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The Myopic Order-Up-To Policy with a Proportional Feedback Controller

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Abstract

We develop a discrete control theory model of a myopic Order-Up-To (OUT) policy reacting to a stochastic demand pattern with Auto Regressive and Moving Average (ARMA) components. We show that the bullwhip effect arises with such a policy despite the fact that it is optimal when the ordering cost is linear. We then derive a set of z-transform transfer functions of a modified OUT policy that allows us to avoid the bullwhip problem by incorporating a proportional controller into the inventory position feedback loop. With this technique, the order variation can always be reduced to the same level as the demand variation. However, bullwhip-effect avoidance always comes at the cost of holding extra inventory. When the ordering cost is piece-wise linear and increasing, we compare the total cost per period under the two types of control policies: with and without bullwhip-effect reduction. Numerical examples reveal that the cost saving can be substantial if the order variance is reduced by using the proportional controller.

Keywords: Bullwhip effect, Inventory, Order-Up-To policy, Control theory

1. Introduction

The purpose of an ordering policy is to control production or distribution in such a way that supply is matched to demand, inventory levels are maintained within acceptable levels and capacity requirements are kept to a minimum. In doing so however, the bullwhip effect may arise (Lee, Padmanabhan and Whang, 1997a,b). The bullwhip effect is measured by the ratio of demand variance faced by the ordering system and the variance of replenishment orders that the system issues over time. The bullwhip effect refers to the phenomenon where the variance of the demand signal amplifies up along the supply chain. The bullwhip phenomenon and its effects on production and distribution have been popularized by playing the beer game (Sterman, 1989) and the use of the Barilla case (Hammond, 1994) throughout business schools

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worldwide. It has been estimated that the economic consequences of the bullwhip effect can be as much as 30% of factory gate profits (Metters, 1997). Lee, Padmanabhan and Whang, (1997b) have further summarised the negative impacts of bullwhip as follows;

- Excessive inventory investments throughout the supply chain to cope with the increased demand variability
- Reduced customer service due to the inertia of the production/distribution system
- Lost revenues due to shortages
- Reduced productivity of capital investment
- Increased investment in capacity
- Inefficient use of transport
- Production schedules missed more frequently

In this paper, we consider a production-inventory system with the periodic demand following the Auto Regressive and Moving Average process (ARMA, Box and Jenkins, 1970). Such a demand process is commonly seen in practice. For example, Disney, Farasyn, Lambrecht, Towill and van de Velde (2003) observed ARMA(1,1) demand processes at a house goods manufacturer; Lee, So and Tang (2000) reported that the AR(1) demand process was found to match the sales patterns of 150 SKU's in a supermarket. Gilbert (2205) considers a general class of ARMA models.

The stock levels are controlled by an Order-Up-To (OUT) policy. Using a simple control engineering principle, we are able to mitigate the bullwhip effect and reduce the total costs produced by the ordering policy. Specifically, we have modified the classical OUT policy using a linear proportional controller to eliminate the bullwhip problem for the general ARMA demand pattern. The OUT policy is a standard ordering algorithm in many MRP systems used to balance the customer service, inventory and capacity trade-off (Gilbert, 2005). This policy is often adopted by companies to coordinate orders for multiple items from the same supplier, where setup costs may be reasonably ignored. At least 60% of the sales value of two of the four largest UK grocery retailers is controlled by this policy. Furthermore, one of these grocery retailers has developed its own software system which one of the authors has modified to incorporate a proportional controller as described herein. This was piloted in 2001 and has since been rolled out across their entire UK business.

Conceptually, the OUT policy is very easy to understand; the system's inventory position (on-hand inventory + outstanding orders – backorders) is reviewed every period and an "**order**" is issued to bring the inventory position "**up-to**" a defined level. It has long been noted that the OUT policy combined with the conditional expectation forecasting mechanism minimises the total inventory related cost over time (Johnson and Thomson, 1975). Recently, it was also found that the OUT policy based on conditional expectation forecasting can avoid the bullwhip effect for certain instances of the ARMA demand pattern (Alwyn, 2001). However, for the general ARMA demand pattern this optimal forecasting technique still cannot avoid bullwhip effect in all instances, while our proposed method can.

To illustrate the advantage of the proposed bullwhip reduction method, we consider the case of a manufacturer whose the production cost is convex but piece-wise linear in the production quantity, and expected holding plus shortage cost is convex in inventory position. The piece-

wise linear cost captures the typical production mode: it costs less to produce within the regular capacity than to use overtime or outsourcing. Though in this case the optimal stock replenishment rule may no longer be the simple OUT policy (Karlin, (1960)), we use it because of its simplicity.

Our scientific contribution herein is to compare the total inventory and production costs between two cases: the simple OUT policy without bullwhip reduction and the modified OUT policy that incorporates a proportional controller to *reduce the bullwhip effect*. The downside of our proposed method is that the bullwhip reduction comes at the cost of holding extra inventory. However, via numerical examples, we demonstrate that our bullwhip reduction method results in lower total cost than the simple OUT policy *without bullwhip reduction,* where the total cost includes both inventory and production costs. Although we can not claim any of methodological steps herein are truly unique, we believe this is the first paper to explicitly integrate and investigate ARMA(1,1) demand processes, conditional expectation forecasting, proportional feedback controllers and the OUT policy. We establish closed form expressions for the variances and use these probability density functions to obtain exact solutions for the costs incurred.

The paper is organized as follows. Section 2 reviews the literature. Section 3 introduces the classical ARMA demand model, the OUT policy and the linear controller. Section 4 then derives expressions for bullwhip and the variance of the inventory levels. Section 5 presents numerical examples to illustrate the advantages of bullwhip reduction method. Section 6 concludes the paper. For ease of exposition, technical details and proofs are provided in the Appendix.

