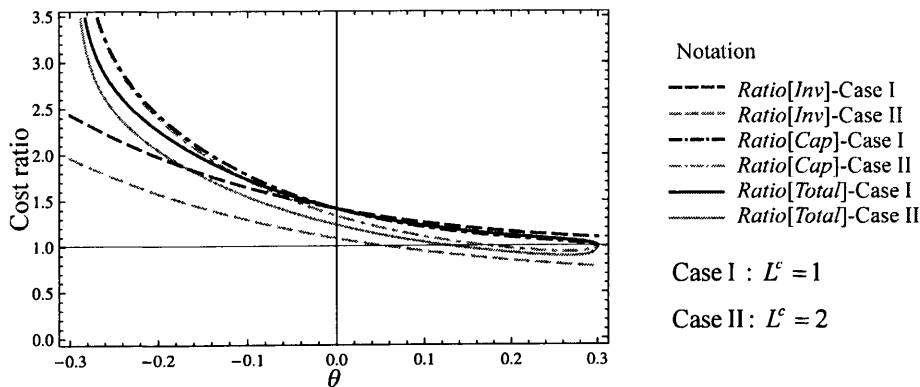
Figure 6-13 Impact of the centralised DC's lead-times (L^c) on the cost ratios when $\phi = 0.7$ (for the simple model with $n = 2$ and $l_i = l_i^c = L_i = 1$)
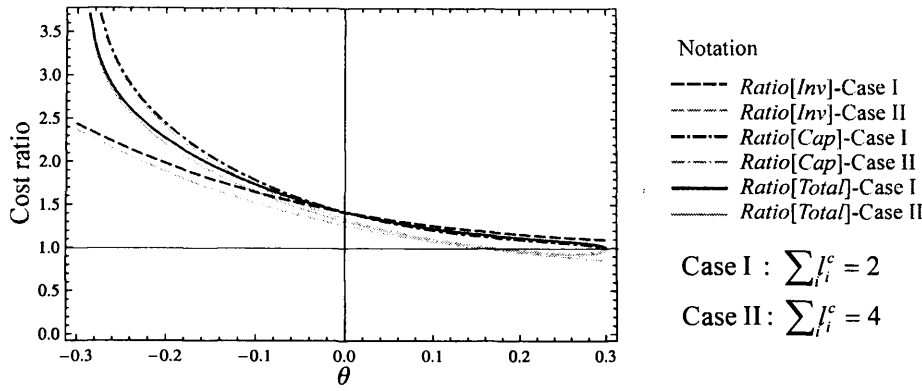
Figure 6-14 Impact of the centralised retailers' total lead-time $\left(\sum_i l_i^c\right)$ *on the cost ratios when* $\phi = 0.7$ *(for the simple model with* $n = 2$ *and* $l_i = L_i = L^c = 1)$

The crucial factors that determine whether consolidation of a distribution network is beneficial are the lead-times of the participants in the centralised system and the cross-correlation coefficient. When the lead-times in the centralised system are higher than in the decentralised system, the benefit from consolidation is low. The values of the cross-correlation coefficient, again, strengthen or weaken the consolidation benefit. The consolidation of a distribution network brings most benefit when customer demands are highly negatively cross-correlated. It must be noted again that this result is for when $\phi = 0.7$. As the ϕ is generally found to be highly positive in consumer products, the result is shown for $\phi = 0.7$ in this section. For negative ϕ, the conclusions to be drawn are somewhat more complex as the result depends very much on the specific values of the auto- and cross- correlation coefficients.

6.4.4 Impact of the number of the decentralised DCs

Figure 6-15 presents the impact of the number of decentralised DCs (n) on the cost ratios. Consolidations of larger n provide greater cost ratios. The magnitude of the cost ratios, however, depends on the demand pattern. Note that the stability area is reduced when n is greater and this is shown by the shaded area under each line. Figure 5-2 shows how the spans of the shaded area are constructed.

The Square Root Law was found when the cross-correlation of the demand is equal to zero. This is a similar situation that has been investigated in Section 6.4.1 but with $n \geq 2$.

It is shown that at $\theta = 0$, the ratios are equal to the square root of n. That is the Square Root Law for inventory and Bullwhip hold (Ratanachote and Disney 2008).
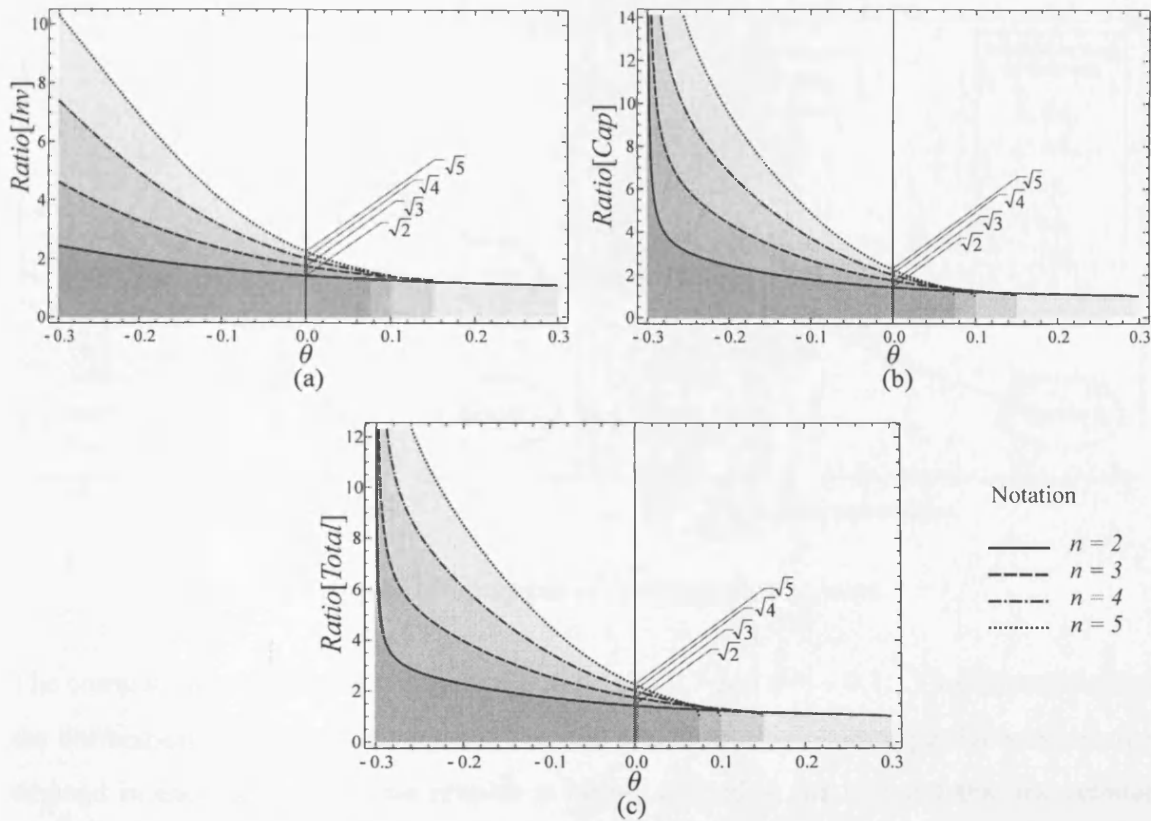
Figure 6-15 Impact of number of decentralised DC's (n) on the cost ratios when $\phi = 0.7$
(for the simple model with $l_i = l_i^c = L_i = L^c = 1$)

6.5 Example of consolidation scenarios

This section presents a consolidation scenario that is guaranteed to be cost effective. Again, the simple model is applied. The situation is depicted in Figure 6-16. In the decentralised system, there are two retailers (with lead-time l_1 and l_2 respectively) and two DCs (with lead-time L_1 and L_2 respectively). It is assumed that L_1 is less than or equal to L_2. For the centralised scenario, the two decentralised DCs are consolidated into a single centralised DC which is at the original DC 1 location. This is due to the advantage of lower lead-time. Thus, the centralised DC's lead-time is the same as the lead-time of DC 1 ($L^c = L_1$). The lead-time of retailer 1 stays the same after consolidation ($l_1^c = l_1$).

The lead-time of retailer 2 in the centralised system is, however, increased or stays the same ($l_2^c \geq l_2$) as it has to take delivery from the centralised DC instead of DC 2.

Before consolidation   After consolidation
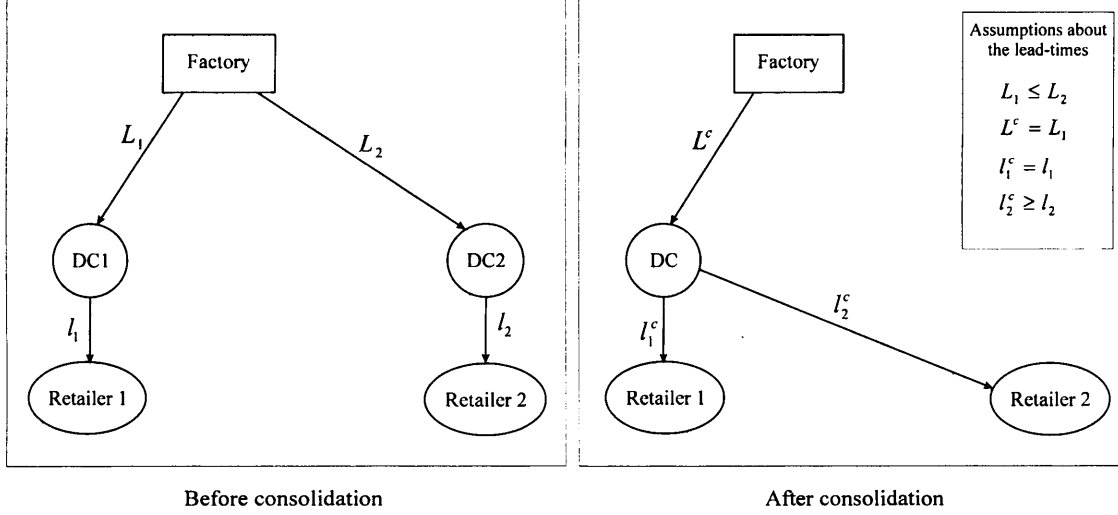
Figure 6-16 Example of consolidation scheme

The correlation coefficients are assumed to be $\phi = 0.7$ and $\theta = -0.1$. This assumption of the correlation coefficients represents the real situation of consumer products where the demand in each period for one retailer is highly related to the demand that the retailer observed in the previous period (Erkip et al. 1990; Lee et al. 2000; Raghunathan 2003) and there is a small negative influence on the demand of the other retailer. The influence of the previous value of the demand on itself is higher than that from other retailers; that is $-\theta < \phi$.

Although the closed form of the cost ratios for this case can be obtained (as shown in Appendix C), it is still tedious to evaluate the problem analytically. The assumptions about the lead-times have simplified the problem but they also create some conditions that should be carefully investigated. The impact of the lead-times presented in Section 6.4 can be applied in this analysis. Under the assumption that $\phi = 0.7$ and $\theta = -0.1$, the cost ratios generally increase with the decentralised locations' lead-times and decrease with the centralised locations' lead-times. However, the assumptions about the lead-times for this scenario force the lead-times to simultaneously change. The impact of the concurrent changes of lead-times on the cost ratios is harder to predict. To deal with this

108

problem, the graphical method is used to identify the worst possible cases that produce the minimum $Ratio[Inv]$ and $Ratio[Cap]$.

The following graphs represent the cost ratios under a particular set of assumptions about lead-times. Figure 6-17 shows the cost ratios under the assumptions that $L_1 \leq L_2$ and $L^c = L_1$. From the assumption that $L_1 \leq L_2$, let $L_2 = L_1 + k_1$ where k_1 represents the difference of the lead-times between DC 2 and DC 1. It is shown in Figure 6-17 that the worst cases for the $Ratio[Inv]$ and $Ratio[Cap]$ are when L_1 is small and when $k_1 = 0$.

(a) Ratio[Inv]   *(b) Ratio[Cap]*
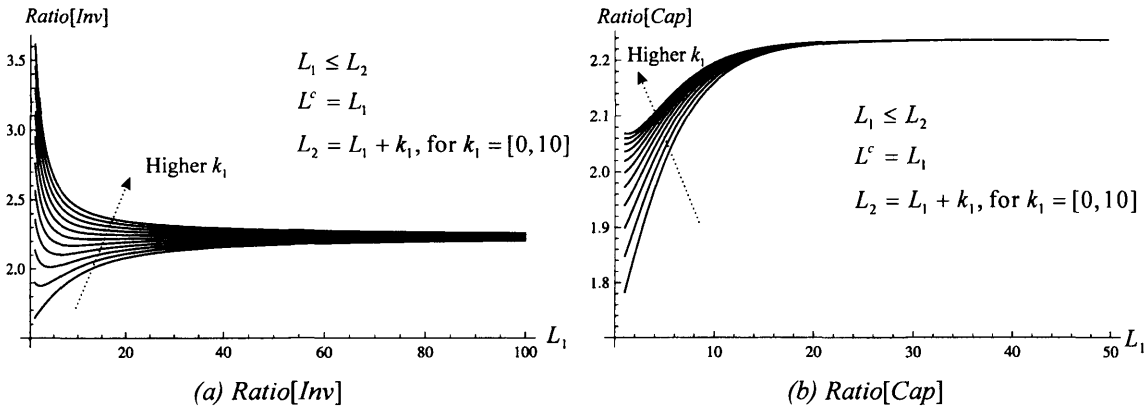
Figure 6-17 The decentralised DC's lead-time plots against the cost ratios

for the simple model with $n = 2$ and $l_1 = l_2 = l_1^c = l_2^c = 1$

Figure 6-18 shows the cost ratios under the assumptions that $l_2^c \geq l_2$. In a similar manner, let $l_2^c = l_2 + k_2$ where k_2 represents the difference of the lead-times of retailer 2 before and after the consolidation. The graph shows that the worst case is when l_2 is smallest and when k_2 is highest.

109
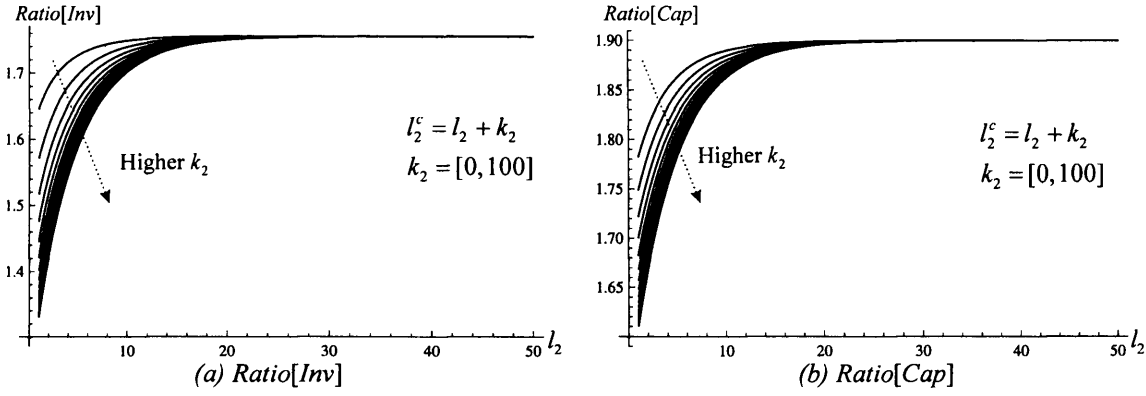
Figure 6-18 The decentralised and centralised retailers' lead-times plots against the cost ratios for the simple model with $n = 2$ and $l_1 = l_1^c = L_1 = L_2 = L^c = 1$

Figure 6-19 shows the cost ratios under the assumptions that $l_1^c = l_1$. From the graph, it can be seen that the worst case is when l_1 is smallest.
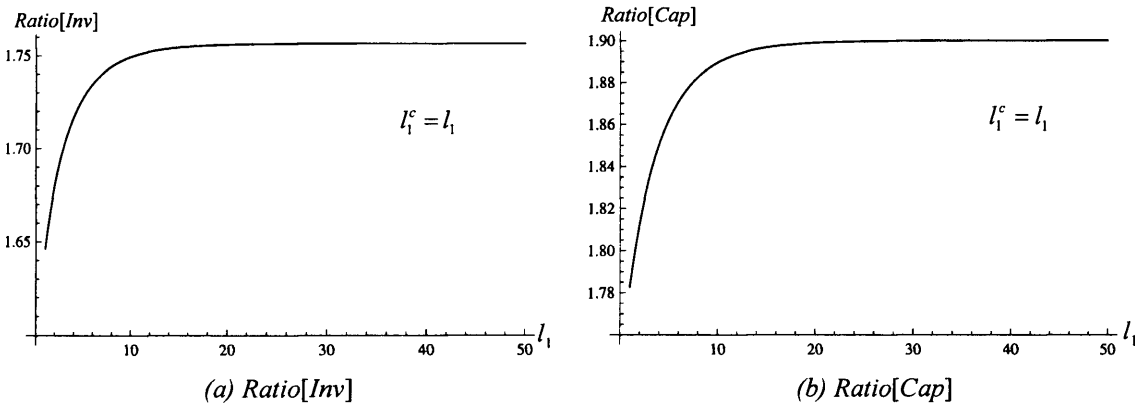
Figure 6-19 The decentralised and centralised retailers' lead-times plots against the cost ratios for the simple case with $n = 2$ and $l_2 = l_2^c = L_1 = L_2 = L^c = 1$

From the above result, the worst case is when l_1 and l_2 are smallest, k_2 is highest and L_1 and k_1 are small. Thus, the values of the cost ratios are examined when $l_1 = l_2 = 1$, $k_2 = 100$, $L_1 = 1$ and $k_1 = 0$. The test showed that the lowest value of the *Ratio[Inv]* is 1.33074 and the lowest value of the *Ratio[cap]* is 1.61093. As a result, it can now be concluded that when $\phi = 0.7$, $\theta = -0.1$, $L_1 \le L_2$, $L^c = L_1$, $l_1^c = l_1$ and $l_2^c \ge l_2$ the consolidation of the distribution network is always attractive.

110

The above examination is for a specific demand pattern that is when $\phi = 0.7$ and $\theta = -0.1$. Another approach is to plot a graph as in Figure 6-20 to investigate the worst cases of other demand patterns under the same lead-time criteria. Only the plot of the demand with positive ϕ is displayed as the nature of the worst case of the demand with negative ϕ is not the same as with positive ϕ; see Section 6.3.4 for more information. Figure 6-20 shows that some demand patterns do not guarantee that the consolidation of the distribution network will be economical. These demand patterns are presented by the shaded area.

6.6 Summary

This chapter has demonstrated the application of the model introduced in Chapter 5. Relatively simple models have been investigated. These models assume that the auto-correlation coefficients are the same for all retailers and the cross-correlation coefficients are the same between all pairs of retailers. Insights about the variances of the inventory level and the order rate have been obtained. These insights include the symmetric property of the variances at the decentralised locations, the constancy property of the variances at the centralised locations, the influence of the summation of the local and downstream lead-times on the order variance at the decentralised locations and the Square Root Law for Inventory and Bullwhip. These properties are very useful to understand the behaviour of the system for different demand patterns, different lead-time settings and different number of locations. The Square Root Law for Inventory and Bullwhip will be discussed in more detail in the next chapter.
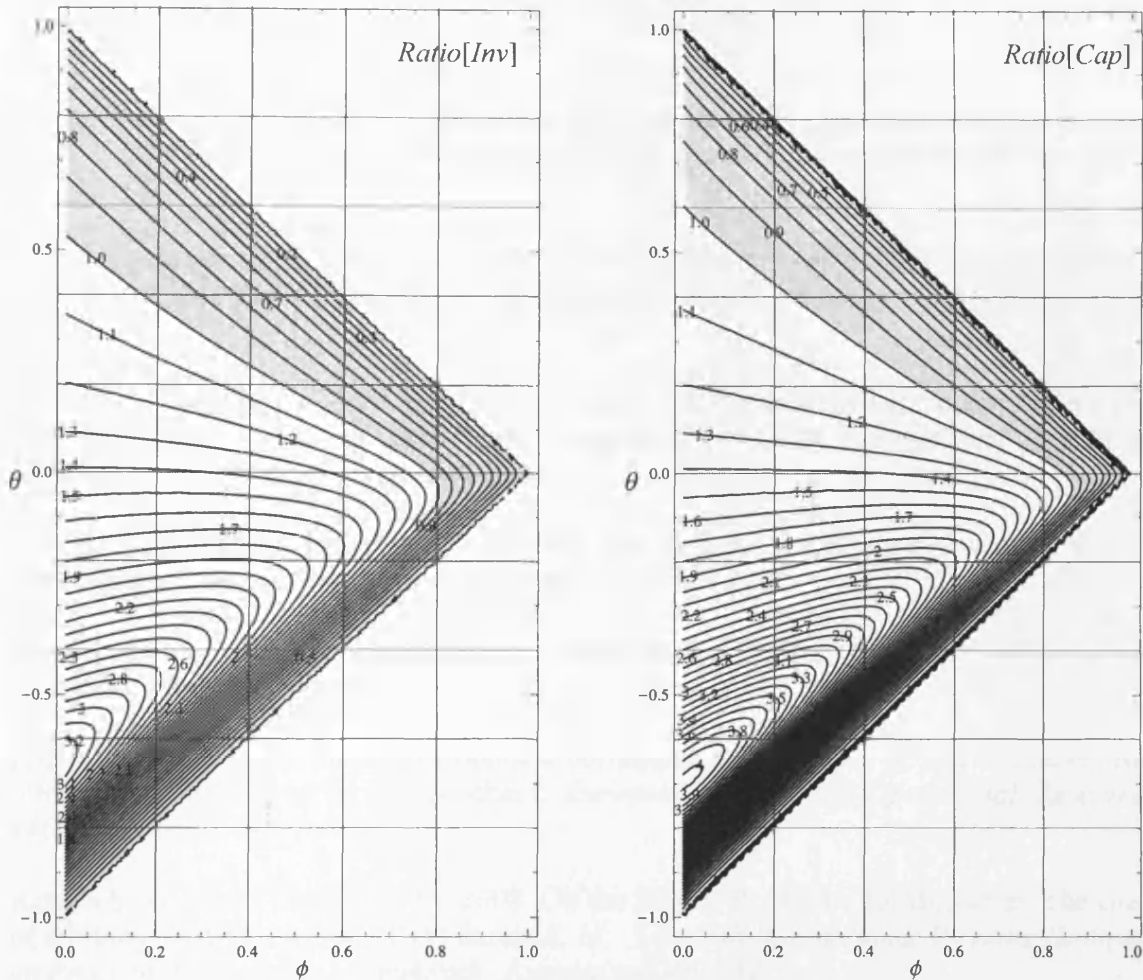
Figure 6-20 Contour plots of the worst cases of the cost ratios

The *Ratio*[*Inv*] and *Ratio*[*Cap*] have been used to measure the benefit of the network consolidation. These measures present the ratios of costs between the decentralised and centralised systems. The results indicated that the consolidation decision should use information about both auto-correlation and the cross-correlation in demands. Disregarding this information could possibly lead to wrong decisions about network consolidation as the cost ratios depend very much on the demand patterns. The magnitude of the cost ratios from the consolidation of multi-distribution centres also depends on the demand pattern. These facts are often ignored in DND methodologies. Moreover, it is important that the lead-times in the centralised system should be closely managed to assure the benefit from the consolidation of a distribution network can actually be realised. A consolidation scenario that is always cost effective, given a specific demand pattern, has also been presented.

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Chapter 7

The Square Root Law for Inventory and Bullwhip

7.1   Introduction

It has been seen from Sections 6.4.1 and 6.4.4 that in certain circumstances the ratios of costs between the decentralised and centralised systems are equal to the square root of the number of the decentralised locations. In this chapter, this will be investigated further and a formal proof for this characteristic of the cost ratios will be provided.

The Square Root Law for Inventory has been introduced by Maister (1976). Quoting directly from Maister,

"If the inventories of a single product (or stock keeping unit) are originally maintained at a number (n) of field locations (referred to as the decentralised system) but are then consolidated into one central inventory (referred to as the centralised system), then the ratio

$$\frac{\text{Decentralised system inventory}}{\text{Centralised system inventory}} = \sqrt{n} \qquad (7.1)$$

exists" (Maister 1976).

Maister (1976) provided a proof of the Square Root Law for cycle stock under the assumptions that the Economic Order Quantity (EOQ) controls the inventory system. The

114

result shows that the Square Root Law is precise when the demand at each decentralised location has independent and identical stochastic properties. It is also a good approximation otherwise (Maister 1976). Since the ratio is always greater than one (as $n \in \mathbf{N} \geq 2$ by definition), this result suggests that consolidation of inventories reduces costs.

The Square Root Law has been extensively generalised by many scholars. Zinn et al. (1989) introduced a measurement, which was developed from Maister's inventory ratio in Equation (7.1), for the savings in inventory from consolidation of inventories as,

$$\text{Portfolio Effect} = 1 - \frac{\text{Centralised system inventory}}{\text{Decentralised system inventory}}. \tag{7.2}$$

Notice, the second term of the above formula is the reciprocal of the ratio in Equation (7.1). Zinn et al. (1989)'s model, which uses the standard deviation of demand at each decentralised location, is shown to be more general than that of Maister (1976). Zinn et al. (1989) showed that the portfolio effect depends only on the correlation of demands between decentralised locations and the magnitude of the standard deviations of demand. However, in their model, they assumed that lead-times were known and identical at all locations. They applied the Pearson product-moment correlation coefficient to define the correlation of demands between locations as do the rest of the papers in this field. Evers and Beier (1993) extended the model developed by Zinn et al. (1989) to include variable lead-times. Tallon (1993) developed a model that allows variable lead-times but treated lead-times at the centralised location differently from Evers and Beier (1993). Tallon (1993) required the correlation of demand "during the lead-time" between decentralised locations to be known; whilst Evers and Beier (1993) did not require this to be known. Evers and Beier (1998) compared the practical issues in the Evers and Beier (1993) and Tallon (1993) approaches. Their study also presents an empirical analysis and provides some managerial tools for inventory consolidation. These studies paid attention to the Square Root Law, the portfolio effect, the cycle stock and the safety stock. However, they did not consider the time dimension of the demand and of the state variables.

Recently, Disney et al. (2006) have studied the Square Root Law by assuming independently and identically distributed (i.i.d.) demands and unit lead-times. Their work was concerned not only with the inventory costs but also the capacity costs. The capacity

costs related directly to the variation of orders in the OUT policy. Disney et al. (2006) showed that, as well as the Square Root Law for Inventory, the Square Root Law for Bullwhip also exists in the OUT policy. This insight was demonstrated by a numerical example. However, their i.i.d. demand and unit lead-times assumptions limit the generality of the result. Ratanachote and Disney (2008) investigated the Square Root Law for Bullwhip further by considering a two-level supply chain with the first-order autoregressive, AR(1), demands and arbitrary lead-times at each location. They provided exact analytical expressions for the variance components which were not explicitly presented in Disney et al. (2006). They used an OUT replenishment policy with the MMSE at the retailer's echelon and a Base Stock replenishment policy at the DC's echelon. Their model considers the time aspect which has not been considered by previous papers. The AR(1) demand represents the relationship of the values of current demand with its last value. However, in their paper the cross-correlation between decentralised locations is ignored although such a relationship exists in real demand data (Erkip et al. 1990). Thus, in this chapter the Square Root Law for Inventory and Bullwhip when the demands are correlated both to the previous values of itself and to the demands of other locations will be investigated. Note that this section extends the model presented in Ratanachote and Disney (2009). They also investigated the Square Root Law for Inventory and Bullwhip for VAR(1) demand but their study is restricted to $n = 2$ and unit lead-times at all locations.

7.2 Simple example of the Square Root Law

This section presents a simple example of the Square Root Law when the cross-correlation of demands between retailers is considered. The simple model as described in Table 6-1 is used. In this example, it is assumed that $n = 2$, all lead-times are units, $\phi_{11} = \phi_{22} = \phi$ and $\phi_{12} = \phi_{21} = \theta$. First, the variances of the inventory level will be investigated. The variance of the inventory level at the decentralised DC i, $Var[I_i]$, is given by

$$Var[I_i] = (\theta + 2\theta\phi)^2 + (1 + \theta^2 + \phi + \phi^2)^2 + (1 + \phi + \phi^2 + \phi^3 + \theta^2(1 + 3\phi))^2 + \theta^2(1 + \theta^2 + \phi(2 + 3\phi))^2, \forall i = \{1, 2\}.$$

(7.3)

The variance of the inventory level at the centralised DC, $Var[I^c]$, is given by

$$Var[I^c] = 2((1 + \theta + \theta^2 + \phi + 2\theta\phi + \phi^2)^2 + (1 + \theta + \phi)^2(1 + (\theta + \phi)^2)^2).$$ (7.4)

The covariance term resulting from the aggregation of inventory variances of two decentralised locations can be found by subtracting the sum of the inventory variances of the two decentralised locations from the inventory variance of the centralised location;

$$Var[I_1] + Var[I_2] - Var[I^c] = 2\left(\begin{array}{c} (\theta + 2\theta\phi)^2 + (1 + \theta^2 + \phi + \phi^2)^2 + (1 + \phi + \phi^2 + \\ \phi^3 + \theta^2(1 + 3\phi))^2 + \theta^2(1 + \theta^2 + \phi(2 + 3\phi))^2 \end{array} \right) -$$

$$2\left(\begin{array}{c} (1 + \theta + \theta^2 + \phi + 2\theta\phi + \phi^2)^2 + \\ (1 + \theta + \phi)^2(1 + (\theta + \phi)^2)^2 \end{array} \right)$$

$$= -4\theta\left(\begin{array}{c} 2 + \theta^4(1 + 3\phi) + \theta^2(3 + 2\phi(4 + 5\phi(1 + \phi))) + \\ \phi(6 + \phi(9 + \phi(8 + \phi(5 + 3\phi)))) \end{array} \right).$$

(7.5)

If $\theta = 0$, Equation (7.5) will reduce to zero. This means the variance of the inventory level of the centralised DC is equal to the sum of the variances of the inventory level of the two decentralised DCs. Note that the inventory variances of the two decentralised locations are equal. This is because of the assumptions of the simple model and the unit lead-times make the two decentralised locations identical. As a result, the *Ratio[Inv]* as shown in Equation (6.3) will be reduced to the square root of two when $\theta = 0$.

Proof of the Square Root Law for Inventory for this simple example:

When the decentralised DCs are identical, $Var[I_1] = Var[I_2] = \sigma_I^2$.

When $\theta = 0$, $Var[I_1] + Var[I_2] = Var[I^c]$. Thus, $Var[I^c] = 2\sigma_I^2$.

The optimal inventory cost per period for location i ($I_{\pounds:i}^*$) as given in Equation (4.23) is

$$I_{\pounds:i}^* = \sigma_{I:i}(B + H)\varphi\left[\Phi^{-1}\left[\frac{B}{B + H} \right] \right], \forall i = \{1, 2\},$$

where $\sigma_{I:i}$ is the standard deviation of the inventory level at location i.

Let $Y_I = (B + H)\varphi\left[\Phi^{-1}\left[\frac{B}{B + H} \right] \right]$. Thus $I_{\pounds:i}^* = \sigma_{I:i}Y_I, \forall i = \{1, 2\}$.

Thus the ratio of the inventory cost, *Ratio[Inv]*, can be given by

$$Ratio[Inv] = \frac{(\sqrt{Var[I_1]})Y_I + (\sqrt{Var[I_2]})Y_I}{(\sqrt{Var[I^c]})Y_I} = \frac{\sqrt{\sigma_I^2} + \sqrt{\sigma_I^2}}{\sqrt{2\sigma_I^2}} = \frac{2\sqrt{\sigma_I^2}}{\sqrt{2\sigma_I^2}} = \sqrt{2}.$$

The contour plot of *Ratio[Inv]* for this case is shown in Figure 7-1 (a). The bold lines represent the Square Root Law for Inventory where the ratio is equal to $\sqrt{2}$. It shows that the Square Root Law also holds for some parameter sets when $\theta \neq 0$. This will be discussed later in this section.

(a) Ratio[Inv]   *(b) Ratio[Cap]*
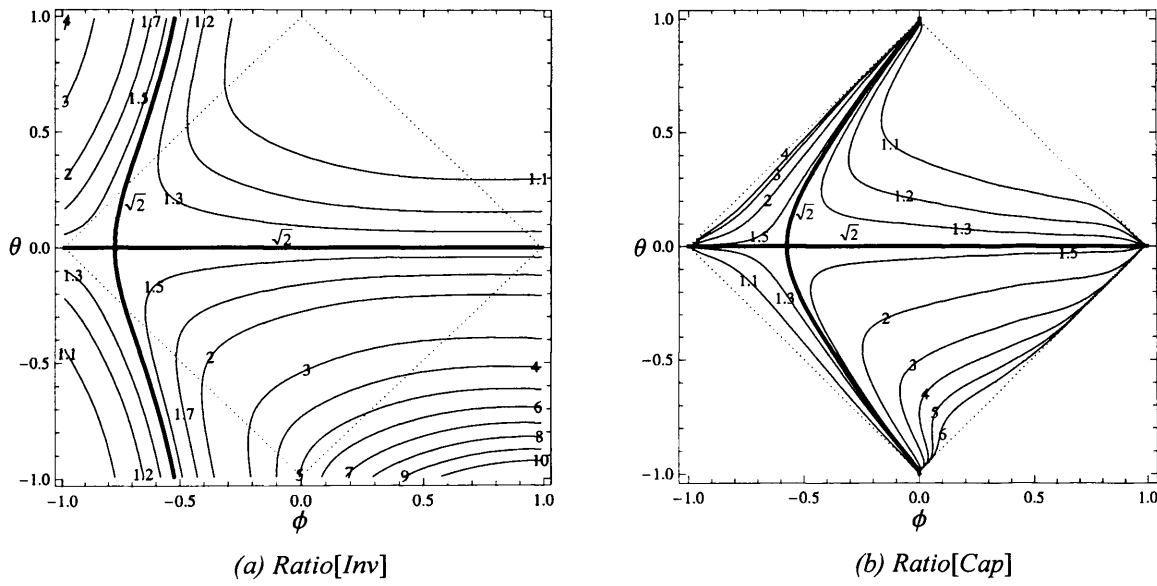
Figure 7-1 Contour plots of Ratio[Inv] and Ratio[Cap] illustrating the Square Root Law for Inventory and Bullwhip for n = 2

A similar result can be found in the ratio of capacity costs. The variances of the order rate, which directly relate to the capacity cost, will now be considered. The variance of the order rate for the decentralised DC i, $Var[O_i]$, is given by

$$Var[O_i] = \frac{1}{4}\begin{pmatrix} 8\theta^6(1 + 7\phi) + 8\phi(1 + \phi^2)(1 + \phi + \phi^2 + \phi^3 + \phi^4) \\ -\dfrac{4(\theta^2 + \phi^2 - 1)}{\theta^4 + (\phi^2 - 1)^2 - 2\theta^2(\phi^2 + 1)} + 8\theta^4(2 + 5\phi(2 + \phi(3 + 7\phi))) \\ + 8\theta^2(1 + \phi(6 + \phi(12 + \phi(20 + 3\phi(5 + 7\phi))))) \end{pmatrix}, \forall i = \{1, 2\}. \quad (7.6)$$

The variance of the order rate for the centralised DC, $Var[O^c]$, is given by

$$Var[O^c] = 2(1 + \theta(1 + \theta + \theta^2 + \theta^3) + \phi + \theta(2 + \theta(3 + 4\theta))\phi + (1 + 3\theta + 6\theta^2)\phi^2 + (1 + 4\theta)\phi^3 + \phi^4)^2 - \frac{2(\theta + \phi)^{10}}{(\theta + \phi - 1)(\theta + \phi + 1)}. \tag{7.7}$$

The covariance term can be found by subtracting the sum of the order variances of the two decentralised locations from the order variance of the centralised location and is given by

$$Var[O_1] + Var[O_2] - Var[O^c] = \frac{1}{2} \left(\begin{array}{c} 8\theta^6(1 + 7\phi) + 8\phi(1 + \phi^2)(1 + \phi + \phi^2 + \phi^3 + \phi^4) \\ -\dfrac{4(\theta^2 + \phi^2 - 1)}{\theta^4 + (\phi^2 - 1)^2 - 2\theta^2(\phi^2 + 1)} \\ + 8\theta^4(2 + 5\phi(2 + \phi(3 + 7\phi))) \\ + 8\theta^2(1 + \phi(6 + \phi(12 + \phi(20 + 3\phi(5 + 7\phi))))) \end{array} \right)$$

$$- \left(\begin{array}{c} 2(1 + \theta(1 + \theta + \theta^2 + \theta^3) + \phi + \theta(2 + \theta(3 + 4\theta))\phi \\ + (1 + 3\theta + 6\theta^2)\phi^2 + (1 + 4\theta)\phi^3 + \phi^4)^2 \\ -\dfrac{2(\theta + \phi)^{10}}{(\theta + \phi - 1)(\theta + \phi + 1)} \end{array} \right) \tag{7.8}$$

$$= \frac{1}{2} \left(\begin{array}{c} -8\theta^7 - 8\theta^5(2 + 3\phi(2 + 7\phi)) \\ -\dfrac{8\theta\phi}{\theta^4 + (\phi^2 - 1)^2 - 2\theta^2(\phi^2 + 1)} \\ - 8\theta^3(2 + \phi(8 + 5\phi(4 + \phi(4 + 7\phi)))) \\ - 8\theta(1 + \phi(2 + \phi(6 + \phi(8 + \phi(10 + \phi(6 + 7\phi)))))) \end{array} \right).$$

Again, if $\theta = 0$, Equation (7.8) will reduce to zero. This means the variance of the order rate of the centralised DC is then equal to the sum of the variances of the order rate of the two decentralised DCs. The order variances of the two decentralised locations are also equal due to the assumptions on the correlation coefficients and the unit lead-times. Thus, the $Ratio[Cap]$ as shown in Equation (6.5) will be reduced to the square root of two when $\theta = 0$.

119

Proof of the Square Root Law for Bullwhip for this example:

When the decentralised DCs are identical, $Var[O_1] = Var[O_2] = \sigma_O^2$.

When $\theta = 0$, $Var[O_1] + Var[O_2] = Var[O^c]$. Thus, $Var[O^c] = 2\sigma_O^2$.

The optimal capacity cost per period for location i ($C^*_{\pounds;i}$) as given in Equation (4.29) is

$$C^*_{\pounds;i} = \sigma_{O;i}(N + P)\varphi\left[\Phi^{-1}\left[\frac{P}{N+P}\right]\right], \ \forall \ i = \{1, 2\},$$

where $\sigma_{O;i}$ is the standard deviation of the order rate of location i.

Let $Y_O = (N + P)\varphi\left[\Phi^{-1}\left[\frac{P}{N+P}\right]\right]$. Thus, $C^*_{\pounds;i} = \sigma_{O;i}Y_O$, $\forall \ i = \{1, 2\}$.

The ratio of the capacity costs is given by

$$Ratio[Cap] = \frac{(\sqrt{Var[O_1]})Y_O + (\sqrt{Var[O_2]})Y_O}{(\sqrt{Var[O^c]})Y_O} = \frac{\sqrt{\sigma_O^2} + \sqrt{\sigma_O^2}}{\sqrt{2\sigma_O^2}} = \frac{2\sqrt{\sigma_O^2}}{\sqrt{2\sigma_O^2}} = \sqrt{2}.$$

The contour plot of *Ratio[Cap]* for this case is shown in Figure 7-1 (b). The bold lines represent the Square Root Law for Bullwhip where the ratio is equal to $\sqrt{2}$. Note that throughout this chapter the unit costs for holding (H), backlog (B), lost capacity (N) and overtime (P) for all locations in the distribution network are assumed to be the same.

Although it may be argued that the results shown in this section can be deduced directly from basic statistics analysis, the methodology used in this study could provide a closed form of the expression of variances which the other methods can not. The analytical analysis of the closed form of the variance expression can be performed so that the impact of constituent parts can be properly recognised. A proper graphical presentation of the result can also be achieved by using the closed form expression.

Investigating both the plots of *Ratio[Inv]* and *Ratio[Cap]* in Figure 7-1, the bold upright and horizontal lines separate the demand pattern into 4 areas. Referring to the intersection of the bold lines, the 4 areas include upper-right, upper-left, lower-right and lower-left areas. The cost ratios for the demand patterns in the upper-right and lower-left areas are always lower than $\sqrt{2}$ and decrease towards the stability borders. In contrast, the cost ratios for the demand patterns in the lower-right and upper-left areas are always higher than $\sqrt{2}$ and increase towards the stability borders.

7.3 The Square Root Law for Inventory

The simple example in Section 7.2 (in which $n = 2$) has shown that if the decentralised locations are identical (by a set of assumptions on lead-times and correlation coefficients), the variance of a state variable (such as the inventory level and the order rate) of all locations will be the same. Moreover, if $\theta = 0$, the variances of the centralised DC will be equal to the sum of the variances of all decentralised locations. Thus, this knowledge can be applied to achieve the Square Root Law for Inventory and Bullwhip when n decentralised locations are consolidated into a single centralised location.

Let us consider n decentralised DCs. The variance of the inventory level for decentralised DC i, $Var[I_i]$. Under the conditions that the demand processes are the same for each customer (that is all $\phi_{ii} = \phi$, all $\phi_{ij} = \theta$ when $i \neq j$ and all $\sigma^2_{\varepsilon;i} = 1$, $\forall\ i, j$), all retailers' lead-times are the same (that is, if all $l_i = l_i^c = l$) and all DCs' lead-times are the same (that is, if all $L_i = L^c = L$) then the variances of the inventory level of all decentralised DCs, $Var[I_i]$, will be the same and are denoted by $Var[I]$. The total inventory cost for all DCs in the decentralised system is

$$\text{Inventory costs in the decentralised system} = n\left(\sqrt{Var[I]}\ Y_I\right), \qquad (7.9)$$

and the inventory cost for the centralised DC is

$$\text{Inventory costs in the centralised system} = \sqrt{n(Var[I])}\ Y_I. \qquad (7.10)$$

Dividing Equation (7.9) by Equation (7.10) reveals the Square Root Law for Inventory

$$\frac{\text{Inventory costs in the decentralised system}}{\text{Inventory costs in the centralised system}} = \frac{n\left(\sqrt{Var[I]}\ Y_I\right)}{\sqrt{n(Var[I])}\ Y_I} = \sqrt{n}. \qquad (7.11)$$

7.4 The Square Root Law for Bullwhip

The Square Root Law for Bullwhip will be investigated in a similar manner to that in Section 7.3. Again, under the conditions that all the demand processes are the same for each customer, all retailers' lead-times are the same and all DCs' lead-times are the same then the variances of the order rate of all decentralised DCs, $Var[O_i]$, will be the same and are denoted by $Var[O]$. Thus, the capacity costs for all DCs in the decentralised system are

$$\text{Capacity costs in the decentralised system} = n\left(\sqrt{Var[O]}\ Y_O\right), \tag{7.12}$$

and the capacity cost for the centralised DC is

$$\text{Capacity costs in the centralised system} = \sqrt{n(Var[O])}\ Y_O. \tag{7.13}$$

Dividing Equation (7.12) by Equation (7.13) reveals the Square Root Law for Bullwhip

$$\frac{\text{Capacity costs in the decentralised system}}{\text{Capacity costs in the centralised system}} = \frac{n\left(\sqrt{Var[O]}\ Y_O\right)}{\sqrt{n(Var[O])}\ Y_O} = \sqrt{n}. \tag{7.14}$$

Figures 7-2 to 7-4 show the Square Root Law for Inventory and Bullwhip for the simple model, where all lead-times are assumed to be unity, for cases where $n = 3$, 4 and 5. The bold lines represent the Square Root Law for Inventory and Bullwhip where the ratio is equal to $\sqrt{n}$. Again for both the plots of $Ratio[Inv]$ and $Ratio[Cap]$, the bold upright and horizontal lines separate the demand pattern into 4 areas as has been discussed for cases where $n = 2$ in Section 7.2. The cost ratios for the demand patterns in the upper-right and lower-left areas are always lower than $\sqrt{n}$ and decrease towards the stability borders. In contrast, the cost ratios for the demand patterns in the lower-right and upper-left areas are always higher than $\sqrt{n}$ and increase the stability borders. This investigation is very useful to get an initial idea of the economic performance of a consolidated DC.
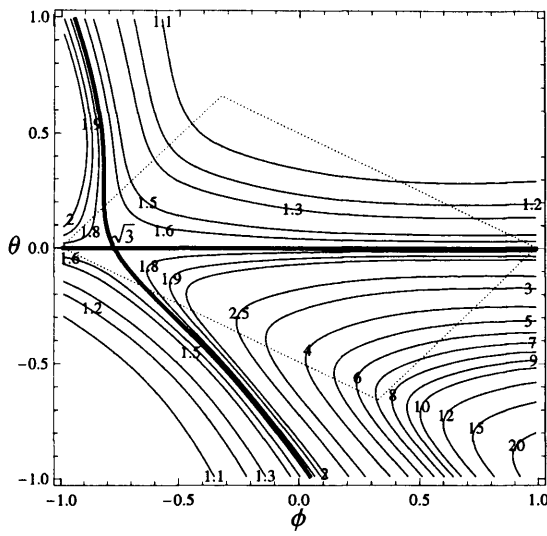
(a) Ratio[Inv]
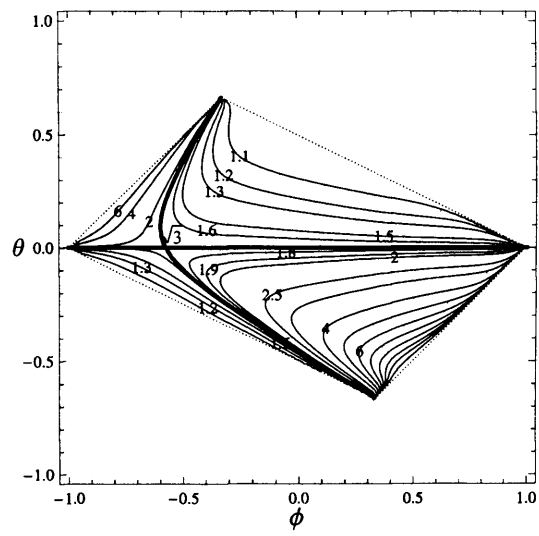
(b) Ratio[Cap]

Figure 7-2   Contour plots of Ratio[Inv] and Ratio[Cap] illustrating the Square Root Law
for Inventory and Bullwhip for n = 3
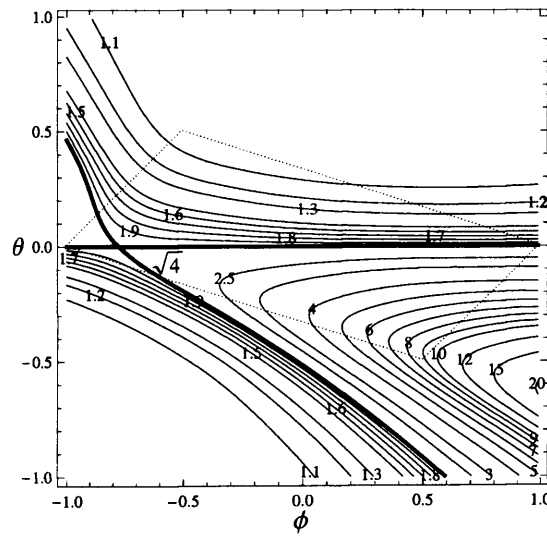
(a) Ratio[Inv]
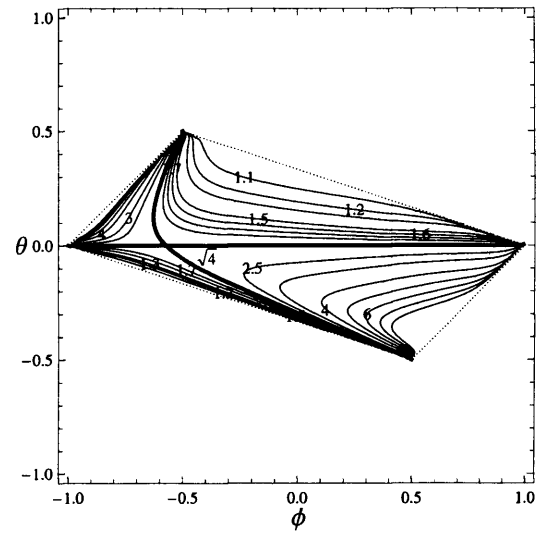
(b) Ratio[Cap]

Figure 7-3   Contour plots of Ratio[Inv] and Ratio[Cap] illustrating the Square Root Law
for Inventory and Bullwhip for n = 4

(a) Ratio[Inv]   *(b) Ratio[Cap]*

Figure 7-4 Contour plots of Ratio[Inv] and Ratio[Cap] illustrating the Square Root Law for Inventory and Bullwhip for n = 5

7.5 The impact of the cost model on the Square Root Law for Bullwhip

In this section, the impact of different cost models on the Square Root Law for Bullwhip are examined. The different capacity cost model developed by Hosoda and Disney (2010) will be considered. This capacity cost model involves normal-time and over-time working. The normal capacity is set to be $(\mu_d + S)$ where μ_d is the mean demand and S is spare capacity above (or below) the mean demand. If the order quantity is smaller than the normal capacity, the capacity cost is at a normal rate. If the order is larger than the normal capacity, it is paid by a premium; either for overtime capacity or subcontractors. The normal-time and over-time costs are assumed to be piece-wise linear and convex.

$$\text{Capacity cost for period } t = \overbrace{N(\mu_d + S)}^{\text{Normal capacity cost}} + \begin{cases} 0, \text{ when } O_t \le (\mu_d + S), \\ \underbrace{P(O_t - (\mu_d + S))}_{\text{Over-time cost}}, \text{ when } O_t > (\mu_d + S), \end{cases} \quad (7.15)$$

where O_t is the order rate at time t, N and P are the unit costs of normal-time and over-time working respectively. Figure 7-5 illustrates this alternative capacity cost model.

124

The same procedure used to obtain the optimum capacity cost as in Section 4.4 will be applied to this alternative cost. The expected normal-time cost is directly given as

$$\text{Normal-time cost} = N(\mu_d + S) \qquad (7.16)$$
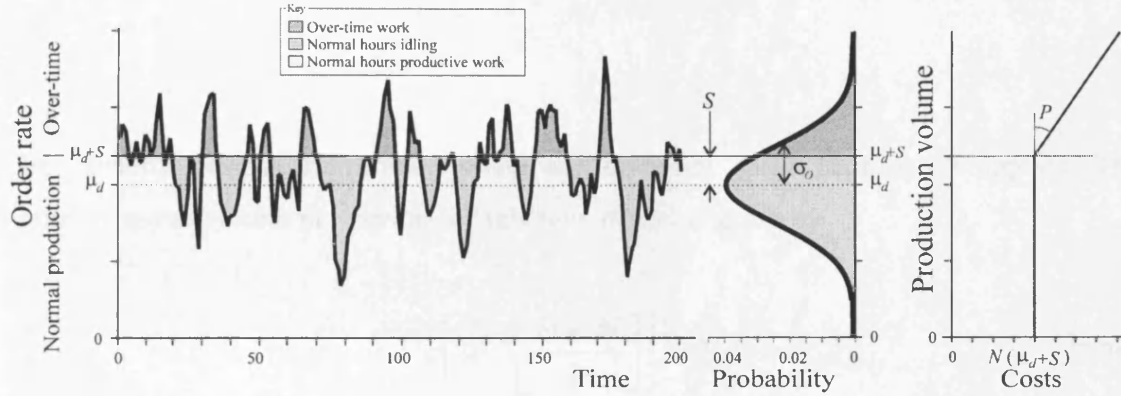
Figure 7-5 How the alternative capacity costs are generated over time

(Hosoda and Disney, 2010)

The expected over-time cost per period is given by

$$
\begin{aligned}
\text{Over - time cost} &= \frac{P}{\sigma_O \sqrt{2\pi}} \int_0^\infty \exp\left[-\frac{(x+S)^2}{2\sigma_O^2}\right] x \, dx \\
&= \frac{P}{2}\left(e^{-\frac{S^2}{2\sigma_O^2}}\sqrt{\frac{2}{\pi}}\sigma_O - S + S\,\text{erf}\left[\frac{S}{\sqrt{2}\sigma_O}\right]\right),
\end{aligned}
\qquad (7.17)
$$

where σ_O is the standard deviation of the order rate. The expected capacity cost is the sum of the costs of normal-time and over-time working. It is given by

$$\text{Capacity cost} = N(\mu_d + S) + \frac{P}{2}\left(e^{-\frac{S^2}{2\sigma_O^2}}\sqrt{\frac{2}{\pi}}\sigma_O - S + S\,\text{erf}\left[\frac{S}{\sqrt{2}\sigma_O}\right]\right). \qquad (7.18)$$

125

The capacity cost in (7.18) is partially differentiated with respect to S. Then it is solved for zero gradient to obtain the optimum spare capacity (S^*) which minimises this capacity cost. The optimum spare capacity is given by

$$S^* = \sigma_O \sqrt{2}\, \text{erf}^{-1}\left[\frac{P-2N}{P}\right] = \sigma_O \Phi^{-1}\left[\frac{P-N}{P}\right].$$

(7.19)

The capacity cost is minimised when the optimum spare capacity is applied. The optimum capacity cost per period for this new model is given by

$$C_\pounds^* = N\mu_d + \sigma_O P\varphi\left[\Phi^{-1}\left[\frac{P-N}{P}\right]\right] = N\mu_d + \sigma_O Y_O,$$

(7.20)

where $Y_O = P\varphi\left[\Phi^{-1}\left[\frac{P-N}{P}\right]\right]$.