2. Literature Review

The use of transform techniques in production and inventory control was initiated by the Nobel Laureate, Herbert Simon in 1952. He treated time as continuous and exploited the Laplace transform. Vassian (1954) quickly replicated this approach in discrete time with the z-transform. However, Vassian sadly died before completing his research and his colleague, John Magee, published the work on his behalf. The first book that used z-transforms in a production and inventory control context appears to be Brown (1963). Magee (1956) incorporated two proportional controllers into a stationary OUT policy and, without giving details of its derivation, studies variance amplification. Deziel and Eilon (1967) used a variant of the OUT policy z-transforms and the "sum of the squares" technique to study variance amplification. They used a single controller in both the WIP (work-in-progress) and net stock feedback loops, as we do here, but their model considered a different order of events that resulted in an inventory drift problem.

Popplewell and Bonney (1987) developed a novel method based on the convolution of power-series representations of z-transforms to capture the impact of stochastic demands and the product structure (via the bill-of-materials) in MRP and re-order point systems. Their method has been implemented in computer software, which is designed to aid the user in the inversion of the z-transforms. This approach was further exploited in Bonney and Popplewell (1988) where the MRP and re-order point systems were investigated from other dynamical

perspectives. A useful feature of our approach herein is that we use the powerful concept of the transfer functions' "Inners", due to Jury (1974). This allows us to completely avoid the inversion of z-transforms, which can be a difficult task for the inexperienced analyst. Grubbström (1998) applied the z-transform to study MRP systems and the net present value of cash flows represented by transfer functions.

The use of a proportional controller to contain bullwhip was first exploited by Magee (1956). Sterman (1989) showed that human players of the Beer Game could be mimicked by an OUT model with proportional controllers based on a set of 2000 Beer Game results. John, Naim and Towill (1994) studied this scenario in continuous time with the Laplace transform. Disney, Farasyn, Lambrecht, Towill and van de Velde (2003) considered the customer service level implications of the proportional controller.

The OUT policy also contains a forecasting mechanism which influences the dynamics of the system. This appears to have been first noticed by Adelson (1966). Common forecasting techniques to exploit include moving average and exponential smoothing. Chen, Drezner, Ryan and Simchi-Levi (2000) studied this using a statistical technique; Dejonckheere, Disney, Lambrecht and Towill (2003a) used control theory. Kim and Ryan (2003) investigated the cost implications of using different forecasting mechanisms within the OUT policy.

There are numerous studies that place the OUT policy within different supply chain structures. For instances, Burns and Sivazlian (1978) addressed a four-echelon supply chain model using signal flow diagrams and z-transforms; Disney (2001) considered a VMI (Vendor Managed Inventory) scenario with simulation and z-transforms. Chen, Drezner, Ryan and Simchi-Levi (2000) studied the case of sharing end consumer demand with suppliers in a multi-stage supply chain, as did Dejonckheere, Disney, Lambrecht and Towill (2003b). Lee, So and Tang (2000) considered the case of AR(1) demand and different levels of information sharing. Interestingly, Hosoda and Disney (2005) showed that the AR(1) demand process is converted into the an ARMA(1,1) process as it passes through the Order-Up-To policy.

3. The Production / Inventory Model

The ARMA Demand Pattern

We have chosen the ARMA demand pattern for our analysis as it is mathematically tractable yet sufficiently general to represent real demand patterns. We have elected to use the mean centred ARMA demand pattern without loss of generality. It is commonly expressed as a difference equation (1) as follows:

$$\left.\begin{aligned} D_0^{ARMA} &= \varepsilon_t + \mu \\ D_t^{ARMA} &= \rho(\, D_{t-1}^{ARMA} - \mu\,) - \theta\varepsilon_{t-1} + \varepsilon_t + \mu \end{aligned}\right\} \qquad (1)$$

where, D_t = the ARMA demand at time, t; μ = unconditional mean of the ARMA demand sequence; ρ = the Auto Regressive constant; θ = the Moving Average constant; ε_t =

random shock. For ease of exposition, we assume ε_t to be a white noise process, that is, a normally distributed independently and identically distributed stochastic variable with zero mean and unit variance, $\sigma_\varepsilon^2=1$. Let $\hat{D}_t$ be the conditional forecast for period t based on information in the previous period (D_{t-1} and ε_{t-1}). We assume $\mu \geq 4\sigma_{ARMA}$ so that the probability of negative demand is negligible (see Johnson and Thompson (1975)). Note that the forecast error $\varepsilon_t = D_t - \hat{D}_t$, hence the variance of the one period ahead forecast error, σ_D^2, is obviously unity.

We may express (1) as a block diagram using standard techniques from discrete linear control theory as shown below in Figure 1. For a general introduction of control theory we refer readers to Nise (1995).

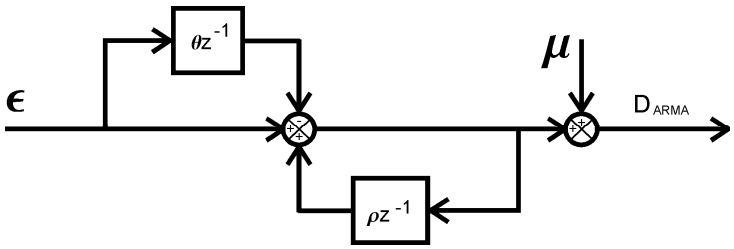

Figure 1. Block diagram of the ARMA demand generator

Re-arranging the block diagram, using common techniques, relinquishes the ARMA demand transfer function (2),

$$\frac{D_{ARMA}(z)}{\varepsilon(z)} = \frac{z-\theta}{z-\rho} \tag{2}$$

where z is the z-transform operator, $F(z) = \sum_{t=0}^{\infty} f(t) z^{-t}$. The variance of the ARMA demand is given by $\sigma_{ARMA}^2 = \dfrac{1+\theta^2-2\theta\rho}{1-\rho^2}$ (see the Appendix for more details).