If the demand processes and the lead-times are under the same conditions as before, then all decentralised DCs are identical and have the same order variances, $VAR[O]$. It is assumed that the mean demands for all locations are the same (that is, if all $\mu_i = \mu_d$, $\forall\ i = 1, ..., n$) then the capacity costs for all DCs in the decentralised system are

Capacity costs in the decentralised system $= n\mu_d N + n\left(\sqrt{Var[O]}\ Y_O\right)$, (7.21)

It is assumed further that the mean demand for the centralised DC is the sum of the mean demands of all decentralised DCs. If $\theta = 0$, the variance of the order rate at the centralised DC will be equal to $n(Var[O])$. Thus, the capacity cost in the centralised system can be given by

Capacity costs in the centralised system $= n\mu_d N + \sqrt{n(Var[O])}\ Y_O$. (7.22)

126

Dividing Equation (7.21) by Equation (7.22), the ratio of capacity cost is obtained as

$$\frac{\text{Capacity costs in the decentralised system}}{\text{Capacity costs in the centralised system}} = \frac{\overbrace{n\mu_d N}^{a} + \overbrace{n(\sqrt{Var[O]}\,Y_O)}^{b}}{\underbrace{n\mu_d N}_{a} + \underbrace{\sqrt{n(Var[O])}\,Y_O}_{b'}}. \qquad (7.23)$$

This result shows that, under the same conditions as applied in Sections 7.3 and 7.4, the Square Root Law for Bullwhip does not hold for this new capacity cost model. Although it does hold when either the mean demand (μ_d) or the unit normal-time working cost (N) is zero, these conditions are not likely to happen in a real situation. Furthermore, as $b > b'$, $a \geq 0$ then $\frac{a+b}{a+b'} < \frac{b}{b'}$. This shows that the consolidation benefit under the cost function given by Equation (7.15) is always less than when the cost function is given by Equation (4.24).

7.6 Summary

This chapter has provided a proof for the Square Root Law for Inventory and Bullwhip when the VAR(1) demand process and the OUT replenishment rule are applied. Although a number of conditions are required to make it hold, the Square Root Law is a special character and can be used to intuitively estimate the benefit of network consolidation. The Square Root Law is useful for practitioners as a simple rule of thumb.

It has also been shown that the Square Root Law is sensitive to the cost model. It will generally hold when the cost of a particular state variable depends on its variance in linear fashion.

The Square Root Law can be generalised to the degree that any demand processes and inventory replenishment rules can be applied. That is the Square Root Law will hold as long as all decentralised locations are identical by having the same lead-time and serving the same demand processes and the lead-time is maintained at the same value after the centralisation.

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Chapter 8

Model applications with real demand data

8.1 Introduction

This chapter presents the applications of the analytical model to real demand data in order to demonstrate the model capability and limitations. Two data sets will be used in this demonstration. They include sales data from the retailers' level and the shipment data that a supplier dispatches to the DCs. The nature of these two data sets differs by the level of supply chain that they represent and the size of the distribution networks.

8.2 Data set I: VAR demand faced by the retailers

Electronic Point of Sales (EPOS) data of a selected food product are obtained from two local stores in the same city. The two stores belong to a global grocery company. They will be referred to as Store 1 and Store 2. The daily EPOS data are aggregated into weekly sales data with a length of 70 weeks running from January 2004 to May 2005. Figure 8-1 shows the plot of the weekly sales data.

8.2.1 Identifying the demand process

Eviews, a statistical software package, is used to fit a model and to estimate the parameters of the model of the sales data. The selection of the estimation output is shown in Table 8-1 and Table 8-2. A complete output of the test of the assumptions applied to

the model of the sales data is presented in Appendix E.1. Eviews provides two information criteria for determining the lag length of the VAR model. This includes the Akaike Information Criterion (AIC) and the Schwarz Criterion (SC). A model with smaller values of the information criterion is preferred. The result in Table 8-1 shows that the real sales data can be appropriately modelled as either a VAR(1) or a VAR(2) process. Thus, it is realistic to assumed that the demand faced by the multi-retailers has a VAR(1) process as applied in this study. This result suggests that the VAR demand can possibly be of a higher order. Thus, a future study may use the VAR(p) demand processes to satisfy this fact.
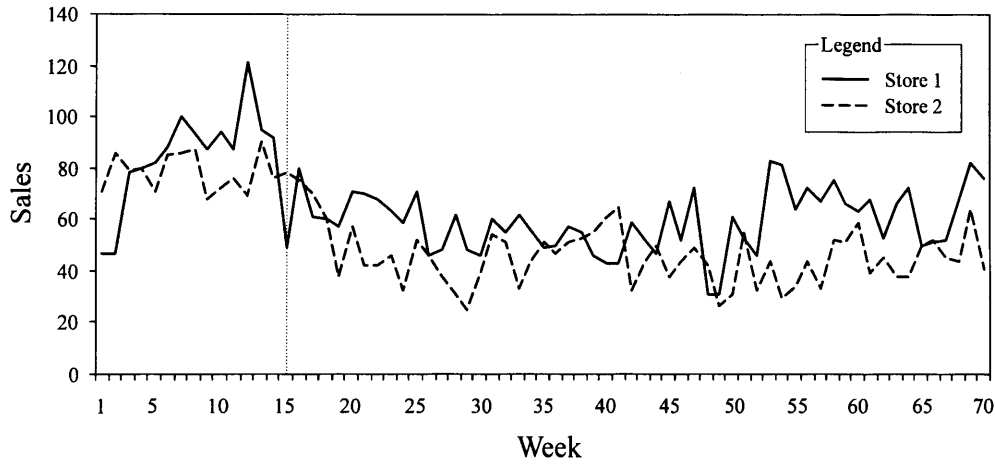
Figure 8-1 Bivariate time-series of the sales figures of a selected product

The result of the model estimation in Table 8 -2 shows a fairly high positive auto-correlation coefficient at both stores and positive cross-correlation coefficients between the demands of the two stores. The t-statistics of all coefficients are statistically significant at the 5% level except for the constant of Store 2. The drop of the sales volume after week 15 may be the cause of the high variation of the constant and thus resulted in low t-statistic of the estimated constant of Store 2. The VAR(1) model of the sales data can be represented by the following process

$$\begin{bmatrix} d_{1,t} \\ d_{2,t} \end{bmatrix} = \begin{bmatrix} \overbrace{20.502}^{\mu_1} \\ \underbrace{5.838}_{\mu_2} \end{bmatrix} + \begin{bmatrix} \overbrace{0.488}^{\phi_{11}} & \overbrace{0.242}^{\phi_{12}} \\ \underbrace{0.199}_{\phi_{21}} & \underbrace{0.638}_{\phi_{22}} \end{bmatrix} \cdot \begin{bmatrix} d_{1,t-1} \\ d_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}, \tag{8.1}$$

130

where $d_{i,t}$ is the demand of Store i at time t and $\varepsilon_{i,t}$ is the error term of Store i at time t.

Notice that the demand model in Equation (8.1) is not a mean-centred version of VAR processes. The modelling technique presented in Chapter 5 is, however, applicable to both non-mean-centred and mean-centred versions. From Table 8-2, the R-squared values show that this model can explain 41.45% and 57.88% of the variances in the sales data of Store 1 and Store 2, respectively. If the VAR model with a higher VAR lag length is used to represented the demand process, its R-squared value will be higher than that of VAR(1). However, its t-statistics of the estimated coefficients will be low and statistically insignificant (see Appendix E.2 for a complete figure). This result confirms that the VAR(1) process is the most appropriate process to represent the sales data.

Table 8-1 Vector autoregression estimation of data set I

Model	Information Criteria	
	AIC	SC
VAR(1)	15.73705	15.93132*
VAR(2)	15.63914*	15.96554
VAR(3)	15.68689	16.14757
VAR(4)	15.81533	16.41251
VAR(5)	15.91737	16.65332
VAR(6)	15.97216	16.84921

* indicate the minimum value of each information criteria

Table 8-2 Estimation output of data set I

Vector Autoregression Estimates
Included observations: 69 after adjustments
Standard errors in ( ) & t-statistics in [ ]

	Store 1	Store 2
Store 1 (-1)	0.488283	0.199499
	(0.10749)	(0.09035)
	[4.54271]	[2.20814]
Store 2 (-1)	0.242379	0.637903
	(0.10833)	(0.09106)
	[2.23742]	[7.00568]
Constant	20.50165	5.837734
	(6.65435)	(5.59323)
	[3.08094]	[1.04371]
R-squared	0.414472	0.578845
Sum square residuals	11636.89	8221.495

8.2.2 Identifying the structure of the order process

The modelling technique presented in Chapter 5 is used to identify the process of the orders that are placed by the stores with the DCs and of the orders that are placed by the DCs with the manufacturers. Both decentralised and centralised systems will be examined by assuming that the two distribution systems have a structure as presented in Figure 8-2. All locations are assumed to employ the OUT replenishment policy with the MMSE forecasting. This assumption is practical according to the action based research done by Potter et al. (2003) on a major UK grocery. The replenishment lead-time between the store and the DC is unity in both decentralised and centralised system.
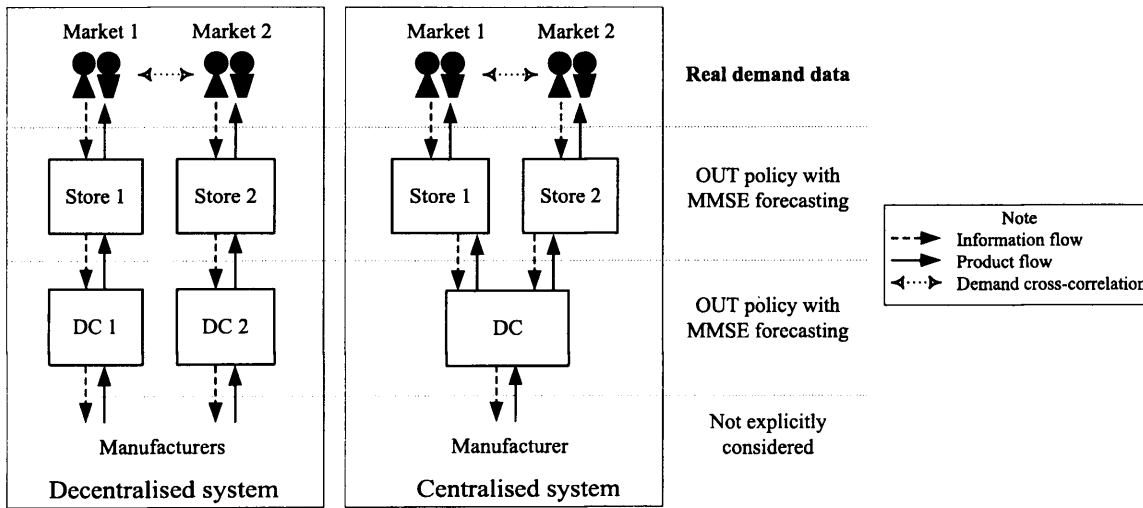
Figure 8-2 The structure of the distribution networks for the analysis of data set I

8.2.2.1 The order process of the decentralised system

The result presented in Section 5.6.1 is used to identify the theoretical process of the orders that are placed by the stores onto the decentralised DCs. From the result in Equation (5.28), the store's order will be a VARMA(1,1) process and can be formulated as

$$\begin{bmatrix} o_{1,t} \\ o_{2,t} \end{bmatrix} = \begin{bmatrix} 20.502 \\ 5.838 \end{bmatrix} + \begin{bmatrix} 0.488 & 0.242 \\ 0.199 & 0.638 \end{bmatrix} \cdot \begin{bmatrix} o_{1,t-1} \\ o_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon'_{1,t} \\ \varepsilon'_{2,t} \end{bmatrix} - \begin{bmatrix} 0.404 & 0.147 \\ 0.121 & 0.492 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon'_{1,t-1} \\ \varepsilon'_{2,t-1} \end{bmatrix}, \quad (8.2)$$

where $o_{i,t}$ is an order placed by store i at time t and $\begin{bmatrix} \varepsilon'_{1,t} \\ \varepsilon'_{2,t} \end{bmatrix} = \begin{bmatrix} 1.775 & 0.515 \\ 0.424 & 2.093 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$ in

which $\varepsilon_{i,t}$ is the error term of store i at time t. Note that the auto-correlation coefficient keeps the original values of the demand process.

The lead-times between the DCs and the manufacturers are also unity. The DCs' orders will also be a VARMA(1,1) process and is given by

$$\begin{bmatrix} O_{1,t} \\ O_{2,t} \end{bmatrix} = \begin{bmatrix} 20.502 \\ 5.838 \end{bmatrix} + \begin{bmatrix} 0.488 & 0.242 \\ 0.199 & 0.638 \end{bmatrix} \begin{bmatrix} O_{1,t-1} \\ O_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon''_{1,t} \\ \varepsilon''_{2,t} \end{bmatrix} - \begin{bmatrix} 0.454 & 0.194 \\ 0.160 & 0.574 \end{bmatrix} \begin{bmatrix} \varepsilon''_{1,t-1} \\ \varepsilon''_{2,t-1} \end{bmatrix}, \qquad (8.3)$$

where $O_{i,t}$ is an order placed by DC i at time t and $\begin{bmatrix} \varepsilon''_{1,t} \\ \varepsilon''_{2,t} \end{bmatrix} = \begin{bmatrix} 2.113 & 0.962 \\ 0.791 & 2.707 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \end{bmatrix}$. Once

more, the auto- and cross-correlation coefficients keep the original values of the demand process and the retailer's order process while the absolute values of the moving average parameters are higher in the upstream level of the supply chain.

8.2.2.2 The order process of the centralised system

For the centralised system, it is assumed that the lead-time between the store and the DC can be maintained at unity. An aggregated order that the stores placed onto the centralised DC, will be identical to the demand faced by the centralised DC (D_t^c). From the result in Section 5.6.2, D_t^c is a column-wise sum of VARMA(1,1) processes and can be expressed as

$$D_t^c = \sum_{i=1}^{2} o_{i,t}^c$$

$$= 26.339 + (0.688)o_{1,t-1}^c + (0.880)o_{2,t-1}^c + \varepsilon'_{1,t} + \varepsilon'_{2,t} - (0.545)\varepsilon'_{1,t-1} - (0.617)\varepsilon'_{2,t-1}, \qquad (8.4)$$

where D_t^c is the demand faced by the centralised DC at time t, $o_{i,t}^c$ is the order placed by Store i at time t, $\varepsilon'_{1,t} = (2.199)\varepsilon_{1,t}$ and $\varepsilon'_{2,t} = (2.608)\varepsilon_{2,t}$.

The order placed by the DC onto the manufacturer will also be a column-wise sum of VARMA(1,1) processes. In order to formulate the expression for the order placed by the centralised DC, pseudo-decentralised DCs are used. Let $\mathbf{O}_t^{c:DC}$ denote the orders placed by the pseudo-decentralised DCs. The detail of this method is presented in Section 5.6.2. From Equation (5.33), the $\mathbf{O}_t^{c:DC}$ is given by

$$\mathbf{O}_t^{c:DC} = \begin{bmatrix} O_{1,t}^{c:DC} \\ O_{2,t}^{c:DC} \end{bmatrix} + \begin{bmatrix} 0.338 & 0.446 \\ 0.367 & 0.613 \end{bmatrix} \cdot \begin{bmatrix} d_{1,t} - d_{1,t-1} \\ d_{2,t} - d_{2,t-1} \end{bmatrix}, \tag{8.5}$$

where $O_{i,t}^{c:DC}$ is the order placed by the pseudo-decentralised DC i at time t.

Utilising the description of $\mathbf{O}_t^{c:DC}$ from Equation (8.5), the order placed by the centralised DC onto the manufacturer at time t, O_t^c, can be expressed as

$$O_t^c = 26.339 + (0.688)O_{1,t-1}^{c:DC} + (0.880)O_{2,t-1}^{c:DC} + \varepsilon_{1,t}'' + \varepsilon_{2,t}'' - (0.656)\varepsilon_{1,t-1}'' - (0.727)\varepsilon_{2,t-1}'', \tag{8.6}$$

where $\varepsilon_{1,t}'' = (2.905)\varepsilon_{1,t}$ and $\varepsilon_{2,t}'' = (3.668)\varepsilon_{2,t}$.

8.2.3 The benefit from consolidation of the distribution network

Again, the modelling technique the previous chapters presented is exploited to evaluate the benefit of the consolidation of the distribution network. The Ratio[Inv], Ratio[Cap] and Ratio[Total] are used in this evaluation. The unit costs are assumed to be the same at all locations in both decentralised and centralised systems. Assumptions about the values of the unit costs adopted by Disney et al. (2006) are used; that is the unit holding cost (H) = 1, backlog cost (B) = 9, lost capacity (or normal-time working) cost (N) = 4 and over-time working (or subcontracting) cost (P) = 6. This assumption is reasonable as the backlog cost is much higher than the holding cost in a supply chain as there is a chance of lost sales and customers when the customer experiences stock out. Also, a cost for over-time working that is 150% of normal-time working is usual in practice. The resulted cost ratios are presented in Table 8-3. Both of the two models of the capacity costs, which

presented in Sections 4.4 and 7.5, are considered. As a result, there are two versions of the *Ratio[Total]* corresponding to the two types of the *Ratio[Cap]*.

Table 8-3 Cost ratios for the real demand data

| | *Ratio[Inv]* | *Ratio[Cap]* | | *Ratio[Total]* | |
		(a) from Section 4.4	(b) from Section 7.5	(a)	(b)
	1.14201	1.09659	1.00863	1.11198	1.01863
% Saving	12.43%	8.81%	0.86%	10.07%	1.83%

Table 8-3 shows that if the inventory cost is considered alone, the inventory cost will reduce by 12.43% when a centralised DC is operated. However, if both inventory and capacity costs are considered the benefit will be less; especially, when the alternative capacity cost model presented in Section 7.5 is applied. As a result, the total cost will reduce by only 1.83%. The consolidation decision, therefore, depends on the type of cost being considered, the cost model being applied and the benchmark of expected savings set by the company.

8.3 Data set II: VARMA orders placed by the DCs

The second data set is obtained from the DC's level. It is shipment data of a detergent product that a supplier dispatches to five distribution centres in the UK. These DCs are managed by a worldwide grocery company. The DCs will be referred according to their locations as Magor, Welham Green, Weybridge, Middlewich and Crick. There are actually seven DCs in total. The other two DCs in Dundee and Antrim are removed from consideration because their locations are remote from the rest of the DCs and additional assumptions would have to be made about lead-time changes. Figure 8-3 shows a plot of weekly shipments received by the DCs. The length of the data is 52 weeks starting from mid August 1998. This data will be used as orders placed by the DCs as it is assumed that each shipment is equal to an order placed by the DC.
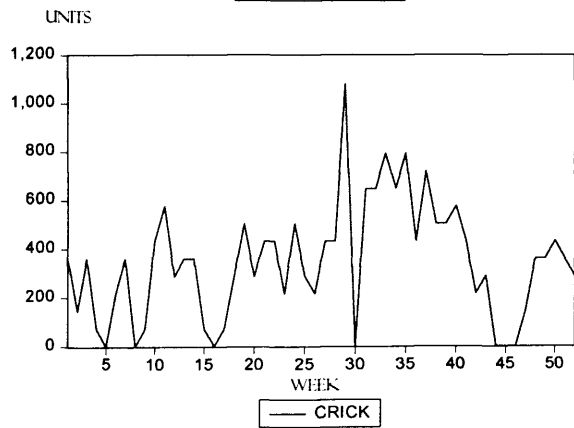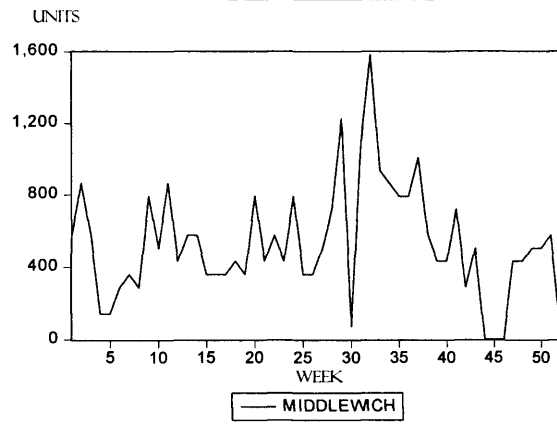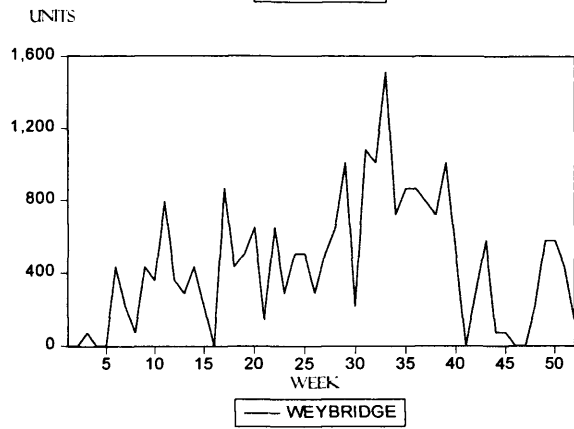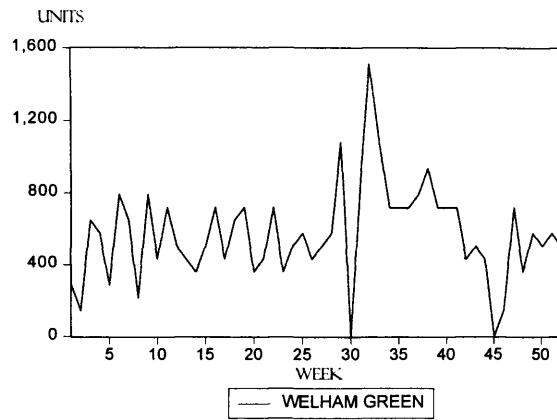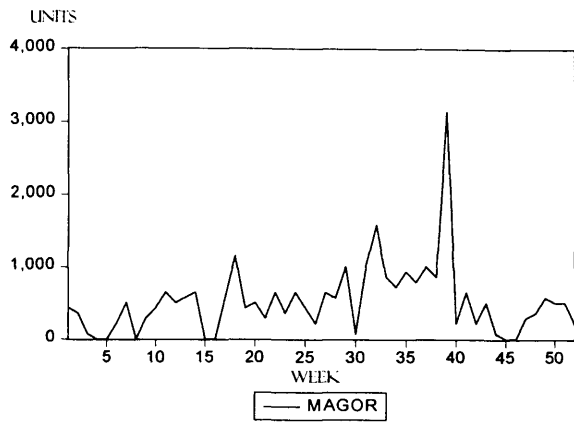
Figure 8-3 Weekly shipment from a supplier to different DCs

136

8.3.1 Identifying the order process at the DCs

Based on the analytical technique, the demand process at the DC level is assumed to follow the VARMA process. Unlike the univariate time series (Box and Jenkins 1976), a standard strategy for specification and testing the adequacy of VARMA models has not been universally established (Lütkepohl 1993). Thus, a logic described by Lütkepohl (1993) is applied to examine the lags of the VAR and the MA components. Then, a module available in SAS is used to determine the values of the coefficient of the model.

According to Lütkepohl (1993), the order of the VARMA(p, q) model that includes an n time series can be investigated by considering each time series as an ARMA(p, q) model. A statistical software called JMulTi (available for downloading at http://www.jmulti.de/) is used to find the optimal lags of the ARMA(p, q) model for each time series (each DC's order data). This is decided under three information criteria including the Akaike Information Criterion (AIC), the Hannan-Quinn Criterion (HQC) and the Schwarz Criterion (SC). The result in Table 8-4 shows that the time series for Magor, Welham Green, Weybridge, and Middlewich are likely to have ARMA(1,1) processes. Again, for convenience, Crick is removed from consideration as the result shows that its time series are more likely to be the AR(2) process. Thus the common optimal lags for the order processes of the four DCs are $p = 1$ and $q = 1$.

Table 8-4 ARMA lags determination for data set II

Time series for	Optimal lags (p, q) (searched all combinations where max (p, q) ≤ 3)			Chosen lags (p, q)
	AIC	HQC	SC	
Magor	(3, 3)	(1, 1)	(1, 1)	(1, 1)
Welham Green	(3, 0)	(3, 0)	(1, 1)	(1, 1)
Weybridge	(1, 1)	(1, 1)	(1, 1)	(1, 1)
Middlewich	(1, 1)	(1, 1)	(1, 1)	(1, 1)

The coefficients of the VARMA model for the remaining four DCs, which include Magor, Welham Green, Weybridge, and Middlewich, are now estimated. According to the optimal lags p and q chosen earlier, the first VAR order and the first MA order are evaluated. The VARMAX procedure available in SAS, a computer program for statistical analysis, is used to estimate the optimal parameters. A complete result from the analysis

is shown in Appendix E.3. The estimated VARMA(1,1) model for data set II can be expressed as

$$\begin{bmatrix} O_{MG,t} \\ O_{WG,t} \\ O_{WB,t} \\ O_{MW,t} \end{bmatrix} = \begin{bmatrix} 524 \\ 570 \\ 440 \\ 531 \end{bmatrix} + \begin{bmatrix} 0.150 & 0.010 & 0.562 & 0.408 \\ 0.032 & 0.329 & 0.156 & 0.541 \\ 0.293 & 0.074 & 0.173 & 0.358 \\ 0.040 & 0.102 & 0.476 & 0.405 \end{bmatrix} \cdot \begin{bmatrix} O_{MG,t-1} \\ O_{WG,t-1} \\ O_{WB,t-1} \\ O_{MW,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{MG,t} \\ \varepsilon_{WG,t} \\ \varepsilon_{WB,t} \\ \varepsilon_{MW,t} \end{bmatrix} +$$

$$\begin{bmatrix} 0.488 & -1.114 & -0.122 & 1.334 \\ 0.110 & 0.195 & 0.188 & 0.430 \\ 0.258 & -0.196 & 0.401 & -0.016 \\ 0.130 & 0.180 & 0.530 & -0.058 \end{bmatrix} \cdot \begin{bmatrix} \varepsilon_{MG,t-1} \\ \varepsilon_{WG,t-1} \\ \varepsilon_{WB,t-1} \\ \varepsilon_{MW,t-1} \end{bmatrix}, \tag{8.7}$$

where $O_{MG,t}$, $O_{WG,t}$, $O_{WB,t}$ and , $O_{MW,t}$ is the order at time t and $\varepsilon_{MG,t}$, $\varepsilon_{WG,t}$, $\varepsilon_{WB,t}$ and $\varepsilon_{MW,t}$ is the error term at time t at Magor, Welham Green, Weybridge, and Middlewich respectively. The VAR coefficient matrix is stable according to the result from its reverse characteristic polynomial root test shown in Appendix E.3.