The order-up-to policy with unit lead-time

The sequence of events in any period is; the inventory level is reviewed and ordering decision is made at the beginning of the period, then the demand is realized, the order placed earlier is received, and the demand is fulfilled at the end of the period. Thus, it effectively takes one period to receive the order placed. Unmet demand in a period is fully backordered. Two costs are considered at the end of the each period, inventory holding and stock-out. They are proportional functions with cost parameters, h and s, respectively. Piece-wise ordering costs will be considered later. In this section, we assume only a linear ordering cost. The objective is to minimize the long-run average total cost per period.

For such a problem, Johnson and Thompson (1975) have shown that the simple order-up-to (OUT) level policy is optimal. The OUT level is updated every period according to

$$S_t = \hat{D}_t + k\sigma_D \tag{3}$$

where $\hat{D}_t$ is an estimate of mean demand in period t, σ_D is the standard deviation of the forecast error, and k is the safety factor, $k=F^{-1}(s/(s+h))$, where F is the standard normal cumulative distribution and s and h are the inventory shortage and holding costs respectively. This is the so-called myopic OUT policy.

Denote by O_t the order quantity in period t, by NS_t the net inventory level at the beginning of period t. It is easy to see:

$$O_t = \hat{D}_t + k\sigma_D - NS_t . \tag{4}$$

Now we propose to make a modification to the classical OUT policy to provide more freedom in shaping its dynamic response. Our change is that we are going to use a proportional controller in the inventory position feedback loop. Specifically, we introduce a proportional controller, ($1/Ti$), as follows:

$$O_t = \hat{D}_t + \frac{1}{Ti}\left(k\sigma_D - NS_t\right). \tag{5}$$

The new policy will be called the ***modified myopic OUT policy,*** where (6) completes the definition,

$$NS_t = NS_{t-1} + O_{t-1} - D_{t-1} . \tag{6}$$

The feedback proportional controller, ($1/Ti$) that we have adopted here is a common control engineering technique for shaping the dynamic response of a system. There are many other such control techniques available, we refer readers to a good control engineering text like Nise (1995) for more information. However, the proportional controller is probably the most simple technique available. Indeed it also has a long history in production and inventory control, for example it was used by Magee (1956), Deziel and Eilon (1967), Towill (1982) and Matsuyama (1997) to name a few.

The conditional forecast of the demand in the current period (remember our sequence of events) is described by the following transfer function

$$\frac{\hat{D}(z)}{\varepsilon(z)} = \frac{\rho(z-\theta)}{(z-\rho)} - \theta . \tag{7}$$

From Eqs 2, 5–7 and our description we may now develop a block diagram of the ARMA demand and the modified OUT policy as shown in Figure 2. In the block diagram, $k\sigma_D$ is a constant.

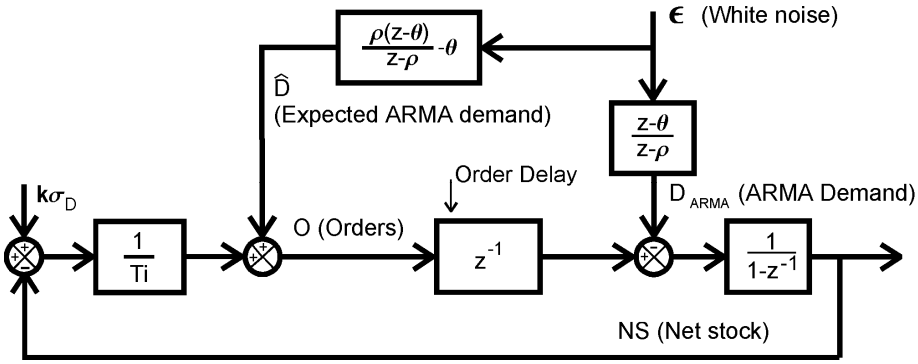

Figure 2. Block diagram of our modified OUT policy with ARMA demand and conditional forecasting

Rearranging Figure 2 for the transfer function that describes the relationship between orders and the white noise process that drives the ARMA demand we have:

$$\frac{O(z)}{\varepsilon(z)} = \frac{(Ti\theta - \theta - Ti\rho)z + (1 - Ti\theta + Ti\rho)z^2}{Ti\rho - \rho + (1 - Ti - Ti\rho)z + Tiz^2}. \tag{8}$$

4. Bullwhip Effect and Inventory Variance in the Modified Myopic OUT Policy

Appendix 1 details our derivation of the long-run variance of the ARMA demand and the replenishment orders. Note that the variance ratios hold regardless of the distribution of the error term, however later in Section 5 we will assume the error terms are normally distributed. Combining these variance expressions together surrenders the bullwhip ratio. It is given by

$$Bullwhip = \frac{\sigma_O^2}{\sigma_D^2} = 1 + \frac{2\left(Ti + \theta - 2Ti\theta + Ti^2(\rho - 1)\right)\left(\rho^2 - 1\right)}{(2Ti - 1)\left(Ti(\rho - 1) - \rho\right)\left(1 + \theta^2 - 2\theta\rho\right)}. \tag{9}$$

When Ti=1 (the classical OUT policy), the bullwhip equation reduces to

$$Bullwhip_{Ti=1} = 1 + \frac{2(\theta - \rho)\left(\rho^2 - 1\right)}{1 + \theta^2 - 2\theta\rho}, \tag{10}$$

which is plotted in Figure 3. From (10) it is easy to see that the classical myopic OUT policy produces unity bullwhip (i.e., no bullwhip) when $\theta = \rho$, and $\rho = \pm 1$. Figure 3 reveals that the classical myopic OUT policy is only able to reduce bullwhip when $\theta > \rho$. This can be confirmed from the fraction of (10), because in the region $-1 < \theta, \rho < 1$,

- the denominator is always positive,
- the last term of the numerator, $\left(\rho^2 - 1\right)$, is always negative,
- the second term of the numerator, ($\rho - \theta$), is negative if $\rho > \theta$.

This last observation means that if $\theta > \rho$, then the fraction becomes negative and hence

bullwhip is avoided. Symmetrically, if $\rho > \theta$, then the fraction is positive, indicating bullwhip is present.

Our modification to the OUT policy (1/*Ti*) however allows us to remove bullwhip for all instances of the ARMA demand pattern, as in the limit of $Ti \to \infty$, bullwhip$\to 0$ in (9). The case of *Ti*=5 in Figure 4 clearly shows the bullwhip reduction properties. The relevant root of the second term of the numerator of (9) determines the minimum *Ti* required to eliminate the bullwhip problem and is shown in (11).

$$TiMin = \frac{1 - 2\theta + \sqrt{1 + 4\theta(\theta - \rho)}}{2 - 2\rho} \tag{11}$$

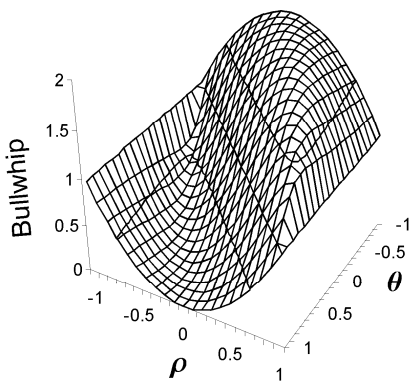

Figure 3. Bullwhip generated by the OUT policy with ARMA demands when Ti=1

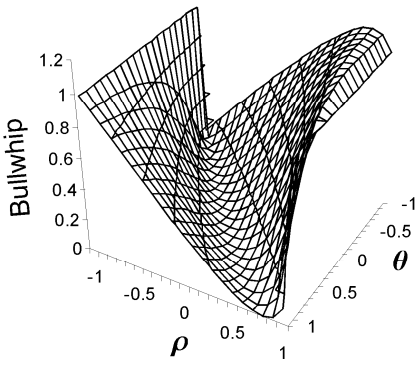

Figure 4. Bullwhip generated by the OUT policy with ARMA demands when Ti=5

Inventory Variance in the Modified Myopic OUT Policy

The transfer function of the inventory level can be found from the block diagram shown in Figure 2. Interestingly, the inventory variance is independent of the demand properties and only depends on the proportional controller, *Ti*, see (12).

$$\frac{NS(z)}{\varepsilon(z)} = \frac{Tiz}{Ti - Tiz - 1}. \tag{12}$$

The inventory variance is given by $\sigma_{NS}^2 = \dfrac{Ti^2}{2Ti - 1}$, which we have elaborated in the Appendix and plotted in Figure 5 below. Here, it is required that *Ti >0.5* for stability. Note that the inventory variance when *Ti=1* is unity, which is minimum for all of the class of ARMA demand patterns.

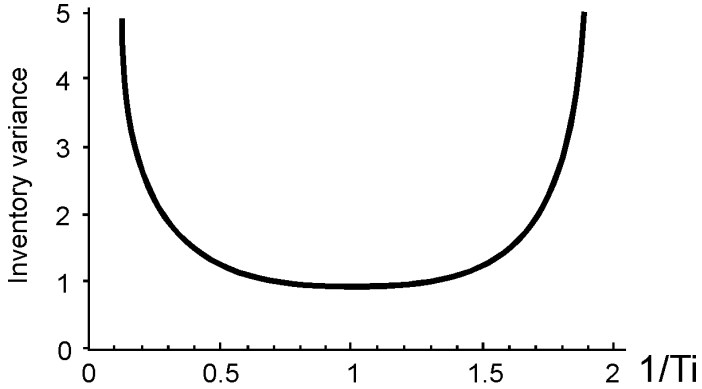

Figure 5. Inventory variance for the modified OUT policy reacting to ARMA demand

5. Expected Cost Per Period: A Comparison Between the Two Myopic OUT Policies

The cost structure that we are studying here is depicted in Figure 6. First consider the inventory related costs. We assume the cost of the warehouse is a fixed (or sunk) cost regardless of storage requirements. However, as it is a constant, we can ignore it in further analysis. The variable inventory holding cost is a linear function of a positive net stock level, and the variable inventory backlog cost is a linear function of a negative net stock level.

The production ordering costs consist of fixed capacity costs and linear material costs, so the fixed capacity costs can be ignored in further analysis. However, we do consider the labour costs (or subcontracting costs) to be piece-wise linear: within the normal production capacity K, the unit cost is c, while it is $c_0 > c$ if the production order size is greater than K. Think of this as incurring an overtime (or sub-contracting) premium. We assume this overtime capacity is practically infinite.

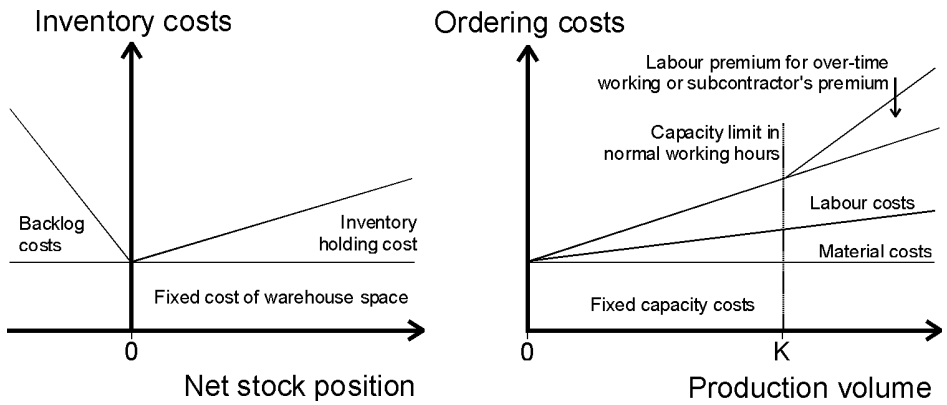

Figure 6. Graphical representation of our considered cost structure

We assume hereafter that the error terms are normally distributed. Thus, with our linear myopic OUT policy, the order quantity in a period is also a normal random variable. If the

order is greater than K, then ordering cost c_O is charged instead of c. Note that the costs c_O is only applied to those items produced in premium / overtime production in that period. We may find the expected amount of ordering costs by studying the probability density function of order levels over time. Holding and backlog costs are calculated similarly. The sum of these costs yields the total expected cost per period. As the probability density function of the normal distribution is essentially non-algebraic, analytic results are difficult to obtain. Hence, we will consider the following numerical scenario.