8.3.2 Identifying the demand process of the market

In this section a reverse logic from Section 8.2 is applied as the available information is from the upstream level of a supply chain. It is assumed that the distribution network has a structure as shown in Figure 8-4. Each DC serves a group of retailers, which again serve a downstream market. The knowledge gained from Section 5.6.1 can be utilised to identify the underlying process of the market demand faced by the retailers. If all participants in a supply chain employ an OUT policy with MMSE forecasting and the order placed by the DCs is a VARMA(1,1) process, the market demand will be a VAR(1) process with the same VAR coefficients. Thus, the VAR coefficient matrix, **A**, for the market demand faced by the groups of retailers can be given by

$$\mathbf{A} = \begin{bmatrix} 0.150 & 0.010 & 0.562 & 0.408 \\ 0.032 & 0.329 & 0.156 & 0.541 \\ 0.293 & 0.074 & 0.173 & 0.358 \\ 0.040 & 0.102 & 0.476 & 0.405 \end{bmatrix}.$$

Market 1   Market 2   Market 3   Market 4

VAR(1) demand

Implied customer demand

Retailers Group 1 | Retailers Group 2 | Retailers Group 3 | Retailers Group 4

OUT policy with MMSE forecasting

VARMA(1,1) orders

Note
---▶ Information flow
—▶ Product flow
◀···▶ Demand cross-correlation

Magor | Welham Green | Weybridge | Middlewich

OUT policy with MMSE forecasting

VARMA(1,1) real orders data

Manufacturers

Not explicitly considered
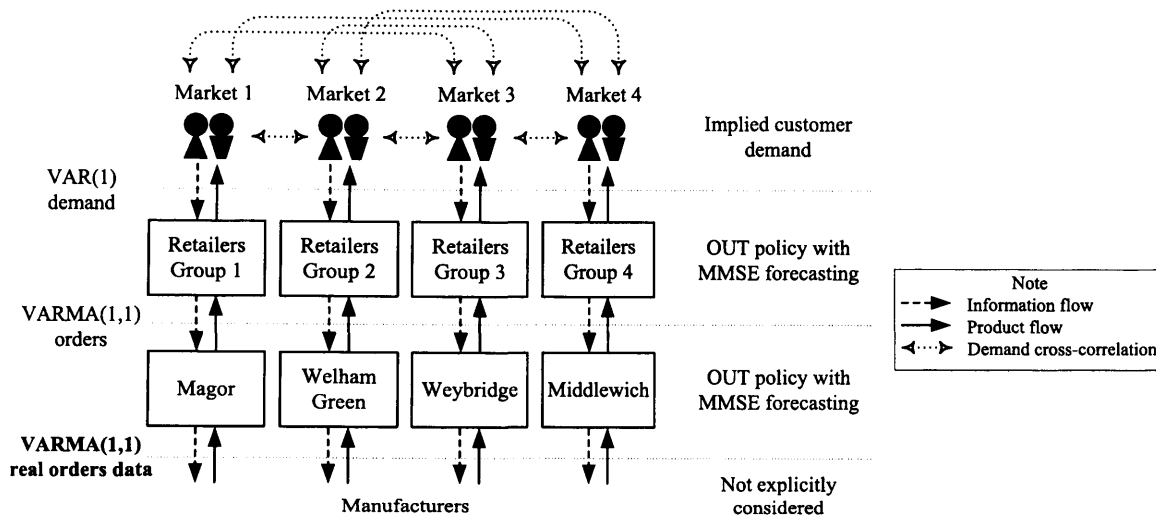
Figure 8-4   The distribution network structure for the analysis of data set II

8.3.3 The benefit from consolidation of the distribution network

The modelling techniques are applied to compare choices of consolidation schemes. Table 8-5 shows choices for consolidating the four DCs, which are simply all possible grouping combinations of the four DCs. Again, it is assumed that the unit costs at all locations are the same, where $H = 1$, $B = 9$, $N = 4$ and $P = 6$ as advocated by Disney et al. (2006). Unit lead-times exist at all location in all scenarios. The VAR coefficient matrix (**A**) is a key component needed for calculating a percentage of cost saving. For Scheme numbers 2 to 11 in Table 8-5, matrix **A** is obtained in a similar manner to that shown in Sections 8.3.1 and 8.3.2.

The result in Table 8-5 shows that Scheme number 2, where three DCs namely Magor, Welham Green and Weybridge are being considered for consolidation, can achieve the highest cost savings in all cost categories. This consolidation also makes sense when their geographical positions and road networks are involved in the analysis. Some schemes are more preferable based on the consideration of a specific cost. For example, if the inventory cost is considered alone, most of the consolidation schemes are preferable as their percentages are all greater than zero. These are especially Schemes number 2, 6 and 9 where over 20% of the inventory cost will be saved. These schemes, where three DCs and two DCs are being considered for consolidation, have percentages of savings even higher than consolidating all four DCs. However, it must be noted that this analysis is based on the assumption that all unit costs are of the same structure. Savings gained from

capacity costs are, in general, lower than the saving from inventory costs alone. This is especially true when the alternative capacity cost model in Section 7.5 is applied as the percent savings are barely higher than zero.

Consolidating two pairs of DCs is also possible such as simultaneously applying Schemes 6 and 11, 7 and 10, or 8 and 9. This example illustrates how flexible the modelling technique presented in this thesis can be.

Table 8-5 Consolidation schemes

Scheme number	DCs to be consolidated				VAR coefficient matrix (A)	Percent cost saving by consolidation				
	Magor	Welham Green	Weybride	Middlewich		Inventory cost	Capacity cost		Total cost	
							(a) from Section 4.4	(b) from Section 7.5	(a)	(b)
1	✓	✓	✓	✓	$\begin{bmatrix} 0.150 & 0.010 & 0.562 & 0.408 \\ 0.032 & 0.329 & 0.156 & 0.541 \\ 0.293 & 0.074 & 0.173 & 0.358 \\ 0.040 & 0.102 & 0.476 & 0.405 \end{bmatrix}$	10.246%	0.471%	0.005%	1.673%	0.031%
2	✓	✓	✓		$\begin{bmatrix} 0.229 & 0.180 & 0.702 \\ -0.422 & 1.051 & 0.441 \\ 0.639 & -0.125 & 0.360 \end{bmatrix}$	26.708%	14.608%	0.087%	17.667%	0.179%
3		✓	✓	✓	$\begin{bmatrix} 0.112 & -0.608 & 1.510 \\ -0.157 & 0.670 & 0.499 \\ -0.198 & -0.058 & 1.252 \end{bmatrix}$	14.487%	5.787%	0.045%	7.972%	0.111%
4	✓		✓	✓	$\begin{bmatrix} 0.565 & 0.605 & -0.085 \\ 0.093 & 0.242 & 0.579 \\ -0.148 & 0.718 & 0.513 \end{bmatrix}$	15.163%	0.286%	0.006%	1.369%	0.054%
5	✓	✓		✓	$\begin{bmatrix} 0.473 & -0.001 & 0.550 \\ -0.252 & 0.416 & 0.888 \\ 0.492 & 0.110 & 0.349 \end{bmatrix}$	12.404%	2.561%	0.016%	4.737%	0.053%

Table 8-5 Consolidation schemes (continued)

Scheme number	DCs to be consolidated				Coefficient matrix $\mathbf{A}$	% cost saving by consolidation				
	Magor	Welham Green	Weybride	Middlewich		Inventory cost	Capacity cost		Total cost	
							(a) from Section 4.4	(b) from Section 7.5	(a)	(b)
6	✓	✓			$\begin{bmatrix} 0.826 & 0.166 \\ -0.018 & 1.016 \end{bmatrix}$	21.456%	0.742%	0.016%	2.555%	0.093%
7	✓		✓		$\begin{bmatrix} 0.530 & 0.547 \\ 0.543 & 0.287 \end{bmatrix}$	3.257%	1.172%	0.005%	1.754%	0.016%
8	✓			✓	$\begin{bmatrix} 0.476 & 0.582 \\ 0.639 & 0.249 \end{bmatrix}$	2.611%	0.662%	0.003%	1.141%	0.011%
9		✓	✓		$\begin{bmatrix} 1.018 & -0.025 \\ 0.154 & 0.807 \end{bmatrix}$	22.021%	1.335%	0.022%	3.716%	0.107%
10		✓		✓	$\begin{bmatrix} 0.701 & 0.311 \\ -0.068 & 1.058 \end{bmatrix}$	15.449%	3.345%	0.028%	5.706%	0.083%
11			✓	✓	$\begin{bmatrix} -0.360 & 1.234 \\ 0.262 & 0.719 \end{bmatrix}$	1.265%	0.491%	0.029%	0.705%	0.007%

8.4 Summary

In Chapter 6 some capabilities of the model developed in this thesis were explored. However, the investigation was mostly limited to the simple model where it was assumed that the auto- and cross- correlations were the same for all locations; that is $\phi_{ii} = \phi$ and $\phi_{ij} = \theta$, $\forall\ i, j$, where $i \neq j$. In this chapter, the two examples with real market demand and supplier shipment data have been presented. The result from statistical analysis on the real market demand has confirmed that the assumption that the market demand has a VAR(1) process is reasonable. The theoretical prediction that the DCs' order has a VARMA(1,1) process has also been proved by the real order data to be logical. The modelling technique presented in this thesis has been shown to be very useful and flexible when applied to the real data. The knowledge about the demand transition has especially been useful for identifying the order process and conversely to identify the demand process from the history of the upstream orders.

The two examples presented in this chapter have been quite limited in terms of access to the complete information about the costs and the lead-times. Although the results have not been properly validated with the real situation, the assumptions about the processes of the market demand and the DC's orders have. The examples presented in Sections 8.2 and 8.3 can also be used as a guideline for practitioners to apply with their distribution networks where the absent information is available.

Throughout this thesis, the unit costs for inventory holding (H), backlog (B), lost capacity or normal-time working (N) and over-time working or subcontracting (P) are assumed to be 1, 9, 4 and 6 respectively and are assumed to be the same at all locations. This is because the main focus of this thesis is the impact of the correlated demand, lead-times and number of DCs. When the model is applied with the real situations, the unit cost at each location can be different. For example, the unit holding costs for the retailers in the decentralised system, the DCs in the decentralised system, the retailers in the centralised system and the centralised DC may be denoted by h_i, H_i, h_i^c and H^c respectively. When the total cost is considered, this difference in unit costs at each location will directly give genuine weights to the inventory and capacity costs.

143

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Chapter 9

Conclusion

9.1 Discussion of the research questions

A discrete stochastic analytical model based on control theory and time series techniques has been used in this study. The model has been employed to investigate the dynamic behaviour of two-echelon distribution networks with correlated demands. To conclude the findings from this study, the research questions addressed at the beginning of this thesis will be answered in this final chapter.

9.1.1 How can we model the distribution network of a supply chain in which the market demand is a correlated multiple time series?

- How can the correlated demand be modelled?
- How can this correlated demand be forecast?
- Which fundamental techniques are useful for modelling the supply chain's distribution network?
- What can we learn from this model compared to previous modelling studies?

The two-level multiple-location distribution network has been modelled as a discrete stochastic analytical model. The system has been described using block diagrams, difference equations and z-transforms. Linear difference equations have been used to show the dynamic relationship of the inventory replenishment and ordering system and of different supply chain levels. The basics of the modelling have been presented in **Chapter 4** and the main model has been shown in **Chapter 5**.

The correlated demand has been modelled as a Vector Auto-Regressive (VAR) process. The demand modelling was introduced in **Section 4.2** and presented in more detail in **Section 5.2**. The VAR model allows a structured presentation of the multiple time series demands that possess both auto-correlation and cross-correlation. The application of the VAR model in representing the demands of multiple retailers has not yet appeared in supply chain studies and, thus, is a unique contribution to this field of study.

An approach to represent the Minimum Mean Square Error (MMSE) forecast of the VAR(1) demands was proposed in **Section 5.3**. This approach can deal with a situation where there are multiple locations in the same supply chain level and each location has different lead-times. Moreover, the MMSE forecasting technique has been explained for both decentralised and centralised distribution networks.

The MMSE forecasting involves obtaining the conditional expected demand during the lead-time and review period. The forecasting becomes very complex in the case of the VAR demands, due to the occurrence of recurring functions. However, this problem has been solved by using a vector notation. By representing the VAR demand in the vector notation, the recurring function can be completely avoided as has been shown in **Section 4.3.2**. The vector notation has also been used to represent the order replenishment system where there are multiple locations in the same supply chain level. This vector notation allows the concise and insightful model of the whole distribution network to be developed.

Another original contribution that has been obtained from the modelling approach is the realisation of demand transition when the VAR(1) demand is passed onto a higher supply chain level as orders. Unlike other related works, this finding is based on the multiple locations in the same supply chain level and the cross-correlation of the demand which exists. The transition of demand has also been investigated for both decentralised and centralised distribution systems. For the decentralised system, the VAR(1) demand evolves into VARMA(1,1) orders for all higher supply chain levels. For the demand transition in the centralised system, the orders at the higher supply chain levels can be described as a column-wise sum of VARMA(1,1) processes. This has been proved in **Section 5.6**.

9.1.2 Under what circumstances should the consolidated distribution network be established?

- *What is the impact of the demand correlation on the consolidation decision?*
- *What is the impact of the lead-times of each player in the distribution network on the consolidation decision?*
- *What is the impact of the number of decentralised locations on the consolidation decision?*
- *What is the impact of the cost function on the consolidation decision?*

The ratios of costs between the decentralised and centralised systems (as presented in **Section 4.4**) have been employed in evaluating the economic performance of the consolidated distribution network. The centralised distribution network will be more cost effective when the ratio is greater than one. In practice, a benchmark for the minimum value of the ratio can be set as a goal post to ensure a desired level of benefit is achieved from the network consolidation.

Two types of costs have been considered in making the consolidation decision: inventory and capacity. The inventory costs (and the same for capacity costs) are linear functions of the standard deviation of the inventory levels (the order rates). The closed form of the variance expression obtained from the analytical model is therefore an important input for the consolidation decision using the ratio of costs. The effect of the demand pattern on the variances has been investigated. The results indicate that a consolidation decision should consider both auto-correlation and cross-correlation to avoid making errors in the design of a distribution network. This has been shown in **Chapter 6**.

The ratio of costs has been investigated analytically and graphically in some 'simple' situations in **Chapter 6**. It has been shown that the ratios of costs are always greater than one if the lead-times at all locations are the same and maintained after consolidation regardless of the demand pattern. This means the centralised system is more attractive under these circumstances (**Section 6.2.2**).

The magnitude of the benefit, however, depends on the demand pattern and lead-times. The cost ratio is highly affected by the specific values of auto- and cross- correlations. The complexity of this relationship does not allow much opportunity for general

conclusions to be obtained. The behaviour of the cost ratio affected by the changing of lead-times is more predictable when ϕ is positive. The following conclusions have also been made for a stylised model with a demand that has a highly positive auto-correlation ($\phi = 0.7$):

- Consolidations of a distribution network should be emphasised more when negative cross-correlations and large lead-times L exist. (**Section 6.4.1**)
- The centralised DC's lead-time is a critical factor affecting the benefit from the consolidation of a distribution network. (**Section 6.4.3**)
- Consolidations of larger n provide greater cost ratios. The magnitude of the cost ratios, however, depends on the demand pattern. (**Section 6.4.4**)

If ϕ is negative, it is recommended to investigate the benefit on a case by case basis using the general model presented in **Chapter 5**.

Consolidation decisions can also be determined by observing the worst cases of the cost ratios under a set of conditions about lead-times. The conditions can be, for example, the lead-time of one of the decentralised DC is shorter than of the other DCs and after consolidation the lead-time of the centralised DC is equal to the shorter lead-time. Contour plots have helped to identify a region of demand patterns where the consolidated system is encouraged. This approach has been shown in **Section 6.5**.

In **Chapter 7**, the Square Root Laws for Inventory and Bullwhip have been proved. This proof is quite different from previous works as it considers both the auto- and cross-correlations of the demand. It has been shown that the ratios of costs will be equal to the square root of n under certain circumstances. The Square Root Law could provide practitioners with an immediate and useful estimation of the benefit of the consolidated distribution system.

The different cost functions have been shown to alter the consolidation decisions. Different cost ratios for inventory costs, capacity costs, total costs and alternative capacity costs all generate different magnitudes of benefits and thus result in different consolidation decisions. This has been shown in **Chapter 6, Section 7.5** and **Chapter 8**.

Although the 'simple' examples in Chapter 6 have been limited to the situation where the auto-correlations (and the cross-correlations) of the demands are the same for all locations, some useful insights about the variances have been learned. These include a symmetric property (shown in **Section 6.3.1**) and constancy property (shown in **Section 6.3.2**). These properties allow us to understand the behaviour of the variances of the system states. More efficient calculations can be achieved by omitting some redundant calculations appreciated from these properties.

Also, in Section 6.3.3 it has been shown that:

> When all participants in the decentralised system use the OUT policy with the MMSE forecasting scheme, the variance of order rates of the decentralised DC will keep the same value under the constraint that the summation of the downstream lead-time and the local lead-time is constant even for cases where the customer demands are cross-correlated. This insight can be useful for assessing the impact of re-allocating lead-times between echelons. (**Section 6.3.3**)

9.1.3 How can the stylised analytical results be related to real world demand data?

In **Chapter 8** the analytical model presented in Chapter 5 has been applied to two sets of real world data including market demand and shipment from supplier. The result has shown that the assumption made about the demand processes in this study is reasonable. The result has also shown that the benefit of centralised DC can be as high as a 12.43% saving in inventory costs in the retail supply chain. The benefit is, however, lower when capacity costs are included in the consideration. The total savings when both inventory and capacity costs are considerably reduced to 10.07% (and to 1.83% for the alternative capacity cost). It has also been shown that the analytical model is a useful tool to analyse a set of consolidation schemes (see **Table 8-5**). In **Section 8.3** the knowledge of the demand transition obtained from Section 5.6 has also been shown to be useful when it is applied to real data. The data of orders that are placed by the DCs has been conveniently used to identify the market demand process.

9.2　Implications for practice

The model developed in this study has shown to be useful for practitioners in evaluating the benefits of different designs for their distribution networks. The pattern of market demand including the auto-correlation and cross-correlation should be first identified as it has shown to have a great impact on the DND decision. The DND for high volume products will be benefit more from the model as such products are corresponding to the assumptions about the inventory policy. Apart from that, the demand data of low volume products are not likely to have VAR process.

Furthermore, if the OUT inventory replenishment policy and the MMSE forecasting are employed, there may be no need for the upstream players to invest in a system to gain point-of-sales data. This is because the data of the orders placed by downstream players contain complete information about the point-of-sales data in which can be obtained from the structural investigation of the demand transition as shown in Section 5.6.

9.3　Limitations and future research opportunities

The analytical model in this study has been limited to simple aggregate flows of products and information so that it could be mathematically tractable. Future research may allow the flows to be more flexible by considering direct shipping between manufacturers and retailers and transhipment of products between locations in the same supply chain level. The role of the distribution facilities such as cross-docking, final-stage manufacturing and pick-up stations may also be taken into consideration.

Referring to the performance measures for DND summarised in Table 2-1, only one out of four components in the cost dimension has been considered in this thesis. Obviously, the consideration of the transportation costs, facility costs and information costs in the model could be another interesting research opportunity. Also, the research would be more complete if it took the components in the customer service dimension into consideration. Apart from that, it was shown in the literature review in Chapter 2 that reverse logistics has received a lot of attention from recent research work. The consideration of both forward and reverse flows of products has become more important

in the current competitive market and environmental awareness. This could provide real research opportunities to relate both forward and reverse flows in DND problems.

As has been discovered in Section 8.2.1, the market demand potentially has the VAR(p) processes with order $p \geq 1$. Therefore, it would be appropriate and interesting for future studies to explore the VAR(p) demand processes when $p > 1$ or the VARMA(p, q) demand processes for a greater generality.

This thesis has considered only cases where the demand is stable. The analysis has been very much limited by the stability concern. In order to have a more comprehensive analysis, research could be extended to cases where the demand is not stable. Other potential extensions that could be considered include analysis of time-variant system, non-linear ordering policies, different forecasting methods, variable lead-times, capacity constraints and finite time horizons.

Finally, the analytical model presented in this thesis can readily be used to investigate the Bullwhip Effect as the expression of the variance of order rates at all supply chain levels has already been obtained from the model approach. The VAR model can also be immediately applied to the case of multiple products where the demands are correlated. The variables used for the retailers in the analytical model can be simply used for the products in the multiple product case.

Appendix A.   Summary of the DND literature

Table A-1   Summary of the DND literature

Author(s) (Year)	Model type	Performance measures	Distribution network design decisions taking into account	Theories, techniques and tools	Demand pattern consideration
Abdinnour-Helm (1999)	Deterministic analytical model	Overall transportation cost	Distribution strategy; Customer allocation; Transhipment	Genetic algorithm; Linear programming; Hub-and-Spoke network design	No
Ambrosino and Grazia Scutellà (2005)	Stochastic analytical model	Facility, warehousing, transportation and inventory costs under certain customer service level	Distribution strategy; Location; allocation; routing; inventory	Integer linear programming	No
Amiri (2006)	Deterministic analytical model	Total costs include shipment costs and costs associate with opening and operating the warehouses and the plants	Numbers, locations and capacities of plants and warehouses	Mixed integer programming; Lagrange relaxation	No
Bottani and Montanari (2010)	Simulation model	Total costs include order, transportation, inventory holding and stock-out, and shipping costs; Bullwhip	Number of supply chain echelons; Re-order and inventory policies; Demand information sharing; Responsiveness of supply chain players	Discrete event simulation	Normal distribution demand with consideration of increases in means

Author(s) (Year)	Model type	Performance measures	Distribution network design decisions taking into account	Theories, techniques and tools	Demand pattern consideration
Camm et al. (1997)	Deterministic analytical model	Total cost	Centralised / decentralised; Manufacturing plants and DCs locations; Customer allocation	Uncapacitated facility-location model; Product sourcing model; Geographic Information System (GIS)	No
Costa et al.(2010)	Deterministic analytical model	Total supply chain costs	Number and location of the manufacturing plants and DCs at each stage of the network; Demand allocation	Heuristics: Genetic Algorithm	No
Croxton and Zinn (2005)	Stochastic analytical model	The sum of the fixed-warehouse cost, the transportation cost and the inventory cost	Centralised / decentralised warehouse; Distribution strategy	Linear programming	No
Ding et al. (2009)	Stochastic analytical model; Simulation model	Financial indicators include investment, production, transportation and inventory holding costs; Logistic indicators include average demand fill-rate, average demand cycle time and probability of on-time delivery	Supply chain configuration; Operational decisions such as order splitting, transportation allocation and inventory control	Multi-objective genetic algorithm; Simulation-based optimization	Stochastic demand
Disney et al. (2006)	Stochastic analytical model	Inventory cost; Capacity cost	Centralised / decentralised DCs	Square root law for Inventory and Bullwhip	Normal distribution demand

Author(s) (Year)	Model type	Performance measures	Distribution network design decisions taking into account	Theories, techniques and tools	Demand pattern consideration
Dong et al. (2010)	Stochastic analytical model	Marginal revenue function	Global facility network under exchange rate uncertainty and responsive pricing; Push / pull Transhipment	Analytical model; Lagrange multipliers	
Du and Evans (2008)	Stochastic analytical model	Fixed costs of installing a repair facility; Transportation cost; Total tardiness of cycle time	Distribution strategy; Centralised / decentralised network structure	Multi objective optimisation; Scatter search; Constraint method; Dual simplex method	Given average demand
Easwaran and Uster (2010)	Deterministic analytical model	Total costs of facility location, processing, and transportation associated with forward and reverse flows in the network	Distribution strategy; Centralised / decentralised network structure	Mixed integer linear programming	Yes
Ferretti et al. (2008)	Stochastic analytical model	Total costs include transportation and facility costs	Determine total number and location of transit points	Linear programming; Non-linear programming	Variation of final customer demand
Guillén et al. (2005)	Stochastic analytical model	Profit over the time horizon (NPV); Resulting demand satisfaction; Financial risk	Number, locations and capacities of plants and warehouses; Production rates of each product at each plant; Flows of materials between the plants and warehouses and between the warehouses and the markets	Stochastic mathematical programming based on a recourse model with two stages	Demand uncertainty

Author(s) (Year)	Model type	Performance measures	Distribution network design decisions taking into account	Theories, techniques and tools	Demand pattern consideration
Hinojosa et al. (2008)	Stochastic analytical model	Total costs include transportation and inventory holding, fixed facility costs	Facility location; Closing and opening a distribution facility	Integer programming; Lagrangian dual; Heuristic algorithm	
Jayaraman and Ross (2003)	Deterministic analytical model	Fixed costs associated with operating open warehouses and cross-docks; Transportation costs; Carrying costs	Distribution strategy (cross-docking); Network configuration for a central manufacturing plant, multiple DC, cross-docking sites, and retailer outlets	Mixed-integer programming; Heuristics: Simulated annealing (SA) methodology	
Lalwani et al. (2006)	Simulation model	Risk due to uncertainties associated with inventory, delivery frequency, changes in demand, transportation	Number and location of DCs	Taguchi methods; Analysis of variance	
Lapierre et al. (2004)	Deterministic analytical model	Total shipment costs	Number and location of transhipment centres; Best transportation alternative (full-truckload, less-than-truckload, parcel or own fleet)	Metaheuristics; Tabu search; Variable neighbourhood search	No

Author(s) (Year)	Model type	Performance measures	Distribution network design decisions taking into account	Theories, techniques and tools	Demand pattern consideration
Lee et al. (2010a)	Deterministic analytical model; Stochastic analytical model	Total costs include fixed facility costs, shipping costs and processing costs of forward and returned products	Type (forward processing, collection or hybrid processing) of facility to build at each potential depot; The quantities of forward and returned products shipped in the transportation links	Stochastic programming; Sample average approximation; Importance sampling	Yes
Lee et al. (2010b)	Deterministic analytical model	The sum of the transportation (routing) cost, variable operating cost, and the fixed cost	Location of facilities; Allocation; Routing	Mixed integer programming; LP-relaxation; Heuristic algorithm	
Longinidis and Georgiadis (2011)	Stochastic analytical model	Economic Value Added (EVA^{TM}) which is a financial figure that expresses the company's net created value	Number, location, and capacity of warehouses and DCs to be setup; Transportation links in the network; Production rates at the warehouse at plants; Inventory levels at warehouses and DCs	Mixed-integer linear programming	Uncertain demands: vary as piecewise constant functions of time over a number of time periods of given duration.