If one ignores the non-linearity of the purchasing cost, the optimal production/replenishment policy would be the myopic OUT policy, which can be determined from (3). This will serve as the benchmark for performance comparison. Now turn to our modified OUT policy. We set the inventory safety factor to achieve the economic stock-out probability using $k\sigma_D = g\mu$, where the safety factor, k, is determined by (3) and the order quantity, O_t, by (5).

The expected total inventory related costs per period are given by

$$E[InvCosts] = s\left(\int_{-\infty}^{0} \frac{e^{-\frac{(-(g\mu)+x)^2}{2\sigma_{NS}^2}}}{\sigma_{NS}\sqrt{2\pi}}(-x)dx\right) + h\left(\int_{0}^{\infty} \frac{e^{-\frac{(-(g\mu)+x)^2}{2\sigma_{NS}^2}}}{\sigma_{NS}\sqrt{2\pi}}(x)dx\right). \tag{13}$$

Minimising (13) with respect to g yields

$$g = \frac{\sqrt{2}\sqrt{\frac{Ti^2}{2Ti-1}}Erf^{-1}\left[\frac{s-h}{s+h}\right]}{\mu}. \tag{14}$$

(14) will minimise the inventory holding and backlog costs for a given Ti. The expected order related costs per period are given by

$$E[OrderCosts] = c_O\left(\int_{0}^{\infty} \frac{e^{-\frac{(K-\mu+x)^2}{2\sigma_O^2}}}{\sigma_O\sqrt{2\pi}}.xdx\right) + c\left(\mu - \int_{0}^{\infty} \frac{e^{-\frac{(K-\mu+x)^2}{2\sigma_O^2}}}{\sigma_O\sqrt{2\pi}}.xdx\right). \tag{15}$$

Numerical Examples

Consider the following scenario. The average ARMA demand, μ, is 5 units per period, the capacity limit, K, is 6 units per period. The cost to produce a unit in normal production, c, is \$100, and in overtime production the unit ordering (production) cost, c_O, is \$200. The inventory holding cost, h, is \$10 per unit per period and the backlog cost, s, is \$50 per unit per

period. With this data set, for the myopic OUT policy, the optimal safety stock gain g=0.193484.

We consider 25 ARMA demand patterns at combinations of $\theta, \rho = 0, \pm 0.475, \pm 0.95$. We note that all solutions when $\theta = \rho$ result in identical curves. This symmetry comes from the fact the ARMA demand is a stochastic i.i.d process under this condition.

Figure 7 reveals that our introduction to the myopic OUT policy, the proportional controller, *Ti*, is capable of reducing total expected costs per period, when compared to the classical myopic policy (when *Ti*=1). We note that the relationship is complex and sometimes, we should use a value of *Ti*>1 in order to minimise total expected costs and at other times we should use *Ti*<1. Our results are completely analytic and exact, although we have yet to fully understand the complete expected costs "solution space" as there are many dimensions to the problem. In general, the controller is much more effective when $\rho > 0$ and when ρ is near to -1. Furthermore, we can see that sometimes 0.5<*Ti*<1 will minimise total costs, but most of the time 1<*Ti*<∞ is required. 0.5<*Ti*<1 is required when demand is strongly negatively correlated. Intuitively, this is because under such as condition the ordering policy will over-react to deviations from the mean demand, in effect, gambling that the supply will match demand. When demand is negatively correlated, this is obviously better achieved with 0.5<*Ti*<1, which means that in some instances of the ARMA demand pattern, bullwhip is actually desirable.

It is interesting to highlight the economic impact of our modification. It is by no means insignificant. For our examples, average demand is 5 units per period. In a perfect world all products would be solely manufactured in normal capacity and there would be no inventory or backlog costs. Therefore we expect at least $500 of unavoidable costs per period. The avoidable costs (that is the over-capacity, inventory and backlog costs) for the classical myopic OUT policy (when *Ti*=1) and our modified OUT policy (where *Ti*>0.5) are shown in Table 1. We can see that we are always able to reduce some of the avoidable costs by "tuning" *Ti* in the ordering policy to the demand pattern. Of course, it means that inventory will have to be slightly increased, but clearly it allows better exploitation of the cost structure.