Author(s) (Year)	Model type	Performance measures	Distribution network design decisions taking into account	Theories, techniques and tools	Demand pattern consideration
Lu and Van Mieghem (2009)	Deterministic analytical model; Stochastic analytical model	Price; Manufacturing cost; Capacity investment cost; Transportation cost	Interplant transhipment; Centralised / decentralised common commodity; Common facility location	Newsvendor network	Deterministic and stochastic demand market correlation
Meepetchdee and Shah (2007)	Stochastic analytical model	Fixed infrastructure cost; Material handling cost; Transportation cost; Cost of secondary links; Network robustness; Network complexity	Logistical network configurations (number of warehouses, warehouse location and assignment of warehouse to serve customers) with desirable robustness levels	Mixed-integer linear programming; Complexity theory	
Mohammadi Bidhandi et al. (2009)	Deterministic analytical model	The sum of all fixed and variable costs for selecting facilities, assigning flows and providing products	Location and capacity choices for suppliers; Plants and warehouses selection; Product range assignment; Production flows	Mixed-integer linear programming; Benders' decomposition; Surrogate constraints	
Nagurney (2010)	Stochastic analytical model	Profit	Supply chain network design problem with oligopolistic firms	Game theory; Nash equilibrium	

Author(s) (Year)	Model type	Performance measures	Distribution network design decisions taking into account	Theories, techniques and tools	Demand pattern consideration
Park et al.(2010)	Stochastic analytical model	System-wide location, transportation and inventory costs	Number and locations of suppliers and DCs; Assignment of each location-fixed DC to a supplier and of each retailer to a DC; Risk pooling	Nonlinear integer programming; Lagrangian relaxation	
Qi et al. (2010)	Stochastic analytical model	The expected costs of location, transportation, and inventory subject to random supply disruptions that may occur at either the supplier or the retailers	Locations of retailers and the assignments of customers to retailers	Mathematical programming	
Ratanachote and Disney (2008)	Stochastic analytical model; Simulation model	Inventory cost; Capacity cost; Ratio of costs between decentralised and centralised systems	Centralised / decentralised DCs	Square root law for Inventory and Bullwhip	AR(1) demand
Romeijn et al. (2007)	Stochastic analytical model	Total transportation costs; Storage holding, replenishment and shortage costs at DCs and retail outlets; Capacity concerns, which may affect operating costs in the form of congestion costs	Number and location of DCs; Distribution and allocation	Set-covering model; Branch-and-price algorithm	

Author(s) (Year)	Model type	Performance measures	Distribution network design decisions taking into account	Theories, techniques and tools	Demand pattern consideration
Sabri and Beamon (2000)	Stochastic analytical model	Cost (production and distribution fixed costs and production, distribution, and transportation variable costs.); Customer service levels (fill rates); Flexibility (plant volume or delivery size)	Number of plant and DCs; DC allocation; Quantity of product generated and flow between locations	Mathematical programming	
Sourirajan et al. (2007)	Stochastic analytical model	The sum of the location and inventory (pipeline and safety stock) costs with relationship between the flows in the network, lead times and safety stocks	Network design and facility location with service levels and lead times	Single-Product Network Design problem with Lead time and Safety stock considerations(SPNDLS); Lagrangian relaxation	
Teo et al. (2001)	Stochastic analytical model	Facility investment and inventory costs	Consolidation of DCs under different demand models	Location-inventory analytical model	Yes i.id, Poisson process
Tiwari et al. (2010)	Stochastic analytical model	Total raw material cost; Total fixed cost; Total variable cost; Total transportation cost; Total inventory holding cost; Total backorder cost	Supply chain design that ensure high customer service levels and allows multiple shipping / transportation options, distributed customer demands with fixed lead times, nonlinear transportation and inventory holding costs and the presence of economies of scale	Hybrid Taguchi–Immune approach (Taguchi technique with Artificial Immune System (AIS))	independent demand

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Appendix B. Summary of the notation for different players used in the model

Table B-1 Summary of the notation for different players used in the model

	Retailers				DCs				Note for Matrix/ Vector
	Decentralised		Centralised		Decentralised		Centralised		
	Individual player	Matrix/ Vector	Individual player	Matrix/ Vector	Individual player	Matrix/ Vector	Individual player	Matrix/ Vector	
Demand	$d_{i,t}$	$\mathbf{D}_t$	$d_{i,t}^c$	$\mathbf{D}_t^c$	$D_{i,t}$	$\mathbf{D}_t^{DC}$	D_t^c	N/A	Column vector
Order	$o_{i,t}$	$\mathbf{O}_t$	$o_{i,t}^c$	$\mathbf{O}_t^c$	$O_{i,t}$	$\mathbf{O}_t^{DC}$	O_t^c	N/A	Column vector
Inventory level	$i_{i,t}$	$\mathbf{I}_t$	$i_{i,t}^c$	$\mathbf{I}_t^c$	$I_{i,t}$	$\mathbf{I}_t^{DC}$	I_t^c	N/A	Column vector
WIP	$w_{i,t}$	$\mathbf{W}_t$	$w_{i,t}^c$	$\mathbf{W}_t^c$	$W_{i,t}$	$\mathbf{W}_t^{DC}$	W_t^c	N/A	Column vector
Target inventory level	$\bar{t}_i$	$\overline{\mathbf{T}}$	$\bar{t}_i^c$	$\overline{\mathbf{T}}^c$	$\bar{T}_i$	$\overline{\mathbf{T}}^{DC}$	$\bar{T}^c$	N/A	Column vector
Expected demand	$\hat{d}_{i,t}$	$\hat{\mathbf{D}}_t$	$\hat{d}_{i,t}^c$	$\hat{\mathbf{D}}_t^c$	$\hat{D}_{i,t}$	$\hat{\mathbf{D}}_t^{DC}$	$\hat{D}_t^c$	N/A	Column vector
Lead-time	l_i	$\mathbf{L}$	l_i^c	$\mathbf{L}^c$	L_i	$\mathbf{L}^{DC}$	L^c	N/A	Diagonal matrix
Coefficient matrix for MMSE forecasting	N/A	$\mathbf{Y}^R$	N/A	$\mathbf{Y}^{c:R}$	N/A	$\mathbf{Y}^{DC}$	N/A	$\mathbf{Y}^{c:DC}$ (pseudo)	Square matrix
Inventory variance	$Var[i_i]$	$Var[\mathbf{I}]$	$Var[i_i^c]$	$Var[\mathbf{I}^c]$	$Var[I_i]$	$Var[\mathbf{I}^{DC}]$	$Var[I^c]$	N/A	Column vector
Order variance	$Var[o_i]$	$Var[\mathbf{O}]$	$Var[o_i^c]$	$Var[\mathbf{O}^c]$	$Var[O_i]$	$Var[\mathbf{O}^{DC}]$	$Var[O^c]$	N/A	Column vector

163

Appendix C. Full formulas for the simple model with $n = 2$

The inventory variance of the retailer (in both the decentralised, $VAR[i_i]$, and the centralised, $VAR[i_i^c]$, systems) is given by

$$VAR[i_i] = VAR[i_i^c] = \frac{1}{2\left(\theta^2 - (-1+\phi)^2\right)^3 \left(\theta^2 - (1+\phi)^2\right)}$$

$$\{-2\theta^7\left((-\theta+\phi)^{l_i} - (\theta+\phi)^{l_i}\right)\left(-1+(-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i}\right) + \theta^8\left((-\theta+\phi)^{2l_i} + (\theta+\phi)^{2l_i}\right) -$$

$$\theta^6\left(\begin{array}{c}-4+6(-\theta+\phi)^{l_i} + 6(\theta+\phi)^{l_i} + 2\phi(\theta+\phi)^{2l_i}(-1+2\phi) - (-\theta^2+\phi^2)^{l_i} + \\ (\theta+\phi)^{l_i}\left(-2\phi + (-\theta+\phi)^{l_i}\right) + 2\phi(-\theta+\phi)^{l_i}\left(-1+(-\theta+\phi)^{l_i}(-1+2\phi)\right)\end{array}\right) +$$

$$\theta^4\left(\begin{array}{c}6+4(-\theta+\phi)^{l_i} - (-\theta+\phi)^{2l_i} + 4(\theta+\phi)^{l_i} + 3(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i} - (\theta+\phi)^{2l_i} - \\ 3(-\theta^2+\phi^2)^{l_i} + 6\phi^4\left((-\theta+\phi)^{2l_i} + (\theta+\phi)^{2l_i}\right) - \\ 6\phi^3\left((-\theta+\phi)^{l_i} + (-\theta+\phi)^{2l_i} + (\theta+\phi)^{l_i} + (\theta+\phi)^{2l_i}\right) + \\ \phi^2\left(-4+14(-\theta+\phi)^{l_i} + 14(\theta+\phi)^{l_i} + 3(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i} - 3(-\theta^2+\phi^2)^{l_i}\right) - \\ 2\phi\left(\begin{array}{c}10-2(-\theta+\phi)^{l_i} + 3(-\theta+\phi)^{2l_i} - 2(\theta+\phi)^{l_i} + (-\theta+\phi)^{l_i}(\theta+\phi)^{l_i} + 3(\theta+\phi)^{2l_i} - \\ (-\theta^2+\phi^2)^{l_i}\end{array}\right)\end{array}\right) -$$

$$\theta^2(-1+\phi)$$

$$\left(\begin{array}{c}-3(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i} + 3(-\theta^2+\phi^2)^{l_i} + 2\left(-6+(-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i}\right) + \\ 4\phi^5\left((-\theta+\phi)^{2l_i} + (\theta+\phi)^{2l_i}\right) + \phi^3\left(\begin{array}{c}4+4(-\theta+\phi)^{l_i} - 2(-\theta+\phi)^{2l_i} + 4(\theta+\phi)^{l_i} + \\ 3(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i} - 2(\theta+\phi)^{2l_i} - 3(-\theta^2+\phi^2)^{l_i}\end{array}\right) + \\ \phi\left(-8+12(-\theta+\phi)^{l_i} + 12(\theta+\phi)^{l_i} + (-\theta+\phi)^{l_i}(\theta+\phi)^{l_i} - (-\theta^2+\phi^2)^{l_i}\right) + \\ \phi^2\left(\begin{array}{c}-20+12(-\theta+\phi)^{l_i} - 6(-\theta+\phi)^{2l_i} + 12(\theta+\phi)^{l_i} - (-\theta+\phi)^{l_i}(\theta+\phi)^{l_i} - \\ 6(\theta+\phi)^{2l_i} + (-\theta^2+\phi^2)^{l_i}\end{array}\right) - \\ 2\phi^4\left(3(\theta+\phi)^{l_i} + (\theta+\phi)^{2l_i} + (-\theta+\phi)^{l_i}\left(3+(-\theta+\phi)^{l_i}\right)\right)\end{array}\right) +$$

$$(-1+\phi)^3(1+\phi)$$

$$\left(-2+4\phi - (-\theta+\phi)^{l_i}(\theta+\phi)^{l_i} + (-\theta^2+\phi^2)^{l_i} - 2\phi^3\left((-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i}\right) + \right.$$

$$\left.\phi^4\left((-\theta+\phi)^{2l_i} + (\theta+\phi)^{2l_i}\right) + \phi^2\left(-(-\theta^2+\phi^2)^{l_i} + \left(-2+(-\theta+\phi)^{l_i}\right)\left(-2+(\theta+\phi)^{l_i}\right)\right)\right) +$$

$$2\theta(-1+\phi)^2\phi(2+\phi)\left((-\theta+\phi)^{l_i} - (\theta+\phi)^{l_i}\right)\left(-1+\phi\left(-2+\phi\left(-1+(-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i}\right)\right)\right) +$$

$$2\theta^5\left((-\theta+\phi)^{l_i} - (\theta+\phi)^{l_i}\right)$$

$$\left(2+(-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i} + \phi\left(-2+3\phi\left(-1+(-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i}\right)\right)\right) +$$

$$2\theta^3\left(-(-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i}\right)$$

$$\left(3+\phi\left(-2\left(-2+(-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i}\right) + \phi\left(\begin{array}{c}-2\left(-2+(-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i}\right) + \\ \phi\left(-4+3\phi\left(-1+(-\theta+\phi)^{l_i} + (\theta+\phi)^{l_i}\right)\right)\end{array}\right)\right)\right) +$$

$$2\left(\theta^2 + (-1+\phi)^2\right)\left(-1+\theta-\phi\right)(1+\theta-\phi)(-1+\theta+\phi)(1+\theta+\phi)l_i\}.$$

(C.1)

The order variance of the retailer (in both the decentralised, $VAR[o_i]$, and the centralised, $VAR[o_i^c]$, systems) is given by

$$VAR[o_i] = VAR[o_i^c] = \frac{1}{16}\left\{-\frac{8(-\theta+\phi)^{4+2l_i}}{-1+(\theta-\phi)^2} - \frac{8(\theta+\phi)^{4+2l_i}}{(-1+\theta+\phi)(1+\theta+\phi)} + \right.$$

$$\frac{\left(4\left(\begin{array}{l}-2-(-1+\theta)\theta^2(-\theta+\phi)^{l_i}+(1+\theta-\phi)(\theta+\phi)^{2+l_i}+\\ \phi(2+(-\theta+\phi)^{l_i}(\theta^2+\theta(-2+\phi)+\phi-\phi^2))\end{array}\right)^2\right)}{\left(\theta^2-(-1+\phi)^2\right)^2} +$$

$$\left.\frac{\left(4\left(\begin{array}{l}(-1+\phi)\phi^2((-\theta+\phi)^{l_i}-(\theta+\phi)^{l_i})-\theta^2(1+\phi)((-\theta+\phi)^{l_i}-(\theta+\phi)^{l_i})+\\ \theta^3((-\theta+\phi)^{l_i}+(\theta+\phi)^{l_i})-\theta(2+(-2+\phi)\phi((-\theta+\phi)^{l_i}+(\theta+\phi)^{l_i}))\end{array}\right)^2\right)}{\left(\theta^2-(-1+\phi)^2\right)^2}\right\}.$$

(C.2)

The inventory variance of the decentralised DC, $VAR[I_i]$, is given by

$$VAR[I_i] = \frac{1}{2\left(\theta^2-(-1+\phi)^2\right)^3\left(\theta^2-(1+\phi)^2\right)}$$

$$\left\{-2\theta^9(1+\phi)\left((-\theta+\phi)^{2(l_i+L_i)}-(\theta+\phi)^{2(l_i+L_i)}\right)+\theta^{10}\left((-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{2(l_i+L_i)}\right)+\right.$$

$$2\theta^7\left(\begin{array}{l}-(-\theta+\phi)^{l_i}+(-\theta+\phi)^{2l_i}+3(-\theta+\phi)^{l_i+L_i}+(-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{l_i}-\\ (\theta+\phi)^{2l_i}-3(\theta+\phi)^{l_i+L_i}-(\theta+\phi)^{2(l_i+L_i)}+4\phi^3((-\theta+\phi)^{2(l_i+L_i)}-(\theta+\phi)^{2(l_i+L_i)})\end{array}\right)-$$

$$\theta^2\left(\begin{array}{l}-8\phi+12\phi^2+2\phi^4+10\phi(-\theta+\phi)^{l_i}-10\phi^4(-\theta+\phi)^{l_i}+6\phi^5(-\theta+\phi)^{l_i}-6\phi^2(-\theta+\phi)^{2l_i}+\\ 6\phi^5(-\theta+\phi)^{2l_i}-4\phi^6(-\theta+\phi)^{2l_i}-6\phi(-\theta+\phi)^{l_i+L_i}-12\phi^2(-\theta+\phi)^{l_i+L_i}+\\ 12\phi^4(-\theta+\phi)^{l_i+L_i}+6\phi^5(-\theta+\phi)^{l_i+L_i}-8\phi^6(-\theta+\phi)^{l_i+L_i}+\\ 15\phi^4(-\theta+\phi)^{2(l_i+L_i)}-18\phi^5(-\theta+\phi)^{2(l_i+L_i)}+10\phi(\theta+\phi)^{l_i}-\\ 10\phi^4(\theta+\phi)^{l_i}+6\phi^5(\theta+\phi)^{l_i}-6\phi^2(\theta+\phi)^{2l_i}+6\phi^5(\theta+\phi)^{2l_i}-\\ 4\phi^6(\theta+\phi)^{2l_i}-6\phi(\theta+\phi)^{l_i+L_i}-12\phi^2(\theta+\phi)^{l_i+L_i}+\\ 12\phi^4(\theta+\phi)^{l_i+L_i}+6\phi^5(\theta+\phi)^{l_i+L_i}-8\phi^6(\theta+\phi)^{l_i+L_i}+\\ (-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}-2\phi(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}+\\ 2\phi^2(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}+\phi^4(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}-\\ 6\phi^5(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}+4\phi^6(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}+\\ 15\phi^4(\theta+\phi)^{2(l_i+L_i)}-18\phi^5(\theta+\phi)^{2(l_i+L_i)}-\\ (-1+\phi)^2(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i}(-\theta^2+\phi^2)^{L_i}(1+\phi^2+2\phi^3+4\phi^4)+\\ 2(1+(-\theta+\phi)^{l_i}+(\theta+\phi)^{l_i})+\\ 4\phi^3\left(\begin{array}{l}-2-2(-\theta+\phi)^{l_i}+(-\theta+\phi)^{2l_i}+2(-\theta+\phi)^{l_i+L_i}-\\ 2(\theta+\phi)^{l_i}+(\theta+\phi)^{2l_i}+2(\theta+\phi)^{l_i+L_i}\end{array}\right)+\\ 3\phi^8\left((-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{2(l_i+L_i)}\right)\end{array}\right)-$$

$$2\theta^5\left(\begin{array}{l}2(-\theta+\phi)^{l_i}+(-\theta+\phi)^{2l_i}+2(-\theta+\phi)^{l_i+L_i}-2(-\theta+\phi)^{l_i}-(\theta+\phi)^{2l_i}-2(\theta+\phi)^{l_i+L_i}-\\[4pt]3\phi^2\left(\begin{array}{l}(-\theta+\phi)^{l_i}-(-\theta+\phi)^{2l_i}-3(-\theta+\phi)^{l_i+L_i}+3(-\theta+\phi)^{2(l_i+L_i)}-\\(\theta+\phi)^{l_i}+(\theta+\phi)^{2l_i}+3(\theta+\phi)^{l_i+L_i}-3(\theta+\phi)^{2(l_i+L_i)}\end{array}\right)-\\[4pt]6\phi^4\left((-\theta+\phi)^{2(l_i+L_i)}-(\theta+\phi)^{2(l_i+L_i)}\right)+6\phi^5\left((-\theta+\phi)^{2(l_i+L_i)}-(\theta+\phi)^{2(l_i+L_i)}\right)+\\[4pt]\phi\left(-2(-\theta+\phi)^{l_i}-3(-\theta+\phi)^{2(l_i+L_i)}+2(\theta+\phi)^{l_i}+3(\theta+\phi)^{2(l_i+L_i)}\right)\end{array}\right)-$$

$$2\theta(-1+\phi)^2\phi$$

$$\left(\begin{array}{l}-2(-\theta+\phi)^{l_i}+2(\theta+\phi)^{l_i}+\\[4pt]\phi\left(\begin{array}{l}-5(-\theta+\phi)^{l_i}-4\phi(-\theta+\phi)^{l_i}-\phi^2(-\theta+\phi)^{l_i}+2\phi(-\theta+\phi)^{2l_i}+\\\phi^2(-\theta+\phi)^{2l_i}+3(-\theta+\phi)^{l_i+L_i}+6\phi(-\theta+\phi)^{l_i+L_i}+\\3\phi^2(-\theta+\phi)^{l_i+L_i}-3\phi^3(-\theta+\phi)^{2(l_i+L_i)}-\phi^4(-\theta+\phi)^{2(l_i+L_i)}+\\\phi^5(-\theta+\phi)^{2(l_i+L_i)}+5(\theta+\phi)^{l_i}+4\phi(\theta+\phi)^{l_i}+\phi^2(\theta+\phi)^{l_i}-\\2\phi(\theta+\phi)^{2l_i}-\phi^2(\theta+\phi)^{2l_i}-3(1+\phi)^2(\theta+\phi)^{l_i+L_i}-\\\phi^3(\theta+\phi)^{2(l_i+L_i)}\left(-3+(-1+\phi)\phi\right)\end{array}\right)\end{array}\right)-$$

$$\theta^8\left(\begin{array}{l}(-\theta+\phi)^{2l_i}+2(-\theta+\phi)^{l_i+L_i}+(\theta+\phi)^{2l_i}+2(\theta+\phi)^{l_i+L_i}-\\6\phi\left((-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{2(l_i+L_i)}\right)+3\phi^2\left((-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{2(l_i+L_i)}\right)+\\(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i}\left(-(-\theta+\phi)^{L_i}(\theta+\phi)^{L_i}+(-\theta^2+\phi^2)^{L_i}\right)\end{array}\right)+$$

$$\theta^6\big[2+6(-\theta+\phi)^{l_i}-4(-\theta+\phi)^{l_i+L_i}-(-\theta+\phi)^{2(l_i+L_i)}+6(\theta+\phi)^{l_i}-4(\theta+\phi)^{l_i+L_i}-$$

$$3(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}-(\theta+\phi)^{2(l_i+L_i)}+3(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i}(-\theta^2+\phi^2)^{L_i}-$$

$$16\phi^3\left((-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{2(l_i+L_i)}\right)+2\phi^4\left((-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{2(l_i+L_i)}\right)+$$

$$4\phi^2\left(\begin{array}{l}(-\theta+\phi)^{2l_i}+2(-\theta+\phi)^{l_i+L_i}+(\theta+\phi)^{2l_i}+2(\theta+\phi)^{l_i+L_i}-\\(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}+(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i}(-\theta^2+\phi^2)^{L_i}\end{array}\right)-$$

$$2\phi\left(\begin{array}{l}(-\theta+\phi)^{l_i}+(-\theta+\phi)^{2l_i}+(-\theta+\phi)^{l_i+L_i}+5(-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{l_i}+(\theta+\phi)^{2l_i}+\\(\theta+\phi)^{l_i+L_i}-(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}+5(\theta+\phi)^{2(l_i+L_i)}+(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i}(-\theta^2+\phi^2)^{L_i}\end{array}\right)\big]+$$

$$\theta^4\left(\begin{array}{l}-2-4\phi-2\phi^2-4(-\theta+\phi)^{l_i}-4\phi(-\theta+\phi)^{l_i}-14\phi^2(-\theta+\phi)^{l_i}+\\6\phi^3(-\theta+\phi)^{l_i}+(-\theta+\phi)^{2l_i}+6\phi(-\theta+\phi)^{2l_i}+6\phi^3(-\theta+\phi)^{2l_i}-\\6\phi^4(-\theta+\phi)^{2l_i}+6(-\theta+\phi)^{l_i+L_i}+12\phi(-\theta+\phi)^{l_i+L_i}+\\12\phi^2(-\theta+\phi)^{l_i+L_i}+6\phi^3(-\theta+\phi)^{l_i+L_i}-12\phi^4(-\theta+\phi)^{l_i+L_i}-\\15\phi^2(-\theta+\phi)^{2(l_i+L_i)}-10\phi^3(-\theta+\phi)^{2(l_i+L_i)}-\\4(\theta+\phi)^{l_i}-4\phi(\theta+\phi)^{l_i}-14\phi^2(\theta+\phi)^{l_i}+6\phi^3(\theta+\phi)^{l_i}+\\(\theta+\phi)^{2l_i}+6\phi(\theta+\phi)^{2l_i}+6\phi^3(\theta+\phi)^{2l_i}-\\6\phi^4(\theta+\phi)^{2l_i}+6(\theta+\phi)^{l_i+L_i}+12\phi(\theta+\phi)^{l_i+L_i}+\\12\phi^2(\theta+\phi)^{l_i+L_i}+6\phi^3(\theta+\phi)^{l_i+L_i}-12\phi^4(\theta+\phi)^{l_i+L_i}+\\3(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}-4\phi(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}+\\5\phi^2(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}-6\phi^3(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}+\\6\phi^4(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}-15\phi^2(\theta+\phi)^{2(l_i+L_i)}-\\10\phi^3(\theta+\phi)^{2(l_i+L_i)}+12\phi^5\left((-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{2(l_i+L_i)}\right)+\\2\phi^6\left((-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{2(l_i+L_i)}\right)+(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i}\\(-\theta^2+\phi^2)^{L_i}\left(-3+\phi\left(4+\phi\left(-5-6(-1+\phi)\phi\right)\right)\right)\end{array}\right)+$$

$$(-1+\phi)^3(1+\phi)$$

$$\left(\left(-2+\phi^2\left(\begin{array}{l}2\left(1+(-\theta+\phi)^{l_i}+(\theta+\phi)^{l_i}\right)-\\2\phi\left(-(-\theta+\phi)^{l_i}+(-\theta+\phi)^{l_i+L_i}-(\theta+\phi)^{l_i}+(\theta+\phi)^{l_i+L_i}\right)+\\\phi^4\left((-\theta+\phi)^{2(l_i+L_i)}+(\theta+\phi)^{2(l_i+L_i)}\right)+(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i}\\\left(-(-\theta+\phi)^{L_i}(\theta+\phi)^{L_i}+(-\theta^2+\phi^2)^{L_i}\right)-\phi^2\\\left(\begin{array}{l}(-\theta+\phi)^{2l_i}+2(-\theta+\phi)^{l_i+L_i}+(\theta+\phi)^{2l_i}+2(\theta+\phi)^{l_i+L_i}-(-\theta+\phi)^{l_i+L_i}(\theta+\phi)^{l_i+L_i}+\\(-\theta+\phi)^{l_i}(\theta+\phi)^{l_i}(-\theta^2+\phi^2)^{L_i}\end{array}\right)\end{array}\right)\right)+\right.$$

$$2\theta^3\left(\begin{array}{l}3\left((-\theta+\phi)^{l_i}-(\theta+\phi)^{l_i}\right)+(\theta+\phi)^{l_i}\\\left(\phi(-4+\phi(2+\phi)(-2+3\phi))+(\theta+\phi)^{L_i}\left(1+\phi(8+4\phi-9\phi^3)\right)\right)+\phi(\theta+\phi)^{2l_i}\\\left(2+\phi\left(2-3\phi^2-(-2+\phi)\phi(\theta+\phi)^{2L_i}(-5+4\phi^3)\right)\right)+\\(-\theta+\phi)^{l_i}\left(\begin{array}{l}(-2+\phi)\phi^3(-\theta+\phi)^{l_i+2L_i}(-5+4\phi^3)+\\\phi\left(4+4\phi-4\phi^2-3\phi^3+(-\theta+\phi)^{l_i}(-2-2\phi+3\phi^3)\right)+\\(-\theta+\phi)^{L_i}\left(-1+\phi(-8-4\phi+9\phi^3)\right)\end{array}\right)\end{array}\right)+$$

$$2(\theta^2+(-1+\phi)^2)(-1+\theta-\phi)(1+\theta-\phi)(-1+\theta+\phi)(1+\theta+\phi)L_i\Big\}.$$

(C.3)