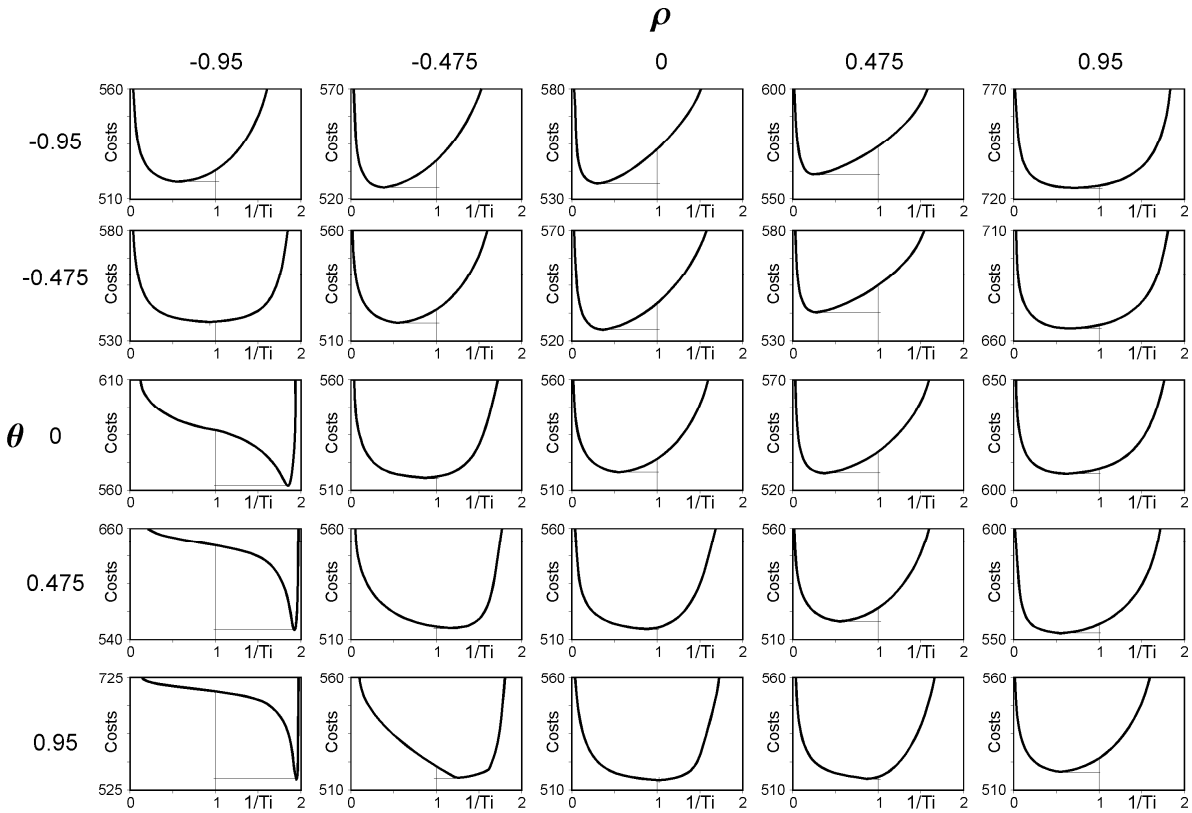

Figure 7. Expected total costs per period by the modified myopic OUT policy in response to ARMA demand

Table 1 highlights that our "tuned" modified OUT policy can always produce a cost saving. Here, in all policies, *g* was set to achieve the economic stockout probability. In the modified OUT policy *Ti* was set to minimise total costs. The savings produced by setting *Ti* to minimise costs is compared to the case of *Ti*=1. On the average, for our considered settings, we can see that the proportional controller, *Ti*, is able to

- reduce bullwhip by 40%,
- realise a economic saving of nearly 20% of avoidable costs.

6. Conclusions

Using the z-transform and the normal distribution probability density function, we have studied the OUT policy with conditional expectation forecasting reacting to the stochastic ARMA demand pattern. We have achieved this in an environment where there are piece-wise linear ordering costs and linear inventory costs. Exploiting the basic control engineering principles, we have made a slight modification to the ordering policy by adding a proportional controller in the inventory feedback loop, which allows us to better exploit the structure of our defined cost function. This OUT policy modified with our bullwhip effect reduction technique, *Ti*, outperforms the myopic OUT policy when the convex ordering cost is considered. In some cases, the savings can be quite substantial.

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ARMA constants		**Myopic OUT policy**		**Modified OUT policy**				**% reduction in**		
θ	ρ	**BW**	**AC**	***Ti***	***g***	**BW**	**AC**	***g***	**BW**	**AC**
$\theta = \rho$		1	23.323	1.757	0.214	0.397	18.128	-10.821	60.237	22.274
-0.95	-0.475	1.735	37.567	2.624	0.246	0.624	25.086	-27.325	64.021	33.223
	0	1.998	52.796	3.401	0.273	0.858	37.012	-41.2	57.043	29.896
	0.475	1.786	74.226	3.921	0.29	1.074	61.088	-49.91	39.848	17.699
	0.95	1.099	226.076	1.394	0.201	1.086	225.142	-4.262	1.23	0.4131
-0.475	-0.95	0.713	38.866	1.086	0.194	0.71	38.804	-0.321	0.44	0.159
	0	1.775	36.863	2.717	0.249	0.653	25.274	-29.037	63.195	31.438
	0.475	1.877	55.125	3.558	0.278	0.988	43.055	-43.883	47.367	21.895
	0.95	1.130	167.171	1.477	0.204	1.105	165.826	-5.664	2.247	0.804
0	-0.95	0.856	87.147	0.538	0.374	0.318	61.872	-93.433	62.802	29
	-0.475	0.264	16.03	1.152	0.195	0.218	15.738	-0.882	17.331	1.821
	0.475	1.735	37.567	2.801	0.252	0.772	27.868	-30.571	55.521	25.817
	0.95	1.185	109.769	1.612	0.209	1.126	107.611	-8.109	4.9947	1.9659
0.475	-0.95	0.869	143.339	0.519	0.509	0.057	50.751	-163.18	93.433	64.593
	-0.475	0.122	15.503	0.896	0.194	0.084	15.236	-0.679	30.741	1.722
	0	0.224	15.564	1.133	0.194	0.144	15.218	-0.703	35.915	2.223
	0.95	1.286	56.847	1.858	0.218	1.084	52.809	-12.755	15.738	7.103
0.95	-0.95	0.9	201.784	0.514	0.581	0.005	45.33	-200.48	99.353	77.535
	-0.475	0.213	20.583	0.776	0.202	0.049	15.782	-4.424	76.858	23.325
	0	0.001	14.991	1	0.193	0.001	14.991	trace		
	0.475	0.264	16.03	1.170	0.195	0.127	15.245	-1.083	51.72	4.897
Average>		**1.001**	**68.912**	**1.710**	**0.26**	**0.547**	**551.326**	**-34.701**	**41.907**	**18.943**

Table 1. Sample economic impact of the modification to the myopic OUT policy
(Key: BW =Bullwhip, AC=Avoidable Costs)

Appendix

Deriving the Variance Ratios

The link between a transfer function and the variance amplification ratio (VR) is described by (A1). The second term in (A1) refers to the statistical definition of VR in the time domain. Here, the VR is the long-term (that is, as $t \to \infty$) ratio of the variance of the output (for example, order rates or inventory levels) by the variance of the input (an i.i.d. white noise random process). The third term is a well-known relationship to control engineers that Dejonckheere, Disney, Lambrecht and Towill (2003a and b) have used in a recent bullwhip investigation. Here, the area under the spectral density curve (the squared frequency response) between frequency w=0 to π radians per period is equal to the variance ratio. The area under the squared frequency response is often called the Noise Bandwidth, W_N (forth term). The fifth term highlights the fact that the area under the time domain squared impulse response $f^2(n)$ is also equal to the VR (Tsypkin (1964) and Deziel and Eilon (1967)). Disney and Towill (2003) exploited this relationship. Finally, taking the contour integral clockwise around the unit circle in the complex plane to enclose the roots of $F(z)F(z^{-1})z^{-1}$ also yields the VR (Grubbström and Andersson (2002)). Grubbström and Andersson (2002) also show how the z-transform multiplication theorem may be used to determine how the VR develops over time. Herein we exploit this last method to determine the VR.

$$VR = \frac{\sigma^2_{Output}}{\sigma^2_{Input}} = \frac{1}{\pi}\int_0^{\pi}\left|F(\sqrt{-1}w)\right|^2 dw = \frac{W_N}{\pi}$$
$$= \sum_{n=0}^{\infty} f^2(n) = \frac{1}{2\pi\sqrt{-1}}\oint F(z)F(z^{-1})z^{-1}dz \qquad \text{(A1)}$$

The use of the contour integral is most appropriate here as the integral can be calculated using a very simple technique due to Åström, Jury and Agniel (1970) that was further refined by Jury (1974). Let's follow the approach of Jury (1974) to derive our variance expressions. We refer readers back to Jury (1974) for any required proof of his approach.

Let the following form

$$F(z) = \frac{B(z)}{A(z)} = \frac{\sum_{i=0}^{n} b_i z^i}{\sum_{i=0}^{n} a_i z^i} \qquad \text{(A2)}$$

describe the transfer function relating input to output of the VR that we require. The coefficients a_i and b_i obviously depend in the transfer function in question. Next construct two matrices, $\mathbf{X_{n+1}}$ and $\mathbf{Y_{n+1}}$, of the co-efficients of $A(z)$ as follows;

$$\mathbf{X_{n+1}} = \begin{bmatrix} a_n & a_{n-1} & a_{n-2} & .. & a_0 \\ 0 & a_n & a_{n-1} & .. & a_1 \\ 0 & 0 & a_n & .. & a_2 \\ : & : & : & : & : \\ 0 & 0 & 0 & 0 & a_n \end{bmatrix}, \qquad \mathbf{Y_{n+1}} = \begin{bmatrix} 0 & 0 & .. & 0 & a_0 \\ 0 & .. & & & : \\ : & 0 & a_0 & .. & a_{n-2} \\ 0 & a_0 & a_1 & .. & a_{n-1} \\ a_0 & a_1 & a_2 & .. & a_n \end{bmatrix}. \tag{A3}$$

Jury (1974) shows that $VR = \dfrac{\left|\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right|_b}{a_n\left|\mathbf{X_{n+1}} + \mathbf{Y}_{n+1}\right|}$, where $\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right]_b = \left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right]$ with the last row replaced by $\left[2b_n b_0, 2\sum b_i b_{i+n-1}, ..., 2\sum b_i b_{i+1}, 2\sum_{i=0}^{n} b_i^2\right]$. Thus, a simple algebraic process will construct a VR expression.

Derivation of the variance of the orders

The transfer function of the order rate was given in (8) from which the coefficients may be easily read as

$$\begin{aligned} b_0 &= 0 & & a_0 = Ti\rho - \rho \\ b_1 &= Ti\theta - \theta - Ti\rho & \text{and} \quad & a_1 = 1 - Ti - Ti\rho\,. \\ b_2 &= 1 - Ti\theta + Ti\rho & & a_2 = Ti \end{aligned} \tag{A9}$$

The required $\mathbf{X_{n+1}}$ and $\mathbf{Y_{n+1}}$ matrices are thus,

$$\mathbf{X_{n+1}} = \begin{bmatrix} a_2 & a_1 & a_0 \\ 0 & a_2 & a_1 \\ 0 & 0 & a_2 \end{bmatrix} = \begin{bmatrix} Ti & 1 - Ti - Ti\rho & Ti\rho - \rho \\ 0 & Ti & 1 - Ti - Ti\rho \\ 0 & 0 & Ti \end{bmatrix} \tag{A10}$$

$$\mathbf{Y_{n+1}} = \begin{bmatrix} 0 & 0 & a_0 \\ 0 & a_0 & a_1 \\ a_0 & a_1 & a_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & Ti\rho - \rho \\ 0 & Ti\rho - \rho & 1 - Ti - Ti\rho \\ Ti\rho - \rho & 1 - Ti - Ti\rho & Ti \end{bmatrix}, \tag{A11}$$

then,

$$\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right] = \begin{bmatrix} Ti & 1 - Ti - Ti\rho & 2\rho(Ti - 1) \\ 0 & Ti(1 + \rho) - \rho & 2(1 - Ti - Ti\rho) \\ \rho(Ti - 1) & 1 - Ti - Ti\rho & 2Ti \end{bmatrix} \tag{A12}$$

$$\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right]_b = \begin{bmatrix} Ti & 1 - Ti - Ti\rho & 2\rho(Ti - 1) \\ 0 & Ti(1 + \rho) - \rho & 2(1 - Ti - Ti\rho) \\ 0 & 2((Ti\theta - \theta - Ti\rho)(1 - Ti\theta + Ti\rho)) & 2\left((Ti\theta - \theta - Ti\rho)^2 + (1 - Ti\theta + Ti\rho)^2\right) \end{bmatrix}.$$