The order variance of the decentralised DC, $VAR[O_i]$, is given by

$$VAR[O_i]=\frac{1}{16}\Big\{-\frac{8(-\theta+\phi)^{2(3+l_i+L_i)}}{-1+(\theta-\phi)^2}-\frac{8(\theta+\phi)^{2(3+l_i+L_i)}}{(-1+\theta+\phi)(1+\theta+\phi)}+$$

$$\frac{\left(4\left(\begin{array}{l}2-(-1+\theta)\theta^3(-\theta+\phi)^{l_i+L_i}+(-1-\theta+\phi)(\theta+\phi)^{3+l_i+L_i}+\\\phi\left(-2+(-\theta+\phi)^{l_i+L_i}\left(-3\theta^2+2\theta^3+\theta(3-2\phi)\phi+(-1+\phi)\phi^2\right)\right)\end{array}\right)^2\right)}{(\theta^2-(-1+\phi)^2)^2}+$$

$$\frac{1}{(\theta^2-(-1+\phi)^2)^2}4\left(\begin{array}{l}\theta^4\left((-\theta+\phi)^{l_i+L_i}-(\theta+\phi)^{l_i+L_i}\right)+3\theta^2\phi\left((-\theta+\phi)^{l_i+L_i}-(\theta+\phi)^{l_i+L_i}\right)-\\(-1+\phi)\phi^3\left((-\theta+\phi)^{l_i+L_i}-(\theta+\phi)^{l_i+L_i}\right)-\\\theta^3(1+2\phi)\left((-\theta+\phi)^{l_i+L_i}+(\theta+\phi)^{l_i+L_i}\right)+\\\theta\left(2+\phi^2(-3+2\phi)\right)\left((-\theta+\phi)^{l_i+L_i}+(\theta+\phi)^{l_i+L_i}\right)\end{array}\right)^2\Big\}.$$

(C.4)

The inventory variance of the centralised DC, $VAR[I^c]$, is given by

$$VAR[I^c]=\frac{1}{2(\theta^2-(-1+\phi)^2)^3(\theta^2-(1+\phi)^2)}$$

$$\Big\{4L^c(-1+\theta-\phi)(1+\theta-\phi)^3(-1+\theta+\phi)(1+\theta+\phi)-4\theta^8(\theta+\phi)^{L^c}\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)-$$

$$3\theta^8(-2+\phi)\phi(\theta+\phi)^{2L^c}\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)^2-$$

$$2\theta^9(1+\phi)\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-\\[4pt](\theta+\phi)^{2(L^c+l_1^c)}-(\theta+\phi)^{2(L^c+l_2^c)}-2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)+$$

$$\theta^{10}\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+\\[4pt](\theta+\phi)^{2(L^c+l_1^c)}+(\theta+\phi)^{2(L^c+l_2^c)}+2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)-$$

$$\theta^8\left(\begin{array}{l}(-\theta+\phi)^{2l_1^c}+(-\theta+\phi)^{2l_2^c}-2(-\theta+\phi)^{l_1^c+l_2^c}+(\theta+\phi)^{2l_1^c}+(\theta+\phi)^{2l_2^c}+2(\theta+\phi)^{l_1^c+l_2^c}+\\[4pt]3(-2+\phi)\phi(-\theta+\phi)^{2L^c}\left((-\theta+\phi)^{l_1^c}-(-\theta+\phi)^{l_2^c}\right)^2\end{array}\right)+$$

$$2\theta(-1+\phi)^2$$

$$\left(\begin{array}{l}4-4\phi^2-4\phi^3-2\phi^3(-\theta+\phi)^{2l_1^c}-\phi^4(-\theta+\phi)^{2l_1^c}+3\phi^5(-\theta+\phi)^{2(L^c+l_1^c)}+\\[4pt]\phi^6(-\theta+\phi)^{2(L^c+l_1^c)}-\phi^7(-\theta+\phi)^{2(L^c+l_1^c)}-2\phi^3(-\theta+\phi)^{2l_2^c}-\phi^4(-\theta+\phi)^{2l_2^c}+\\[4pt]3\phi^5(-\theta+\phi)^{2(L^c+l_2^c)}+\phi^6(-\theta+\phi)^{2(L^c+l_2^c)}-\phi^7(-\theta+\phi)^{2(L^c+l_2^c)}+4\phi^3(-\theta+\phi)^{l_1^c+l_2^c}+\\[4pt]2\phi^4(-\theta+\phi)^{l_1^c+l_2^c}-6\phi^5(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-2\phi^6(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+2\phi^7(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-\\[4pt]10\phi^2(\theta+\phi)^{l_1^c}-8\phi^3(\theta+\phi)^{l_1^c}-2\phi^4(\theta+\phi)^{l_1^c}+2\phi^3(\theta+\phi)^{2l_1^c}+\phi^4(\theta+\phi)^{2l_1^c}-\\[4pt]10\phi^2(\theta+\phi)^{l_2^c}-8\phi^3(\theta+\phi)^{l_2^c}-2\phi^4(\theta+\phi)^{l_2^c}+2\phi^3(\theta+\phi)^{2l_2^c}+\phi^4(\theta+\phi)^{2l_2^c}+\\[4pt]4\phi^3(\theta+\phi)^{l_1^c+l_2^c}+2\phi^4(\theta+\phi)^{l_1^c+l_2^c}-4\phi\left(-1+(\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)+\\[4pt]6(\theta+\phi)^{L^c}(\phi+\phi^2)^2\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)+\\[4pt]\phi^5(\theta+\phi)^{2L^c}\left(-3+(-1+\phi)\phi\right)\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)^2\end{array}\right)+$$

$$2\theta^7\left(\begin{array}{l}(-\theta+\phi)^{2l_1^c}+(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2l_2^c}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{l_1^c+l_2^c}-\\[4pt]2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+2(\theta+\phi)^{l_1^c}-(\theta+\phi)^{2l_1^c}-6(\theta+\phi)^{L^c+l_1^c}-(\theta+\phi)^{2(L^c+l_1^c)}-\\[4pt](-\theta+\phi)^{2l_2^c}\left(1+(\theta+\phi)^{2L^c}\right)-\\[4pt]2(\theta+\phi)^{l_2^c}\left(-1+3(\theta+\phi)^{L^c}+(\theta+\phi)^{l_1^c}+(\theta+\phi)^{2L^c+l_1^c}\right)+\\[4pt]4\phi^3\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-(\theta+\phi)^{2(L^c+l_1^c)}-\\[4pt](\theta+\phi)^{2(L^c+l_2^c)}-2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)\end{array}\right)+$$

$$(-1+\phi)^3(1+\phi)$$

$$\left(\begin{array}{l}-4-4\phi^3\left(-1+(\theta+\phi)^{L^c}\right)\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)+4\phi^2\left(1+(\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)-\\[4pt]\phi^4\left(\begin{array}{l}(-\theta+\phi)^{2l_1^c}+(-\theta+\phi)^{2l_2^c}-2(-\theta+\phi)^{l_1^c+l_2^c}+(\theta+\phi)^{2l_1^c}+\\[4pt]4(\theta+\phi)^{L^c+l_1^c}+(\theta+\phi)^{2l_2^c}+4(\theta+\phi)^{L^c+l_2^c}+2(\theta+\phi)^{l_1^c+l_2^c}\end{array}\right)+\\[4pt]\phi^6\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+(\theta+\phi)^{2(L^c+l_1^c)}+\\[4pt](\theta+\phi)^{2(L^c+l_2^c)}+2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)\end{array}\right)-$$

$$2\theta^5\left(\begin{array}{l}-4-4\phi+(-\theta+\phi)^{2l_1^c}+3\phi^2(-\theta+\phi)^{2l_1^c}-3\phi(-\theta+\phi)^{2(L^c+l_1^c)}-\\[4pt]9\phi^2(-\theta+\phi)^{2(L^c+l_1^c)}-6\phi^4(-\theta+\phi)^{2(L^c+l_1^c)}+6\phi^5(-\theta+\phi)^{2(L^c+l_1^c)}+\\[4pt](-\theta+\phi)^{2l_2^c}+3\phi^2(-\theta+\phi)^{2l_2^c}-3\phi(-\theta+\phi)^{2(L^c+l_2^c)}-\\[4pt]9\phi^2(-\theta+\phi)^{2(L^c+l_2^c)}-6\phi^4(-\theta+\phi)^{2(L^c+l_2^c)}+6\phi^5(-\theta+\phi)^{2(L^c+l_2^c)}-\\[4pt]2(-\theta+\phi)^{l_1^c+l_2^c}-6\phi^2(-\theta+\phi)^{l_1^c+l_2^c}+6\phi(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+\\[4pt]18\phi^2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+12\phi^4(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-12\phi^5(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-\\[4pt]4(\theta+\phi)^{l_1^c}+4\phi(\theta+\phi)^{l_1^c}+6\phi^2(\theta+\phi)^{l_1^c}-(\theta+\phi)^{2l_1^c}-3\phi^2(\theta+\phi)^{2l_1^c}-\\[4pt]4(\theta+\phi)^{l_2^c}+4\phi(\theta+\phi)^{l_2^c}+6\phi^2(\theta+\phi)^{l_2^c}(\theta+\phi)^{2l_2^c}-3\phi^2(\theta+\phi)^{2l_2^c}-\\[4pt]-2(\theta+\phi)^{l_1^c+l_2^c}-6\phi^2(\theta+\phi)^{l_1^c+l_2^c}-\\[4pt]2(\theta+\phi)^{L^c}(2+9\phi^2)\big((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\big)-3\phi(\theta+\phi)^{2L^c}\\[4pt]\big((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\big)^2\big(-1+\phi\big(-3+2(-1+\phi)\phi^2\big)\big)\end{array}\right)+$$

$$\theta^4\left(\begin{array}{l}-4-8\phi-4\phi^2+(-\theta+\phi)^{2l_1^c}+6\phi(-\theta+\phi)^{2l_1^c}+6\phi^3(-\theta+\phi)^{2l_1^c}-6\phi^4(-\theta+\phi)^{2l_1^c}-\\[4pt]15\phi^2(-\theta+\phi)^{2(L^c+l_1^c)}-10\phi^3(-\theta+\phi)^{2(L^c+l_1^c)}+12\phi^5(-\theta+\phi)^{2(L^c+l_1^c)}+2\phi^6(-\theta+\phi)^{2(L^c+l_1^c)}+\\[4pt](-\theta+\phi)^{2l_2^c}+6\phi(-\theta+\phi)^{2l_2^c}+6\phi^3(-\theta+\phi)^{2l_2^c}-6\phi^4(-\theta+\phi)^{2l_2^c}-15\phi^2(-\theta+\phi)^{2(L^c+l_2^c)}-\\[4pt]10\phi^3(-\theta+\phi)^{2(L^c+l_2^c)}+12\phi^5(-\theta+\phi)^{2(L^c+l_2^c)}+2\phi^6(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{l_1^c+l_2^c}-\\[4pt]12\phi(-\theta+\phi)^{l_1^c+l_2^c}-12\phi^3(-\theta+\phi)^{l_1^c+l_2^c}+12\phi^4(-\theta+\phi)^{l_1^c+l_2^c}+30\phi^2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+\\[4pt]20\phi^3(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-24\phi^5(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-4\phi^6(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-8(\theta+\phi)^{l_1^c}-\\[4pt]8\phi(\theta+\phi)^{l_1^c}-28\phi^2(\theta+\phi)^{l_1^c}+12\phi^3(\theta+\phi)^{l_1^c}+(\theta+\phi)^{2l_1^c}+6\phi(\theta+\phi)^{2l_1^c}+\\[4pt]6\phi^3(\theta+\phi)^{2l_1^c}-6\phi^4(\theta+\phi)^{2l_1^c}-8(\theta+\phi)^{l_2^c}-8\phi(\theta+\phi)^{l_2^c}-28\phi^2(\theta+\phi)^{l_2^c}+\\[4pt]12\phi^3(\theta+\phi)^{l_2^c}+(\theta+\phi)^{2l_2^c}+6\phi(\theta+\phi)^{2l_2^c}+6\phi^3(\theta+\phi)^{2l_2^c}-6\phi^4(\theta+\phi)^{2l_2^c}+\\[4pt]2(\theta+\phi)^{l_1^c+l_2^c}+12\phi(\theta+\phi)^{l_1^c+l_2^c}+12\phi^3(\theta+\phi)^{l_1^c+l_2^c}-12\phi^4(\theta+\phi)^{l_1^c+l_2^c}-\\[4pt]12(\theta+\phi)^{L^c}\big(-1+(-1+\phi)\phi\big)\big(1+\phi+2\phi^2\big)\big((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\big)+\\[4pt]\phi^2(\theta+\phi)^{2L^c}\big((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\big)^2\big(-15+2\phi\big(-5+\phi^2(6+\phi)\big)\big)\end{array}\right)-$$

$$\theta^2\left(\begin{array}{l}4\big(1+(\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\big)-\\[6pt]6\phi^2\left(\begin{array}{l}-4+(-\theta+\phi)^{2l_1^c}+(-\theta+\phi)^{2l_2^c}-2(-\theta+\phi)^{l_1^c+l_2^c}+(\theta+\phi)^{2l_1^c}+\\4(\theta+\phi)^{L^c+l_1^c}+(\theta+\phi)^{2l_2^c}+4(\theta+\phi)^{L^c+l_2^c}+2(\theta+\phi)^{l_1^c+l_2^c}\end{array}\right)-\\[14pt]4\phi^6\left(\begin{array}{l}(-\theta+\phi)^{2l_1^c}+(-\theta+\phi)^{2l_2^c}-2(-\theta+\phi)^{l_1^c+l_2^c}+(\theta+\phi)^{2l_1^c}+\\4(\theta+\phi)^{L^c+l_1^c}+(\theta+\phi)^{2l_2^c}+4(\theta+\phi)^{L^c+l_2^c}+2(\theta+\phi)^{l_1^c+l_2^c}\end{array}\right)+\\[14pt]4\phi^3\left(\begin{array}{l}-4+(-\theta+\phi)^{2l_1^c}+(-\theta+\phi)^{2l_2^c}-2(-\theta+\phi)^{l_1^c+l_2^c}-4(\theta+\phi)^{l_1^c}+(\theta+\phi)^{2l_1^c}+\\4(\theta+\phi)^{L^c+l_1^c}-4(\theta+\phi)^{l_2^c}+(\theta+\phi)^{2l_2^c}+4(\theta+\phi)^{L^c+l_2^c}+2(\theta+\phi)^{l_1^c+l_2^c}\end{array}\right)+\\[14pt]3\phi^8\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+\\(\theta+\phi)^{2(L^c+l_1^c)}+(\theta+\phi)^{2(L^c+l_2^c)}+2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)-\end{array}\right)$$

$$\theta^2 \left(6\phi^5 \begin{pmatrix} -(-\theta+\phi)^{2l_1^c} + 3(-\theta+\phi)^{2(L^c+l_1^c)} - (-\theta+\phi)^{2l_2^c} + 3(-\theta+\phi)^{2(L^c+l_2^c)} + \\ 2(-\theta+\phi)^{l_1^c+l_2^c} - 6(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - 2(\theta+\phi)^{l_1^c} - (\theta+\phi)^{2l_1^c} - 2(\theta+\phi)^{L^c+l_1^c} + \\ 3(\theta+\phi)^{2(L^c+l_1^c)} - 2(\theta+\phi)^{l_2^c} - (\theta+\phi)^{2l_2^c} - 2(\theta+\phi)^{L^c+l_2^c} + 3(\theta+\phi)^{2(L^c+l_2^c)} - \\ 2(\theta+\phi)^{l_1^c+l_2^c} + 6(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{pmatrix} + \right.$$

$$\left. \phi^4 \begin{pmatrix} 4 + 15(-\theta+\phi)^{2(L^c+l_1^c)} + 15(-\theta+\phi)^{2(L^c+l_2^c)} - 30(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - 20(\theta+\phi)^{l_1^c} + \\ 24(\theta+\phi)^{L^c+l_1^c} + 15(\theta+\phi)^{2(L^c+l_1^c)} - 20(\theta+\phi)^{l_2^c} + 24(\theta+\phi)^{L^c+l_2^c} + \\ 15(\theta+\phi)^{2(L^c+l_2^c)} + 30(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{pmatrix} \right) -$$

$$\theta^2 \left(-4\phi \left(4 - 5(\theta+\phi)^{l_1^c} + 3(\theta+\phi)^{L^c+l_1^c} + (\theta+\phi)^{l_2^c}\left(-5 + 3(\theta+\phi)^{L^c} \right) \right) \right) +$$

$$\theta^6 \big[4 - (-\theta+\phi)^{2(L^c+l_1^c)} - (-\theta+\phi)^{2(L^c+l_2^c)} + 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + 12(\theta+\phi)^{l_1^c} -$$
$$8(\theta+\phi)^{L^c+l_1^c} - (\theta+\phi)^{2(L^c+l_1^c)} + 12(\theta+\phi)^{l_2^c} - 8(\theta+\phi)^{L^c+l_2^c} - (\theta+\phi)^{2(L^c+l_2^c)} -$$
$$2(\theta+\phi)^{2L^c+l_1^c+l_2^c} +$$

$$4\phi^2 \begin{pmatrix} (-\theta+\phi)^{2l_1^c} + (-\theta+\phi)^{2l_2^c} - 2(-\theta+\phi)^{l_1^c+l_2^c} + (\theta+\phi)^{2l_1^c} + 4(\theta+\phi)^{L^c+l_1^c} + \\ (\theta+\phi)^{2l_2^c} + 4(\theta+\phi)^{L^c+l_2^c} + 2(\theta+\phi)^{l_1^c+l_2^c} \end{pmatrix} -$$

$$16\phi^3 \begin{pmatrix} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + \\ (\theta+\phi)^{2(L^c+l_1^c)} + (\theta+\phi)^{2(L^c+l_2^c)} + 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{pmatrix} +$$

$$2\phi^4 \begin{pmatrix} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + \\ (\theta+\phi)^{2(L^c+l_1^c)} + (\theta+\phi)^{2(L^c+l_2^c)} + 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{pmatrix} -$$

$$2\phi \begin{pmatrix} (-\theta+\phi)^{2l_1^c} + 5(-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2l_2^c} + 5(-\theta+\phi)^{2(L^c+l_2^c)} - \\ 2(-\theta+\phi)^{l_1^c+l_2^c} - 10(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + 2(\theta+\phi)^{l_1^c} + (\theta+\phi)^{2l_1^c} + \\ 2(\theta+\phi)^{L^c+l_1^c} + 5(\theta+\phi)^{2(L^c+l_1^c)} + (\theta+\phi)^{2l_2^c}\left(1 + 5(\theta+\phi)^{2L^c} \right) + 2(\theta+\phi)^{l_2^c} + \\ \left(1 + (\theta+\phi)^{L^c} + (\theta+\phi)^{l_1^c} + 5(\theta+\phi)^{2L^c+l_1^c} \right) \end{pmatrix} \big] -$$

$$2\theta^3 \big[8 + 6(\theta+\phi)^{l_1^c} - 2\phi^3$$

$$\begin{pmatrix} 4 + 5(-\theta+\phi)^{2(L^c+l_1^c)} + 5(-\theta+\phi)^{2(L^c+l_2^c)} - 10(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + 4(\theta+\phi)^{l_1^c} - \\ 5(\theta+\phi)^{2(L^c+l_1^c)} + 4(\theta+\phi)^{l_2^c} - 5(\theta+\phi)^{2(L^c+l_2^c)} - 10(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{pmatrix} +$$

$$\phi^4 \begin{pmatrix} -3(-\theta+\phi)^{2l_1^c} + 5(-\theta+\phi)^{2(L^c+l_1^c)} - 3(-\theta+\phi)^{2l_2^c} + 5(-\theta+\phi)^{2(L^c+l_2^c)} + \\ 6(-\theta+\phi)^{l_1^c+l_2^c} - 10(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - 6(\theta+\phi)^{l_1^c} + 3(\theta+\phi)^{2l_1^c} + \\ 18(\theta+\phi)^{L^c+l_1^c} - 5(\theta+\phi)^{2(L^c+l_1^c)} - 6(\theta+\phi)^{l_2^c} + 3(\theta+\phi)^{2l_2^c} + \\ 18(\theta+\phi)^{L^c+l_2^c} - 5(\theta+\phi)^{2(L^c+l_2^c)} + 6(\theta+\phi)^{l_1^c+l_2^c} - 10(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{pmatrix} +$$

$$8\phi^6 \begin{pmatrix} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - \\ (\theta+\phi)^{2(L^c+l_1^c)} - (\theta+\phi)^{2(L^c+l_2^c)} - 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{pmatrix} -$$

$$4\phi^7 \begin{pmatrix} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - \\ (\theta+\phi)^{2(L^c+l_1^c)} - (\theta+\phi)^{2(L^c+l_2^c)} - 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{pmatrix} -$$

$$2\left((\theta+\phi)^{L^c+l_1^c} + (\theta+\phi)^{l_2^c}\left(-3 + (\theta+\phi)^{L^c} \right) \right) +$$

$$2\phi\left(\begin{array}{l}-4+(-\theta+\phi)^{2l_1^c}+(-\theta+\phi)^{2l_2^c}-2(-\theta+\phi)^{l_1^c+l_2^c}+4(\theta+\phi)^{l_1^c}-(\theta+\phi)^{2l_1^c}+\\4(\theta+\phi)^{l_2^c}-(\theta+\phi)^{2l_2^c}-2(\theta+\phi)^{l_1^c+l_2^c}-8(\theta+\phi)^{L^c}\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)\end{array}\right)+$$

$$2\phi^2\left(\begin{array}{l}4+(-\theta+\phi)^{2l_1^c}+(-\theta+\phi)^{2l_2^c}-2(-\theta+\phi)^{l_1^c+l_2^c}+4(\theta+\phi)^{l_1^c}-(\theta+\phi)^{2l_1^c}+\\4(\theta+\phi)^{l_2^c}-(\theta+\phi)^{2l_2^c}-2(\theta+\phi)^{l_1^c+l_2^c}-4(\theta+\phi)^{L^c}\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)\end{array}\right)\Big]\Big\}.$$

(C.5)

The order variance of the centralised DC, $VAR[O^c]$, is given by

$$VAR[O^c]=\frac{1}{\left(\theta^2-(-1+\phi)^2\right)^2\left(\theta^2-(1+\phi)^2\right)}$$

$$\left\{\theta^8(1+3\phi)\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+\\(\theta+\phi)^{2(L^c+l_1^c)}+(\theta+\phi)^{2(L^c+l_2^c)}+2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)+$$

$$\theta^9\left(\begin{array}{l}-(-\theta+\phi)^{2(L^c+l_1^c)}-(-\theta+\phi)^{2(L^c+l_2^c)}+2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+\\(\theta+\phi)^{2(L^c+l_1^c)}+(\theta+\phi)^{2(L^c+l_2^c)}+2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)-$$

$$\theta^7\left(\begin{array}{l}-(-\theta+\phi)^{2(L^c+l_1^c)}-(-\theta+\phi)^{2(L^c+l_2^c)}+2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+\\2(\theta+\phi)^{L^c+l_1^c}+(\theta+\phi)^{2(L^c+l_1^c)}+2(\theta+\phi)^{L^c+l_2^c}+(\theta+\phi)^{2(L^c+l_2^c)}+2(\theta+\phi)^{2L^c+l_1^c+l_2^c}+\\4\phi\left((-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-\\(\theta+\phi)^{2(L^c+l_1^c)}-(\theta+\phi)^{2(L^c+l_2^c)}-2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\right)\end{array}\right)+$$

$$\theta^5\phi\Big[2\left(\begin{array}{l}3(-\theta+\phi)^{2(L^c+l_1^c)}+3(-\theta+\phi)^{2(L^c+l_2^c)}-6(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-2(\theta+\phi)^{L^c+l_1^c}-\\3(\theta+\phi)^{2(L^c+l_1^c)}-2(\theta+\phi)^{L^c+l_2^c}-3(\theta+\phi)^{2(L^c+l_2^c)}-6(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)+$$

$$3\phi\left(\begin{array}{l}3(-\theta+\phi)^{2(L^c+l_1^c)}+3(-\theta+\phi)^{2(L^c+l_2^c)}-6(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+2(\theta+\phi)^{L^c+l_1^c}-\\3(\theta+\phi)^{2(L^c+l_1^c)}+2(\theta+\phi)^{L^c+l_2^c}-3(\theta+\phi)^{2(L^c+l_2^c)}-6(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)+$$

$$4\phi^2\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-\\(\theta+\phi)^{2(L^c+l_1^c)}-(\theta+\phi)^{2(L^c+l_2^c)}-2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)+$$

$$6\phi^3\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}-\\(\theta+\phi)^{2(L^c+l_1^c)}-(\theta+\phi)^{2(L^c+l_2^c)}-2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)\Big]-$$

$$(-1+\phi)^2(1+\phi)$$

$$\left(\begin{array}{l}2+2\phi-2\phi^3(\theta+\phi)^{L^c}\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)-2\phi^4(\theta+\phi)^{L^c}\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)+\\\phi^6\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+l_1^c)}+(-\theta+\phi)^{2(L^c+l_2^c)}-2(-\theta+\phi)^{2L^c+l_1^c+l_2^c}+\\(\theta+\phi)^{2(L^c+l_1^c)}+(\theta+\phi)^{2(L^c+l_2^c)}+2(\theta+\phi)^{2L^c+l_1^c+l_2^c}\end{array}\right)\end{array}\right)+$$

$$\theta^3\Big[8\phi^2(\theta+\phi)^{L^c}\left((\theta+\phi)^{l_1^c}+(\theta+\phi)^{l_2^c}\right)+2\left(2+(\theta+\phi)^{L^c+l_1^c}+(\theta+\phi)^{L^c+l_2^c}\right)+$$

$$4\phi\left(-1+3(\theta+\phi)^{L^c+l_1^c}+3(\theta+\phi)^{L^c+l_2^c}\right)+$$

171

$$4\phi^3 \left(\begin{array}{l} 5(-\theta+\phi)^{2(L^c+l_1^c)} + 5(-\theta+\phi)^{2(L^c+l_2^c)} - 10(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + 2(\theta+\phi)^{L^c+l_1^c} - \\ 5(\theta+\phi)^{2(L^c+l_1^c)} + 2(\theta+\phi)^{L^c+l_2^c} - 5(\theta+\phi)^{2(L^c+l_2^c)} - 10(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) +$$

$$4\phi^5 \left(\begin{array}{l} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - \\ (\theta+\phi)^{2(L^c+l_1^c)} - (\theta+\phi)^{2(L^c+l_2^c)} - 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) -$$

$$8\phi^6 \left(\begin{array}{l} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - \\ (\theta+\phi)^{2(L^c+l_1^c)} - (\theta+\phi)^{2(L^c+l_2^c)} - 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) +$$

$$\phi^4 \left(\begin{array}{l} -5(-\theta+\phi)^{2(L^c+l_1^c)} - 5(-\theta+\phi)^{2(L^c+l_2^c)} + 10(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - 6(\theta+\phi)^{L^c+l_1^c} + \\ 5(\theta+\phi)^{2(L^c+l_1^c)} - 6(\theta+\phi)^{L^c+l_2^c} + 5(\theta+\phi)^{2(L^c+l_2^c)} + 10(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) \Big] -$$

$$\theta^6 \Big[(-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + 4(\theta+\phi)^{L^c+l_1^c} +$$

$$(\theta+\phi)^{2(L^c+l_1^c)} + 4(\theta+\phi)^{L^c+l_2^c} + (\theta+\phi)^{2(L^c+l_2^c)} + 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} -$$

$$4\phi^2 \left(\begin{array}{l} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + \\ (\theta+\phi)^{2(L^c+l_1^c)} + (\theta+\phi)^{2(L^c+l_2^c)} + 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) +$$

$$8\phi^3 \left(\begin{array}{l} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + \\ (\theta+\phi)^{2(L^c+l_1^c)} + (\theta+\phi)^{2(L^c+l_2^c)} + 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) +$$

$$\phi \left(\begin{array}{l} 5(-\theta+\phi)^{2(L^c+l_1^c)} + 5(-\theta+\phi)^{2(L^c+l_2^c)} - 10(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + 2(\theta+\phi)^{L^c+l_1^c} + \\ 5(\theta+\phi)^{2(L^c+l_1^c)} + 2(\theta+\phi)^{L^c+l_2^c} + 5(\theta+\phi)^{2(L^c+l_2^c)} + 10(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) \Big] +$$

$$\theta^4 \Big[2 + 4(\theta+\phi)^{L^c+l_1^c} + 4(\theta+\phi)^{L^c+l_2^c} + 4\phi(\theta+\phi)^{L^c} \left((\theta+\phi)^{l_1^c} + (\theta+\phi)^{l_2^c} \right) -$$

$$10\phi^4 \left(\begin{array}{l} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + \\ (\theta+\phi)^{2(L^c+l_1^c)} + (\theta+\phi)^{2(L^c+l_2^c)} + 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) +$$

$$6\phi^5 \left(\begin{array}{l} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + \\ (\theta+\phi)^{2(L^c+l_1^c)} + (\theta+\phi)^{2(L^c+l_2^c)} + 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) -$$

$$\phi^3 \left(\begin{array}{l} 5(-\theta+\phi)^{2(L^c+l_1^c)} + 5(-\theta+\phi)^{2(L^c+l_2^c)} - 10(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - 6(\theta+\phi)^{L^c+l_1^c} + \\ 5(\theta+\phi)^{2(L^c+l_1^c)} - 6(\theta+\phi)^{L^c+l_2^c} + 5(\theta+\phi)^{2(L^c+l_2^c)} + 10(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) -$$

$$\phi^2 \left(\begin{array}{l} 15(-\theta+\phi)^{2(L^c+l_1^c)} + 15(-\theta+\phi)^{2(L^c+l_2^c)} - 30(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - 8(\theta+\phi)^{L^c+l_1^c} + \\ 15(\theta+\phi)^{2(L^c+l_1^c)} - 8(\theta+\phi)^{L^c+l_2^c} + 15(\theta+\phi)^{2(L^c+l_2^c)} + 30(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) \Big] +$$

$$\theta^2 \phi \Big[-8 + 6(\theta+\phi)^{L^c+l_1^c} + 6(\theta+\phi)^{L^c+l_2^c} + 12\phi(\theta+\phi)^{L^c} \left((\theta+\phi)^{l_1^c} + (\theta+\phi)^{l_2^c} \right) +$$

$$4\phi^5 \left(\begin{array}{l} (-\theta+\phi)^{2(L^c+l_1^c)} + (-\theta+\phi)^{2(L^c+l_2^c)} - 2(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + \\ (\theta+\phi)^{2(L^c+l_1^c)} + (\theta+\phi)^{2(L^c+l_2^c)} + 2(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) +$$

$$3\phi^4 \left(\begin{array}{l} 3(-\theta+\phi)^{2(L^c+l_1^c)} + 3(-\theta+\phi)^{2(L^c+l_2^c)} - 6(-\theta+\phi)^{2L^c+l_1^c+l_2^c} - 2(\theta+\phi)^{L^c+l_1^c} + \\ 3(\theta+\phi)^{2(L^c+l_1^c)} - 2(\theta+\phi)^{L^c+l_2^c} + 3(\theta+\phi)^{2(L^c+l_2^c)} + 6(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) -$$

$$\phi^3 \left(\begin{array}{l} 15(-\theta+\phi)^{2(L^c+l_1^c)} + 15(-\theta+\phi)^{2(L^c+l_2^c)} - 30(-\theta+\phi)^{2L^c+l_1^c+l_2^c} + 4(\theta+\phi)^{L^c+l_1^c} + \\ 15(\theta+\phi)^{2(L^c+l_1^c)} + 4(\theta+\phi)^{L^c+l_2^c} + 15(\theta+\phi)^{2(L^c+l_2^c)} + 30(\theta+\phi)^{2L^c+l_1^c+l_2^c} \end{array} \right) \Big] -$$

$$\theta(-1+\phi)\big[-4+\phi$$

$$\left(\begin{array}{l} -8+ \\ \phi\Big(-4+6(\theta+\phi)^{L^c+I_1^c}+6(\theta+\phi)^{L^c+I_2^c}+10\phi(\theta+\phi)^{L^c}\Big((\theta+\phi)^{I_1^c}+(\theta+\phi)^{I_2^c}\Big)+ \\ 2\phi^2(\theta+\phi)^{L^c}\Big((\theta+\phi)^{I_1^c}+(\theta+\phi)^{I_2^c}\Big)+ \\ 2\phi^3\left(\begin{array}{l}3(-\theta+\phi)^{2(L^c+I_1^c)}+3(-\theta+\phi)^{2(L^c+I_2^c)}-6(-\theta+\phi)^{2L^c+I_1^c+I_2^c}-(\theta+\phi)^{L^c+I_1^c}- \\ 3(\theta+\phi)^{2(L^c+I_1^c)}-(\theta+\phi)^{L^c+I_2^c}-3(\theta+\phi)^{2(L^c+I_2^c)}-6(\theta+\phi)^{2L^c+I_1^c+I_2^c}\end{array}\right)+ \\ \phi^4\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+I_1^c)}+(-\theta+\phi)^{2(L^c+I_2^c)}-2(-\theta+\phi)^{2L^c+I_1^c+I_2^c}- \\ (\theta+\phi)^{2(L^c+I_1^c)}-(\theta+\phi)^{2(L^c+I_2^c)}-2(\theta+\phi)^{2L^c+I_1^c+I_2^c}\end{array}\right)- \\ 3\phi^5\left(\begin{array}{l}(-\theta+\phi)^{2(L^c+I_1^c)}+(-\theta+\phi)^{2(L^c+I_2^c)}-2(-\theta+\phi)^{2L^c+I_1^c+I_2^c}- \\ (\theta+\phi)^{2(L^c+I_1^c)}-(\theta+\phi)^{2(L^c+I_2^c)}-2(\theta+\phi)^{2L^c+I_1^c+I_2^c}\end{array}\right)\end{array}\right) \Big]\Big\}.$$

(C.6)

The *Ratio[Inv]* for the simple model when all lead-times are equal to L is

$$Ratio[Inv]=$$

$$\left\{\sqrt{\left[\frac{1}{\left(\theta^2-(-1+\phi)^2\right)^3\left(\theta^2-(1+\phi)^2\right)}\right.}\right.$$

$$\Big(2L\big(\theta^2+(-1+\phi)^2\big)(-1+\theta-\phi)(1+\theta-\phi)(-1+\theta+\phi)(1+\theta+\phi)- \\ 2\theta^9(1+\phi)\big((-\theta+\phi)^{4L}-(\theta+\phi)^{4L}\big)+\theta^{10}\big((-\theta+\phi)^{4L}+(\theta+\phi)^{4L}\big)+$$

$$\theta^6\left(\begin{array}{l}2+6(-\theta+\phi)^L-2\phi(-\theta+\phi)^L-4(-\theta+\phi)^{2L}-4\phi(-\theta+\phi)^{2L}+12\phi^2(-\theta+\phi)^{2L}- \\ (-\theta+\phi)^{4L}-10\phi(-\theta+\phi)^{4L}-16\phi^3(-\theta+\phi)^{4L}+2\phi^4(-\theta+\phi)^{4L}+6(\theta+\phi)^L- \\ 2\phi(\theta+\phi)^L-4(\theta+\phi)^{2L}-4\phi(\theta+\phi)^{2L}+12\phi^2(\theta+\phi)^{2L}-3(-\theta+\phi)^{2L}(\theta+\phi)^{2L}+ \\ 2\phi(-\theta+\phi)^{2L}(\theta+\phi)^{2L}-4\phi^2(-\theta+\phi)^{2L}(\theta+\phi)^{2L}-(\theta+\phi)^{4L}-10\phi(\theta+\phi)^{4L}- \\ 16\phi^3(\theta+\phi)^{4L}+2\phi^4(\theta+\phi)^{4L}+(-\theta+\phi)^L(\theta+\phi)^L(-\theta^2+\phi^2)^L(3-2\phi+4\phi^2)\end{array}\right)-$$

$$\theta^2(-1+\phi)$$

$$\left(\begin{array}{l}-2+6\phi-6\phi^2+2\phi^3-2(-\theta+\phi)^L-12\phi(-\theta+\phi)^L-12\phi^2(-\theta+\phi)^L-4\phi^3(-\theta+\phi)^L+ \\ 6\phi^4(-\theta+\phi)^L+6\phi(-\theta+\phi)^{2L}+24\phi^2(-\theta+\phi)^{2L}+12\phi^3(-\theta+\phi)^{2L}-12\phi^5(-\theta+\phi)^{2L}- \\ 15\phi^4(-\theta+\phi)^{4L}+3\phi^5(-\theta+\phi)^{4L}-2(\theta+\phi)^L-12\phi(\theta+\phi)^L-12\phi^2(\theta+\phi)^L- \\ 4\phi^3(\theta+\phi)^L+6\phi^4(\theta+\phi)^L+6\phi(\theta+\phi)^{2L}+24\phi^2(\theta+\phi)^{2L}+12\phi^3(\theta+\phi)^{2L}- \\ 12\phi^5(\theta+\phi)^{2L}-(-\theta+\phi)^{2L}(\theta+\phi)^{2L}+\phi(-\theta+\phi)^{2L}(\theta+\phi)^{2L}-\phi^2(-\theta+\phi)^{2L}(\theta+\phi)^{2L}- \\ \phi^3(-\theta+\phi)^{2L}(\theta+\phi)^{2L}-2\phi^4(-\theta+\phi)^{2L}(\theta+\phi)^{2L}+4\phi^5(-\theta+\phi)^{2L}(\theta+\phi)^{2L}- \\ 15\phi^4(\theta+\phi)^{4L}+3\phi^5(\theta+\phi)^{4L}-(-1+\phi)(-\theta+\phi)^L(\theta+\phi)^L(-\theta^2+\phi^2)^L \\ (1+\phi^2+2\phi^3+4\phi^4)+3\phi^6\big((-\theta+\phi)^{4L}+(\theta+\phi)^{4L}\big)+3\phi^7\big((-\theta+\phi)^{4L}+(\theta+\phi)^{4L}\big)\end{array}\right)-$$

$$\theta^8\left(\begin{array}{l}3\big((-\theta+\phi)^{2L}+(\theta+\phi)^{2L}\big)-6\phi\big((-\theta+\phi)^{4L}+(\theta+\phi)^{4L}\big)+ \\ 3\phi^2\big((-\theta+\phi)^{4L}+(\theta+\phi)^{4L}\big)+(-\theta+\phi)^L(\theta+\phi)^L\big(-(-\theta+\phi)^L(\theta+\phi)^L+(-\theta^2+\phi^2)^L\big)\end{array}\right)+$$

$$2\theta^7\left(\begin{array}{l}(\theta+\phi)^L-4(\theta+\phi)^{2L}-(\theta+\phi)^{4L}(1+4\phi^3)+\\(-\theta+\phi)^L\left(-1+4(-\theta+\phi)^L+(-\theta+\phi)^{3L}(1+4\phi^3)\right)\end{array}\right)+$$

$$2\theta\phi\left(\begin{array}{l}(-1+\phi)^2\phi^4(\theta+\phi)^{4L}\left(-3+(-1+\phi)\phi\right)-(2+\phi)(\theta+\phi)^L(-1+\phi^2)^2+\\(-1+\phi)^2\phi(\theta+\phi)^{2L}\left(3+4\phi(2+\phi)\right)+(-\theta+\phi)^L\\\left(\begin{array}{l}-(-1+\phi)^2\phi^4(-\theta+\phi)^{3L}\left(-3+(-1+\phi)\phi\right)+(2+\phi)(-1+\phi^2)^2-\\(-1+\phi)^2\phi(-\theta+\phi)^L\left(3+4\phi(2+\phi)\right)\end{array}\right)\end{array}\right)+$$

$(-1+\phi)^3(1+\phi)$

$$\left(\begin{array}{l}-2+\phi^2\left(\begin{array}{l}2\phi(-\theta+\phi)^L-2\phi(-\theta+\phi)^{2L}-3\phi^2(-\theta+\phi)^{2L}+\phi^4(-\theta+\phi)^{4L}+\\2\phi(\theta+\phi)^L-2\phi(\theta+\phi)^{2L}-3\phi^2(\theta+\phi)^{2L}-(-\theta+\phi)^{2L}(\theta+\phi)^{2L}+\\\phi^2(-\theta+\phi)^{2L}(\theta+\phi)^{2L}++\phi^4(\theta+\phi)^{4L}-\\(-\theta+\phi)^L(\theta+\phi)^L(-1+\phi^2)(-\theta^2+\phi^2)^L+2\left(1+(-\theta+\phi)^L+(\theta+\phi)^L\right)\end{array}\right)\end{array}\right)+$$

$$\theta^4\left(\begin{array}{l}-2-4\phi-2\phi^2-4(-\theta+\phi)^L-4\phi(-\theta+\phi)^L-14\phi^2(-\theta+\phi)^L+\\6\phi^3(-\theta+\phi)^L+7(-\theta+\phi)^{2L}+18\phi(-\theta+\phi)^{2L}+12\phi^2(-\theta+\phi)^{2L}+\\12\phi^3(-\theta+\phi)^{2L}-18\phi^4(-\theta+\phi)^{2L}-15\phi^2(-\theta+\phi)^{4L}-10\phi^3(-\theta+\phi)^{4L}-\\4(\theta+\phi)^L-4\phi(\theta+\phi)^L-14\phi^2(\theta+\phi)^L+6\phi^3(\theta+\phi)^L+7(\theta+\phi)^{2L}+\\18\phi(\theta+\phi)^{2L}+12\phi^2(\theta+\phi)^{2L}+12\phi^3(\theta+\phi)^{2L}-18\phi^4(\theta+\phi)^{2L}+\\3(-\theta+\phi)^{2L}(\theta+\phi)^{2L}-4\phi(-\theta+\phi)^{2L}(\theta+\phi)^{2L}+5\phi^2(-\theta+\phi)^{2L}(\theta+\phi)^{2L}-\\6\phi^3(-\theta+\phi)^{2L}(\theta+\phi)^{2L}+6\phi^4(-\theta+\phi)^{2L}(\theta+\phi)^{2L}-15\phi^2(\theta+\phi)^{4L}-\\10\phi^3(\theta+\phi)^{4L}+12\phi^5\left((-\theta+\phi)^{4L}+(\theta+\phi)^{4L}\right)+2\phi^6\left((-\theta+\phi)^{4L}+(\theta+\phi)^{4L}\right)+\\(-\theta+\phi)^L(\theta+\phi)^L(-\theta^2+\phi^2)^L\left(-3+\phi\left(4+\phi(-5-6(-1+\phi)\phi)\right)\right)\end{array}\right)+$$

$$2\theta^5\left(\begin{array}{l}3(\theta+\phi)^{2L}(1+4\phi^2)+(\theta+\phi)^L\left(2-\phi(2+3\phi)\right)+3\phi(\theta+\phi)^{4L}\\\left(-1+\phi(-3+2(-1+\phi)\phi^2)\right)+(-\theta+\phi)^L\\\left(\begin{array}{l}-2+2\phi+3\phi^2-3(-\theta+\phi)^L(1+4\phi^2)-\\3\phi(-\theta+\phi)^{3L}\left(-1+\phi(-3+2(-1+\phi)\phi^2)\right)\end{array}\right)\end{array}\right)+$$

$$2\theta^3\left(\begin{array}{l}-(-2+\phi)\phi^3(\theta+\phi)^{4L}(-5+4\phi^3)+(\theta+\phi)^{2L}\left(1+2\phi(5+3\phi-6\phi^3)\right)+\\(\theta+\phi)^L\left(-3+\phi(-4+\phi(2+\phi)(-2+3\phi))\right)+(-\theta+\phi)^L\\\left(\begin{array}{l}3+4\phi+4\phi^2-4\phi^3-3\phi^4+(-2+\phi)\phi^3(-\theta+\phi)^{3L}(-5+4\phi^3)+\\(-\theta+\phi)^L\left(-1+2\phi(-5-3\phi+6\phi^3)\right)\end{array}\right)\end{array}\right)]\Big\}/$$

$$\left\{\sqrt{\frac{\left(\begin{array}{l}(\theta+\phi)^{6+4L}+(1+L)(-1+\theta+\phi)(1+\theta+\phi)+2(\theta+\phi)^{2+L}(1+\theta+\phi)-\\(\theta+\phi)^{3+2L}(2+3\theta+3\phi)\end{array}\right)}{\left((-1+\theta+\phi)^3(1+\theta+\phi)\right)}}\right\}.$$

$$(C.7)$$

The *Ratio[Cap]* for the simple model when all lead-times are equal to L is

$$Ratio[Cap] = \cfrac{1}{\sqrt{2}\sqrt{\dfrac{1+\theta+\phi+2(\theta+\phi)^{6+4L}-2(\theta+\phi)^{3+2L}(1+\theta+\phi)}{(-1+\theta+\phi)^2(1+\theta+\phi)}}}$$

$$\left\{\sqrt{\left[-\frac{2(-\theta+\phi)^{6+4L}}{-1+(\theta-\phi)^2}-\frac{2(-\theta+\phi)^{6+4L}}{(-1+\theta+\phi)(1+\theta+\phi)}+\frac{1}{\left(\theta^2-(-1+\phi)^2\right)^2}\binom{2-(-1+\theta)\theta^3(-\theta+\phi)^{2L}+(-1-\theta+\phi)(\theta+\phi)^{3+2L}+}{\left(-2+(-\theta+\phi)^{2L}\left(-3\theta^2+2\theta^3+\theta(3-2\phi)\phi+(-1+\phi)\phi^2\right)\right)}^2\right.}\right.$$

$$\left.\left.+\frac{1}{\left(\theta^2-(-1+\phi)^2\right)^2}\binom{\theta^4\left((-\theta+\phi)^{2L}-(\theta+\phi)^{2L}\right)+3\theta^2\phi\left((-\theta+\phi)^{2L}-(\theta+\phi)^{2L}\right)-}{(-1+\phi)\phi^3\left((-\theta+\phi)^{2L}-(\theta+\phi)^{2L}\right)-\theta^3(1+2\phi)\left((-\theta+\phi)^{2L}-(\theta+\phi)^{2L}\right)+}_{\theta(2+\phi^2(-3+2\phi))\left((-\theta+\phi)^{2L}-(\theta+\phi)^{2L}\right)}^2\right]\right\}.$$

(C.8)

175

Appendix D.   Contour Plots for Section 6.2.2
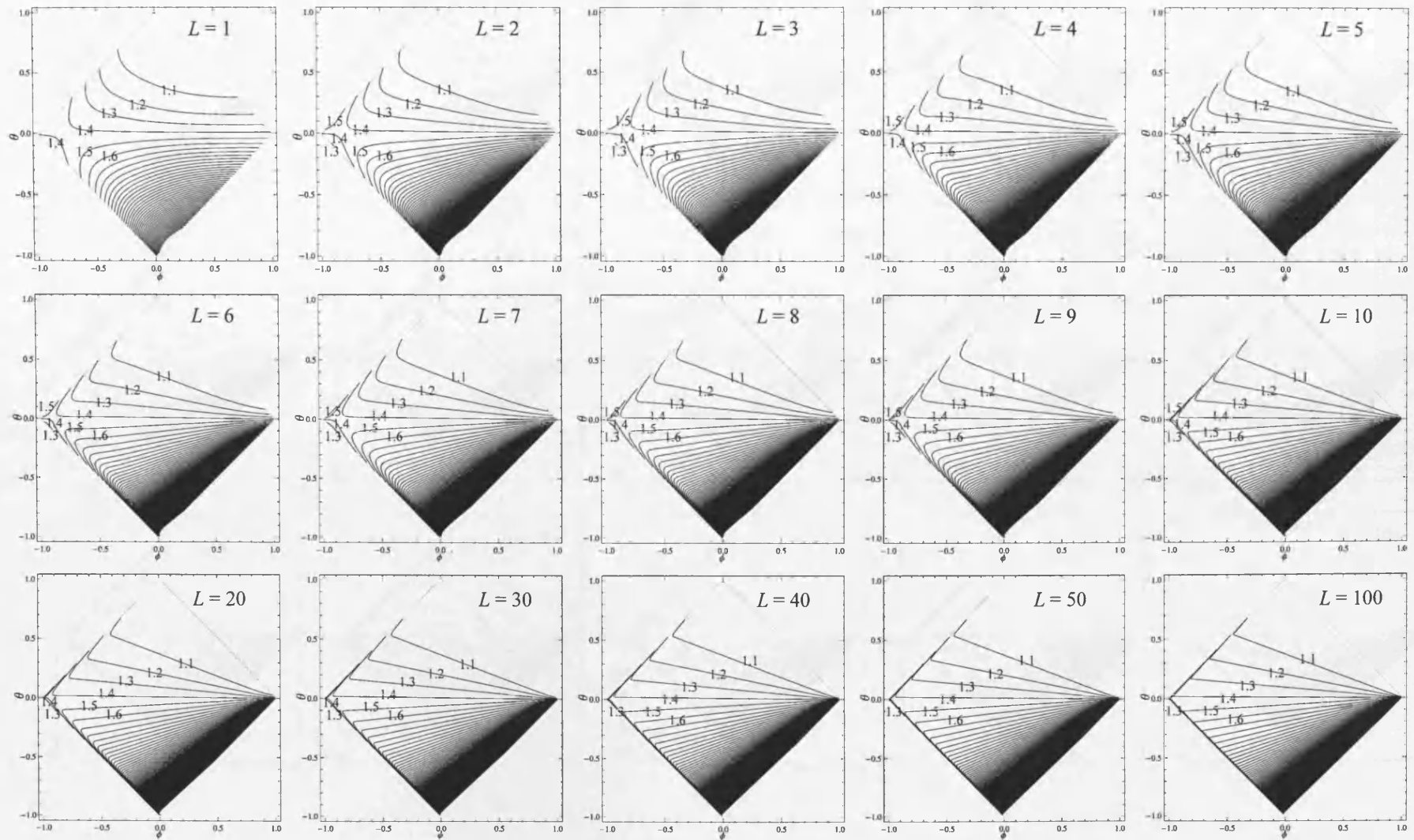
Figure D-1 Contour plots of Ratio[Inv] by different values of lead-time L
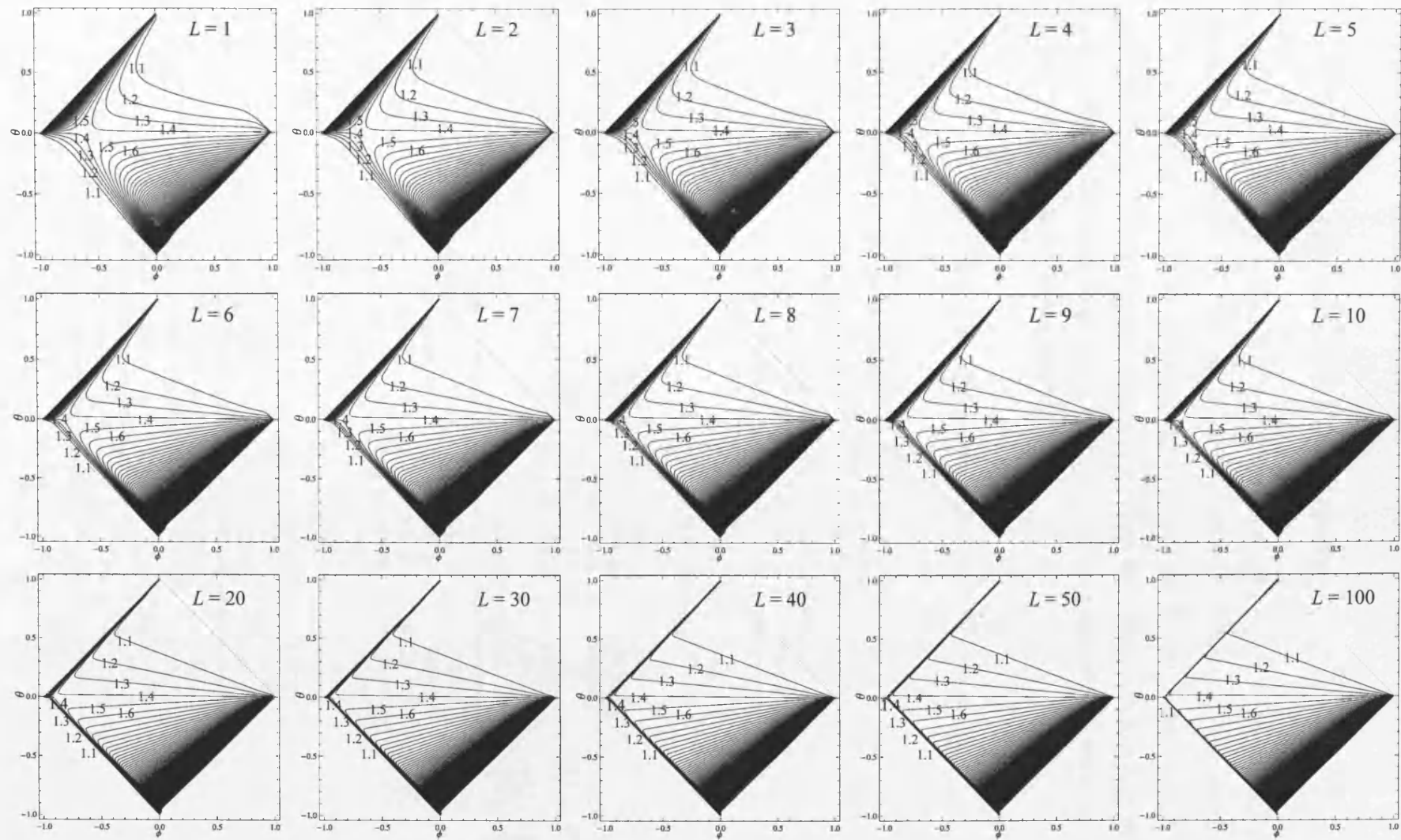
Figure D-2 Contour plots of Ratio[Cap] by different values of lead-time L

Appendix E. Real data analysis

E.1 Data set I: Test result for VAR(1) model

This section presents a complete test result produced by Eviews to validate the result presented in Section 8.2. The bivariate time series in Data set I was estimated as a VAR(1) process. Table E-1 provides the estimated coefficients together with their test statistics. Some statistical characters for the model as a whole are given as well as the result for the model for each store. This result is cross-checked by statistical software called "JMulTi" where the result is identical.