(A13)

The determinants of these matrices are;

$$\left|\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}\right| = 2(2Ti-1)(Ti(\rho-1)-\rho)(\rho^2-1) \tag{A14}$$

$$\left|\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}\right|_b = Ti\begin{pmatrix} 4\theta-4Ti^2(\theta-\rho)^2(\rho-1)-2(1+\theta^2)\rho+ \\ 2Ti(1-\theta(4+\theta)+3\rho+\theta(2+3\theta)\rho-2(1+\theta)\rho^2) \end{pmatrix}, \tag{A15}$$

which lead to the order variance expression:

$$\sigma_O^2 = \frac{\left|\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}\right|_b}{a_n\left|\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}\right|} = \frac{\begin{pmatrix} 2\theta-2Ti^2(\theta-\rho)^2(\rho-1)-(1+\theta^2)\rho+ \\ Ti(1-\theta(4+\theta)+3\rho+\theta(2+3\theta)\rho-2(1+\theta)\rho^2) \end{pmatrix}}{(2Ti-1)(Ti(\rho-1)-\rho)(\rho^2-1)}. \tag{A16}$$

Derivation of the variance of the ARMA demand

Consider the simple case of the variance of the ARMA demand. The transfer function is given by

$$\frac{D(z)}{\varepsilon(z)} = \frac{z-\theta}{z-\rho},$$

which has the following constant coefficients,

$$\begin{matrix} b_0 = -\theta \\ b_1 = 1 \end{matrix} \quad \text{and} \quad \begin{matrix} a_0 = -\rho \\ a_1 = 1 \end{matrix}. \tag{A4}$$

Arranging these coefficients into the $\mathbf{X_{n+1}}$ and $\mathbf{Y_{n+1}}$ matrices yields,

$$\mathbf{X_{n+1}} = \begin{bmatrix} a_1 & a_0 \\ 0 & a_1 \end{bmatrix} = \begin{bmatrix} 1 & -\rho \\ 0 & 1 \end{bmatrix}, \quad \mathbf{Y_{n+1}} = \begin{bmatrix} 0 & a_0 \\ a_0 & a_1 \end{bmatrix} = \begin{bmatrix} 0 & -\rho \\ -\rho & 1 \end{bmatrix}. \tag{A5}$$

Thus the $[\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}]$ and $[\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}]_b$ matrices are

$$[\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}] = \begin{bmatrix} 1 & -2\rho \\ -\rho & 2 \end{bmatrix}, \quad [\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}]_b = \begin{bmatrix} 1 & -2\rho \\ -2\theta & 2(\theta^2+1) \end{bmatrix}. \tag{A6}$$

The determinants of these two matrices are

$$\left|\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}\right| = 2(1-\rho^2), \quad \left|\mathbf{X_{n+1}}+\mathbf{Y_{n+1}}\right| = 2(\theta^2+1)-4\theta\rho. \tag{A7}$$

Assuming that the variance of the random shock is unity, we may determine the variance of the ARMA demand as,

$$\sigma_{ARMA}^2 = \frac{\left|\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right|_b}{a_n\left|\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right|} = \frac{2\left(1+\theta^2 - 2\theta\rho\right)}{2\left(1-\rho^2\right)} = \frac{1+\theta^2 - 2\theta\rho}{1-\rho^2}. \tag{A8}$$

Derivation of the variance of the inventory levels

Now we turn our attention to the long-run variance of the inventory levels. The transfer function is given in (12). The coefficients of this transfer function are

$$\begin{matrix} b_0 = 0 \\ b_1 = Ti \end{matrix} \quad \text{and} \quad \begin{matrix} a_0 = Ti - 1 \\ a_1 = -Ti \end{matrix}. \tag{A17}$$

The $\mathbf{X_{n+1}}$ and $\mathbf{Y_{n+1}}$ matrices then become,

$$\mathbf{X_{n+1}} = \begin{bmatrix} a_1 & a_0 \\ 0 & a_1 \end{bmatrix} = \begin{bmatrix} -Ti & Ti-1 \\ 0 & -Ti \end{bmatrix}, \quad \mathbf{Y_{n+1}} = \begin{bmatrix} 0 & a_0 \\ a_0 & a_1 \end{bmatrix} = \begin{bmatrix} 0 & Ti-1 \\ Ti-1 & -Ti \end{bmatrix}. \tag{A18}$$

The corresponding $\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right]$ and $\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right]_b$ matrices are thus

$$\left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right] = \begin{bmatrix} -Ti & 2(Ti-1) \\ Ti-1 & -2Ti \end{bmatrix}, \quad \left[\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right]_b = \begin{bmatrix} -Ti & 2(Ti-1) \\ 0 & 2Ti^2 \end{bmatrix} \tag{A19}$$

and their determinants are

$$\left|\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right| = 2Ti^2 - 2(Ti-1)^2, \quad \left|\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right| = -2Ti^3. \tag{A20}$$

Therefore, the variance expression is

$$\sigma_{NS}^2 = \frac{\left|\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right|_b}{a_n\left|\mathbf{X_{n+1}} + \mathbf{Y_{n+1}}\right|} = \frac{-2Ti^3}{-2Ti\left(Ti^2 - (Ti-1)^2\right)} = \frac{Ti^2}{2Ti-1}. \tag{A21}$$

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