Table E-1 Estimation output for VAR(1) model

Vector Autoregression Estimates
Sample (adjusted): 2 70
Included observations: 69 after adjustments
Standard errors in ( ) & t-statistics in [ ]

	Store 1	Store 2
Store 1 (-1)	0.488283	0.199499
	(0.10749)	(0.09035)
	[4.54271]	[2.20814]
Store 2 (-1)	0.242379	0.637903
	(0.10833)	(0.09106)
	[2.23742]	[7.00568]
C	20.50165	5.837734
	(6.65435)	(5.59323)
	[3.08094]	[1.04371]
R-squared	0.414472	0.578845
Adj. R-squared	0.396729	0.566083
Sum sq. resids	11636.89	8221.495
S.E. equation	13.27843	11.16101
F-statistic	23.35942	45.35595
Log likelihood	-274.8169	-262.8306
Akaike AIC	8.052663	7.705234
Schwarz SC	8.149798	7.802369
Mean dependent	64.62319	52.26087
S.D. dependent	17.09584	16.94338

Determinant resid covariance (dof adj.)	21510.32
Determinant resid covariance	19680.52
Log likelihood	-536.9283
Akaike information criterion	15.73705
Schwarz criterion	15.93132

The diagnosis of the inverse roots of the characteristic autoregressive polynomial (Lütkepohl 1993) shown in Table E-2 confirms that the estimated VAR(1) model is stable (stationary) as all roots have modulus less than one and lie inside the unit circle. The autoregressive root graph is shown in Figure E-1. It is important that the model is stable otherwise certain results such as impulse response standard errors will be invalid. Note that the algebraic representation of the stability condition is presented in Equation (5.3) and a numerical example for the root calculation is shown in Table 6-1.

Table E-2 AR roots for data set I

Roots of Characteristic Polynomial
Endogenous variables: Store 1   Store 2
Exogenous variables: C
Lag specification: 1 1

Root	Modulus
0.795367	0.795367
0.330820	0.330820

No root lies outside the unit circle.
VAR satisfies the stability condition.

Inverse Roots of AR Characteristic Polynomial
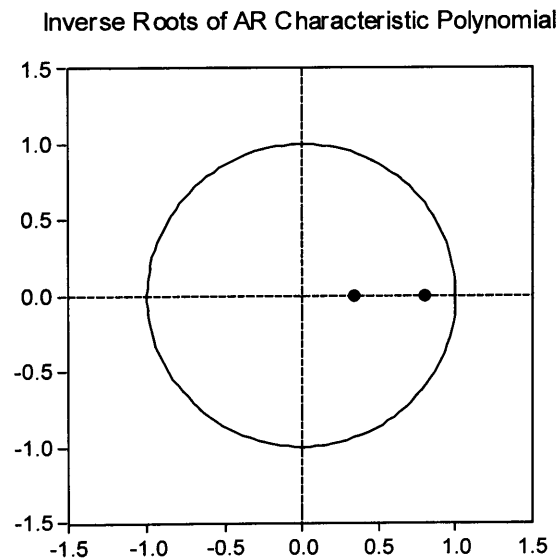
Figure E-1 AR root graph for data set I

Eviews compute various criteria as shown in Table E-3 for selecting the lag length of VAR model. Lütkepohl (1993) discusses about all of the criteria in details. The result shows that all criteria support the choice for VAR of first order.

Table E-3 Lag length criteria

VAR Lag Order Selection Criteria
Endogenous variables: Store 1 Store 2
Exogenous variables: C
Sample: 1 70
Included observations: 64

Lag	LogL	LR	FPE	AIC	SC	HQ
0	-527.8199	NA	53163.18	16.55687	16.62434	16.58345
1	-495.5243	61.56334*	21960.90*	15.67264*	15.87503*	15.75237*
2	-491.6088	7.219330	22029.94	15.67527	16.01260	15.80816
3	-489.4312	3.878759	23347.11	15.73223	16.20448	15.91827
4	-488.8432	1.010601	26025.95	15.83885	16.44604	16.07805
5	-487.0601	2.953271	27980.61	15.90813	16.65024	16.20049
6	-485.1092	3.109332	29967.45	15.97216	16.84921	16.31767

* indicates lag order selected by the criterion
LR: sequential modified LR test statistic (each test at 5% level)
FPE: Final prediction error
AIC: Akaike information criterion
SC: Schwarz information criterion
HQ: Hannan-Quinn information criterion

Another important diagnosis is checking the whiteness of the model's residuals. If the residual is not white noise, the model chosen may not be suitable to represent the data. Eviews provides some tools to confirm the whiteness of the residuals by checking the residual auto-correlations at different lags and by testing for nonnormality. The residuals for the models are visualised in Figure E-2. The residuals are then checked for auto-correlations. Figure E-3 shows the correlograms of the residuals for 12 lags. Under the two standard error intervals, no significant auto-correlations found. Portmanteau Tests for autocorrelations also confirm that there is no significant auto-correlation in the residuals. The test result is in Table E-4. The last test for residual auto-correlation is Lagrange Multiplier (LM) test. From Table E-5, the LM test, again, confirms that no significant residual auto-correlation found. Although the test result for the first lag is quite close but it, however, passes the test at 1% level. Many tools had been applied to check the residual auto-correlations because different criteria examine different features of the data. We

should not rely on single criterion as all different criteria provide useful information for our decision making (Lütkepohl 1993).

Finally, the nonnormaltity of the residuals is investigated. There are tests of skewness, kurtosis and Jarque-Bera test as presented in Table E-6. The residuals of the model are confirmed to be normally distributed as all tests fail to reject that the residuals are multivariate normal at 1% level. This is true for all element-wise and joint tests. More interesting tests for validation of VAR model can be found in Lütkepohl (1993)
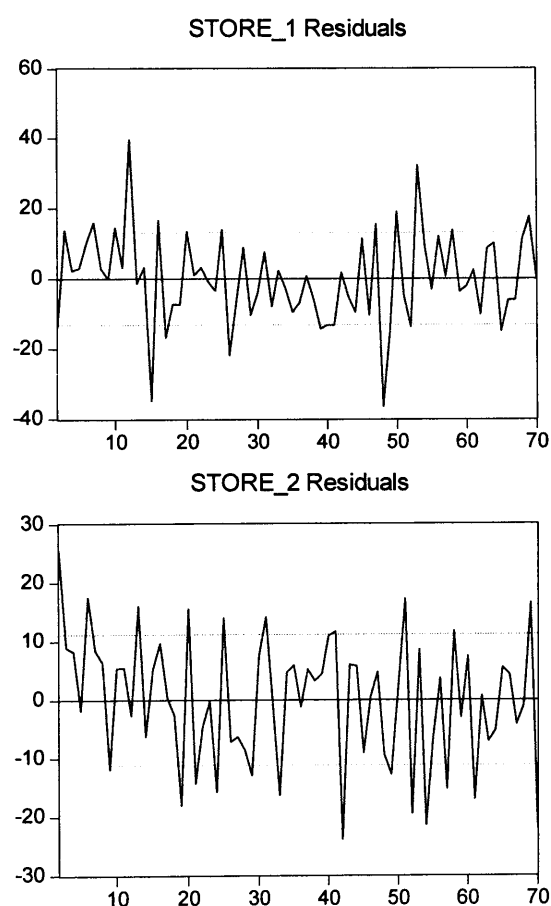
Figure E-2 Residuals plot

Autocorrelations with 2 Std.Err. Bounds
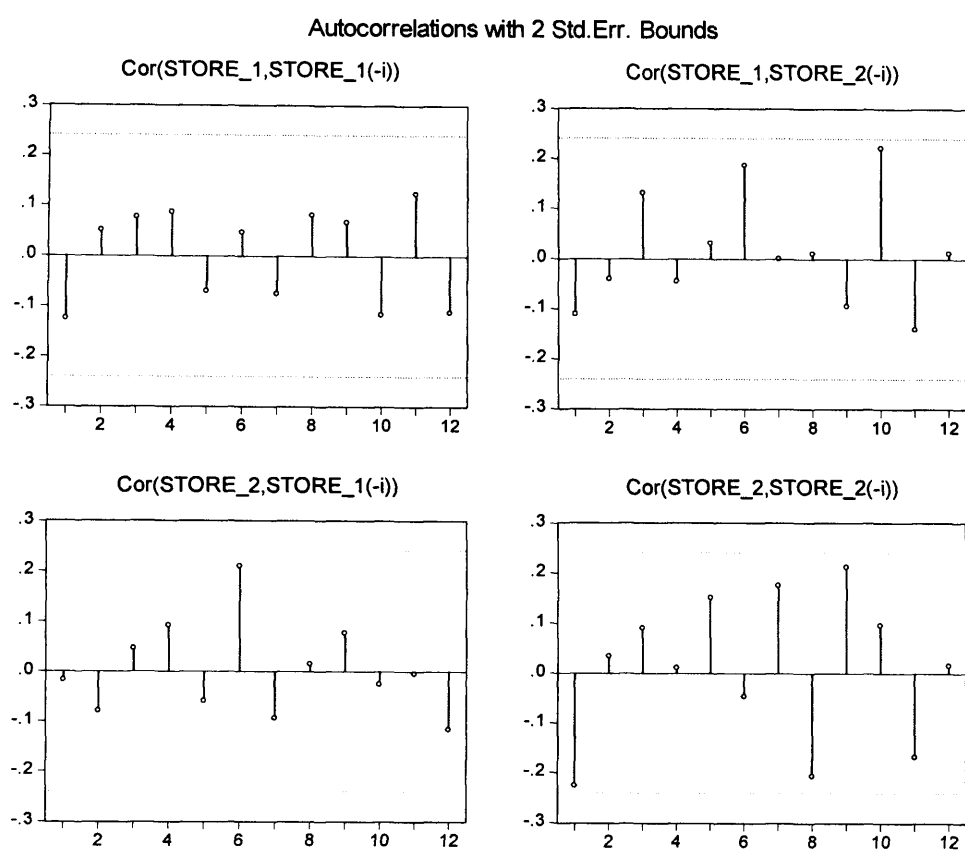
Figure E-3 Correlograms

Table E-4 Portmanteau test

VAR Residual Portmanteau Tests for Autocorrelations
Null Hypothesis: no residual autocorrelations up to lag h
Sample: 1 70
Included observations: 69

Lags	Q-Stat	Prob.	Adj Q-Stat	Prob.	df
1	4.842292	NA*	4.913502	NA*	NA*
2	5.898786	0.5516	6.001533	0.5396	7
3	7.719453	0.7382	7.904958	0.7218	11
4	8.916108	0.8819	9.175253	0.8682	15
5	11.23183	0.9158	11.67190	0.8992	19
6	17.32241	0.7931	18.34253	0.7387	23
7	20.74091	0.7983	22.14699	0.7299	27
8	24.32727	0.7970	26.20368	0.7115	31
9	29.06987	0.7493	31.65768	0.6303	35
10	34.34796	0.6819	37.83036	0.5231	39
11	38.70289	0.6581	43.01123	0.4708	43
12	40.38295	0.7414	45.04498	0.5539	47

*The test is valid only for lags larger than the VAR lag order.
df is degrees of freedom for (approximate) chi-square distribution

Table E-5 Lagrange Multiplier tests

VAR Residual Serial Correlation LM Tests
Null Hypothesis: no serial correlation at
lag order h
Sample: 1 70
Included observations: 69

Lags	LM-Stat	Prob
1	13.22344	0.0102
2	1.050151	0.9021
3	2.037625	0.7288
4	1.186613	0.8803
5	2.356975	0.6704
6	6.457724	0.1675
7	3.745080	0.4416
8	4.047242	0.3997
9	5.186714	0.2687
10	6.134638	0.1893
11	5.106669	0.2765
12	1.763680	0.7791

Probs from chi-square with 4 df.

Table E-6 Test for nonnormality of a VAR process

VAR Residual Normality Tests
Orthogonalization: Residual Covariance (Urzua)
Null Hypothesis: residuals are multivariate normal
Sample: 1 70
Included observations: 69

Component	Skewness	Chi-sq	df	Prob.
1	0.038579	0.018660	1	0.8913
2	-0.242883	0.739603	1	0.3898
Joint		0.758263	2	0.6845

Component	Kurtosis	Chi-sq	df	Prob.
1	4.180117	5.712588	1	0.0168
2	2.556689	0.455898	1	0.4995
Joint		6.168486	2	0.0458

Component	Jarque-Bera	df	Prob.
1	5.731247	2	0.0569
2	1.195502	2	0.5500
Joint	8.198426	9	0.5143

E.2 Data set I: Test result for VAR(p) model

This section provides the test result for the VAR model of higher orders in order to validate the claim that data set I has VAR(1) process.

Table E-7 Estimation output for VAR(2) model

Vector Autoregression Estimates
Included observations: 68 after adjustments
Standard errors in ( ) & t-statistics in [ ]

	STORE_1	STORE_2
STORE_1(-1)	0.378966	0.214301
	(0.12366)	(0.09773)
	[3.06456]	[2.19274]
STORE_1(-2)	0.142930	-0.006247
	(0.12517)	(0.09893)
	[1.14186]	[-0.06314]
STORE_2(-1)	0.117013	0.394299
	(0.15181)	(0.11998)
	[0.77077]	[3.28633]
STORE_2(-2)	0.156638	0.284349
	(0.14533)	(0.11486)
	[1.07779]	[2.47564]
C	16.95399	2.742762
	(7.10669)	(5.61657)
	[2.38564]	[0.48833]
R-squared	0.439386	0.627094
Adj. R-squared	0.403791	0.603417
Sum sq. resids	10965.09	6848.888
S.E. equation	13.19276	10.42653
F-statistic	12.34418	26.48579
Log likelihood	-269.3086	-253.3072
Akaike AIC	8.067900	7.597270
Schwarz SC	8.231099	7.760469
Mean dependent	64.88235	51.76471
S.D. dependent	17.08585	16.55666

Determinant resid covariance (dof adj.)	18435.17
Determinant resid covariance	15823.79
Log likelihood	-521.7308
Akaike information criterion	15.63914
Schwarz criterion	15.96554

Table E-8 Estimation output for VAR(3) model

Vector Autoregression Estimates
Included observations: 67 after adjustments
Standard errors in ( ) & t-statistics in [ ]

	STORE_1	STORE_2
STORE_1(-1)	0.386462	0.218290
	(0.12756)	(0.10038)
	[3.02960]	[2.17469]
STORE_1(-2)	0.152467	-0.001189
	(0.13661)	(0.10750)
	[1.11606]	[-0.01106]
STORE_1(-3)	0.041355	0.038887
	(0.12638)	(0.09945)
	[0.32722]	[0.39103]
STORE_2(-1)	0.001716	0.297818
	(0.16493)	(0.12978)
	[0.01040]	[2.29473]
STORE_2(-2)	0.046336	0.186313
	(0.16844)	(0.13254)
	[0.27509]	[1.40569]
STORE_2(-3)	0.193769	0.170652
	(0.15385)	(0.12106)
	[1.25947]	[1.40961]
C	14.53336	0.637677
	(7.40339)	(5.82566)
	[1.96307]	[0.10946]
R-squared	0.464908	0.635357
Adj. R-squared	0.411398	0.598892
Sum sq. resids	10372.46	6422.612
S.E. equation	13.14817	10.34618
F-statistic	8.688358	17.42405
Log likelihood	-263.9831	-247.9256
Akaike AIC	8.089049	7.609720
Schwarz SC	8.319390	7.840061
Mean dependent	64.68657	51.35821
S.D. dependent	17.13777	16.33615

Determinant resid covariance (dof adj.)		18286.52
Determinant resid covariance		14665.07
Log likelihood		-511.5108
Akaike information criterion		15.68689
Schwarz criterion		16.14757

Table E-9 Estimation output for VAR(4) model

Vector Autoregression Estimates
Included observations: 66 after adjustments
Standard errors in ( ) & t-statistics in [ ]

	STORE_1	STORE_2
STORE_1(-1)	0.378990	0.196692
	(0.13311)	(0.10417)
	[2.84726]	[1.88821]
STORE_1(-2)	0.179600	0.020962
	(0.14540)	(0.11379)
	[1.23518]	[0.18422]
STORE_1(-3)	0.064513	0.046858
	(0.14095)	(0.11030)
	[0.45771]	[0.42481]
STORE_1(-4)	-0.007223	0.035344
	(0.13027)	(0.10195)
	[-0.05544]	[0.34669]
STORE_2(-1)	-0.008410	0.271182
	(0.17190)	(0.13453)
	[-0.04892]	[2.01577]
STORE_2(-2)	0.024395	0.144609
	(0.18051)	(0.14127)
	[0.13514]	[1.02365]
STORE_2(-3)	0.222434	0.149678
	(0.17532)	(0.13721)
	[1.26872]	[1.09090]
STORE_2(-4)	-0.042015	0.048415
	(0.16160)	(0.12647)
	[-0.25999]	[0.38282]
C	14.49037	-0.233303
	(7.79790)	(6.10258)
	[1.85824]	[-0.03823]
R-squared	0.463626	0.625185
Adj. R-squared	0.388345	0.572579
Sum sq. resids	10269.62	6289.631
S.E. equation	13.42269	10.50449
F-statistic	6.158634	11.88437
Log likelihood	-260.2105	-244.0310
Akaike AIC	8.157895	7.667607
Schwarz SC	8.456484	7.966197
Mean dependent	64.45455	50.92424
S.D. dependent	17.16273	16.06746

Determinant resid covariance (dof adj.)	19679.48
Determinant resid covariance	14678.29
Log likelihood	-503.9060
Akaike information criterion	15.81533
Schwarz criterion	16.41251

Table E-10 Estimation output for VAR(5) model

Vector Autoregression Estimates
Included observations: 65 after adjustments
Standard errors in ( ) & t-statistics in [ ]

	STORE_1	STORE_2
STORE_1(-1)	0.379964	0.200088
	(0.13531)	(0.10564)
	[2.80813]	[1.89410]
STORE_1(-2)	0.168152	0.010739
	(0.14906)	(0.11638)
	[1.12805]	[0.09228]
STORE_1(-3)	0.091508	0.061375
	(0.14914)	(0.11644)
	[0.61356]	[0.52711]
STORE_1(-4)	0.046423	0.071772
	(0.14374)	(0.11222)
	[0.32297]	[0.63958]
STORE_1(-5)	-0.127680	-0.098495
	(0.13244)	(0.10340)
	[-0.96404]	[-0.95256]
STORE_2(-1)	-0.016383	0.266265
	(0.17534)	(0.13689)
	[-0.09343]	[1.94505]
STORE_2(-2)	0.006157	0.130153
	(0.18517)	(0.14456)
	[0.03325]	[0.90032]
STORE_2(-3)	0.178897	0.116857
	(0.18483)	(0.14430)
	[0.96790]	[0.80982]
STORE_2(-4)	-0.069894	0.012538
	(0.18168)	(0.14184)
	[-0.38471]	[0.08840]
STORE_2(-5)	0.137805	0.132591
	(0.16392)	(0.12797)
	[0.84070]	[1.03609]
C	15.92216	0.827487
	(8.13136)	(6.34827)
	[1.95812]	[0.13035]
R-squared	0.470889	0.628992
Adj. R-squared	0.372906	0.560287
Sum sq. resids	9965.161	6073.910
S.E. equation	13.58455	10.60565
F-statistic	4.805799	9.154954
Log likelihood	-255.7861	-239.6955
Akaike AIC	8.208802	7.713709
Schwarz SC	8.576775	8.081682
Mean dependent	64.18462	50.61538
S.D. dependent	17.15453	15.99384

Determinant resid covariance (dof adj.)	20650.62
Determinant resid covariance	14252.59
Log likelihood	-495.3146
Akaike information criterion	15.91737
Schwarz criterion	16.65332

Table E-11 Estimation output for VAR(6) model

Vector Autoregression Estimates
Included observations: 64 after adjustments
Standard errors in ( ) & t-statistics in [ ]

	STORE_1	STORE_2
STORE_1(-1)	0.348526	0.206215
	(0.13758)	(0.10658)
	[2.53328]	[1.93480]
STORE_1(-2)	0.189581	-0.017538
	(0.15157)	(0.11742)
	[1.25076]	[-0.14936]
STORE_1(-3)	0.072722	0.056065
	(0.15156)	(0.11742)
	[0.47981]	[0.47749]
STORE_1(-4)	0.086394	0.115169
	(0.15152)	(0.11738)
	[0.57018]	[0.98114]
STORE_1(-5)	-0.070374	-0.075052
	(0.14539)	(0.11263)
	[-0.48404]	[-0.66635]
STORE_1(-6)	-0.107829	0.045783
	(0.13589)	(0.10528)
	[-0.79348]	[0.43488]
STORE_2(-1)	-0.061209	0.291026
	(0.17926)	(0.13887)
	[-0.34145]	[2.09562]
STORE_2(-2)	-0.000411	0.100172
	(0.18813)	(0.14574)
	[-0.00219]	[0.68733]
STORE_2(-3)	0.149770	0.103397
	(0.18779)	(0.14548)

	STORE_1	STORE_2
	[0.79753]	[0.71072]
STORE_2(-4)	-0.127375	-0.015137
	(0.18845)	(0.14599)
	[-0.67591]	[-0.10368]
STORE_2(-5)	0.079462	0.191209
	(0.18289)	(0.14168)
	[0.43449]	[1.34958]
STORE_2(-6)	0.212561	-0.107894
	(0.16727)	(0.12958)
	[1.27077]	[-0.83263]
C	17.26286	0.115015
	(8.46290)	(6.55618)
	[2.03983]	[0.01754]

	STORE_1	STORE_2
R-squared	0.479367	0.623956
Adj. R-squared	0.356865	0.535475
Sum sq. resids	9505.583	5704.819
S.E. equation	13.65225	10.57635
F-statistic	3.913143	7.051864
Log likelihood	-250.8361	-234.4979
Akaike AIC	8.244879	7.734311
Schwarz SC	8.683402	8.172834
Mean dependent	63.81250	50.07813
S.D. dependent	17.02368	15.51784

Determinant resid covariance (dof adj.)	20702.76
Determinant resid covariance	13146.45
Log likelihood	-485.1092
Akaike information criterion	15.97216
Schwarz criterion	16.84921

E.3 Data set II: Test result for VARMA(1,1) model

This section presents the result for VARMA(1,1) coefficient estimation from SAS software. The maximum likelihood estimation is used and the optimal result is shown in Table E-12. The model is acceptable but not perfectly fit. It is found that some MA parameters are not statistically significant. Four out of 48 schematic representations of the residuals lie outside the boundary of ± 2(S.E). In future study, the data needs to have more numbers of observations which may allow the model to be perfectly fit.

Table E-12 Results for VARMA model estimation and model diagnosis

The SAS System

The VARMAX Procedure

Optimization Results
Model Parameter Estimates

Equation	Parameter	Estimate	Standard Error	t Value	Pr > |t|	Variable
MAGOR	AR1_1_1	0.15025	0.00000			MAGOR(t-1)
	AR1_1_2	0.01040	0.00000			GREEN(t-1)
	AR1_1_3	0.56250	0.00000			WEY(t-1)
	AR1_1_4	0.40811	0.00000			MID(t-1)
	MA1_1_1	0.48890	0.26828	1.82	0.0743	e1(t-1)
	MA1_1_2	-1.11494	0.30219	-3.69	0.0005	e2(t-1)
	MA1_1_3	-0.12270	0.14844	-0.83	0.4123	e3(t-1)
	MA1_1_4	1.33459	0.31504	4.24	0.0001	e4(t-1)
GREEN	AR1_2_1	0.03241	0.00000			MAGOR(t-1)
	AR1_2_2	0.32996	0.00000			GREEN(t-1)
	AR1_2_3	0.15640	0.00000			WEY(t-1)
	AR1_2_4	0.54162	0.00000			MID(t-1)
	MA1_2_1	0.11087	0.00000			e1(t-1)
	MA1_2_2	0.19550	0.17394	1.12	0.2663	e2(t-1)
	MA1_2_3	0.18843	0.19978	0.94	0.3500	e3(t-1)
	MA1_2_4	0.43010	0.45490	0.95	0.3489	e4(t-1)
WEY	AR1_3_1	0.29309	0.00000			MAGOR(t-1)
	AR1_3_2	0.07438	0.00000			GREEN(t-1)
	AR1_3_3	0.17391	0.00000			WEY(t-1)
	AR1_3_4	0.35872	0.00000			MID(t-1)
	MA1_3_1	0.25830	0.10637	2.43	0.0187	e1(t-1)
	MA1_3_2	-0.19681	0.02580	-7.63	0.0001	e2(t-1)
	MA1_3_3	0.40194	0.00000			e3(t-1)
	MA1_3_4	-0.01624	0.26677	-0.06	0.9517	e4(t-1)
MID	AR1_4_1	0.04031	0.00000			MAGOR(t-1)
	AR1_4_2	0.10222	0.00000			GREEN(t-1)
	AR1_4_3	0.47695	0.00000			WEY(t-1)
	AR1_4_4	0.40510	0.00000			MID(t-1)
	MA1_4_1	0.13036	0.00000			e1(t-1)
	MA1_4_2	0.18032	0.00000			e2(t-1)
	MA1_4_3	0.53084	0.00000			e3(t-1)

Table E-12 Results for VARMA model estimation and model diagnosis (continued)

```
                    Information
                      Criteria

            AICC    45.14542
            HQC     45.37514
            AIC     44.91195
            SBC     46.12408
            FPEC     3.233E19
```

```
   Schematic Representation of Cross Correlations of Residuals

Variable/
Lag      0    1    2    3    4    5    6    7    8    9   10   11   12

MAGOR  ++++  .... .... .... ....  -... .-.. .... .... .... .... .... ....
GREEN  ++++  .... .... .... .... .... .... +... .... .... .... .... ....
WEY    ++++  .... .... .... .... .... .... .... .... .... .... .... ....
MID    ++++  .... .... .... .... .... .... +... .... .... .... .... ....

    + is > 2*std error,   - is < -2*std error,   . is between
```

```
              Univariate Model White Noise Diagnostics

              Durbin          Normality               ARCH
Variable      Watson     Chi-Square    Pr > ChiSq   F Value    Pr > F

MAGOR        1.81781       73.61        <.0001        0.24     0.6292
GREEN        1.90072        5.37        0.0682        0.53     0.4687
WEY          1.82813        0.20        0.9037        8.89     0.0045
MID          1.91755        9.36        0.0093        3.30     0.0756
```

Table E-13 AR roots for data set II

Roots of Characteristic Polynomial
Endogenous variables: Magor Welham_Green Weybridge
Middlewich

Root	Modulus
0.998208	0.998208
0.319233	0.319233
-0.251324	0.251324
-0.009116	0.009116

No root lies outside the unit circle.
VAR satisfies the stability condition.

References:

Lütkepohl, H. 1993. *Introduction to multiple time series analysis*. 2nd ed. Heidelberg, Germany: Springer-Verlag.