ON INEQUALITIES OF HARDY–SOBOLEV TYPE

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Abstract. Hardy–Sobolev–type inequalities associated with the operator $L := x \cdot \nabla$ are established, using an improvement to the Sobolev embedding theorem obtained by M. Ledoux. The analysis involves the determination of the operator semigroup $\{e^{-tL^*L}\}_{t>0}$.

1. Introduction

The following inequalities of Hardy and Sobolev are well-known to play a fundamental role in Analysis:

Hardy’s inequality

$$\int_{\mathbb{R}^n} |\nabla f|^p dx \geq C_H(n, p) \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^p} dx, \quad f \in C_0^\infty(\mathbb{R}^n \setminus \{0\}),$$

with best possible constant $C_H(n, p) = \{(n - p)/p\}^p$;

Sobolev’s inequality for $1 \leq p < n$ and $p^* := np/(n - p)$,

$$\|f\|_{L^{p^*}(\mathbb{R}^n)} \leq C_S(n, p) \|\nabla f\|_{L^p(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n),$$

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with best possible constant
\[ C_S(n, p) = \pi^{-1/2} n^{-1/p} \left( \frac{p-1}{n-p} \right)^{(p-1)/p} \left\{ \frac{\Gamma(1+n/2)\Gamma(n)}{\Gamma(n/p)\Gamma(1+n-n/p)} \right\}^{1/n}, \]
for \(1 < p < n\), and
\[ C_S(n, 1) = \pi^{-1/2} n^{-1} (\Gamma(1+n/2))^{1/n}. \]

From (1.1) and (1.2) it follows that for \(0 < \delta < C_H(n, p)\), \(1 \leq p < n\),
\[
\| \nabla f \|_{L^p(\mathbb{R}^n)} - \delta \| f / | \|_{L^p(\mathbb{R}^n)}^p \\
\geq \{1 - \delta/C_H(n, p)\} \| \nabla f \|_{L^p(\mathbb{R}^n)}^p \\
\geq \left[ (1 - \delta/C_H(n, p)) / C_S(n, p) \right] \| f \|_{L^p(\mathbb{R}^n)}^p,
\]
and so
\[
\| f \|_{L^p(\mathbb{R}^n)} \leq C \left\{ \| \nabla f \|_{L^p(\mathbb{R}^n)} - \delta \| f / | \|_{L^p(\mathbb{R}^n)}^p \right\}, \tag{1.3}
\]
where \(C \geq C_S(n, p)\{1 - \delta/C_H(n, p)\}^{-1}\). In the case \(p = 2\), Stubbe \[8\] shows that the optimal value of the constant \(C\) is
\[ C_S^2(n, 2)[1 - \delta/C_H(n, 2)]^{-(n-1)/n}. \]

In Theorem 1 below we prove the inequality
\[
\int_{\mathbb{R}^n} |(x \cdot \nabla) f(x)|^p dx \geq (n/p)^p \int_{\mathbb{R}^n} |f(x)|^p dx, \quad f \in C_0^\infty(\mathbb{R}^n), \tag{1.4}
\]
which is satisfied (and non-trivial) for all values of \(n\), including \(n = p\), and show that this implies Hardy’s inequality for \(1 \leq p \leq n\). The above argument leading to (1.3) does not work with the right-hand side \(\| \nabla f \|_{L^p(\mathbb{R}^n)}^p - \delta \| f / | \|_{L^p(\mathbb{R}^n)}^p\) replaced by \(\| (x \cdot \nabla) f \|_{L^p(\mathbb{R}^n)} - \delta \| f \|_{L^p(\mathbb{R}^n)}^p\) since, by scaling considerations, we don’t have a Sobolev–type inequality
\[ \| f \|_{L^q(\mathbb{R}^n)} \leq C \| (x \cdot \nabla) f \|_{L^p(\mathbb{R}^n)} \]
for \(q \neq p\). It is natural to ask if there is some analogue of Stubbe’s inequality, and indeed of the \(L^p\) version (1.3), when \(\| \nabla f \|\) is replaced by \(\| (x \cdot \nabla) f \|\). This was the question which initiated this research. Our investigation makes use of the following result of Ledoux in [7] which, \textit{inter alia}, improves on the standard Sobolev inequality: for every \(1 \leq p < q < \infty\) and every function \(f\) in the Sobolev space \(W^{1,p}(\mathbb{R}^n)\),
\[
\| f \|_{L^q(\mathbb{R}^n)} \leq C \| \nabla f \|_{L^p(\mathbb{R}^n)} \| f \|_{B^\omega_{\infty,\infty}(\mathbb{R}^n)}^{1-\theta}, \tag{1.5}
\]
where \(\theta = p/q\). \(C\) is a positive constant which depends only on \(p, q\) and \(n\), and \(B^\omega_{\infty,\infty}\) is the homogenous Besov space of indices \((\alpha, \infty, \infty)\); see [9]. The latter is the space of tempered distributions for which the norm
\[ \| f \|_{B^\omega_{\infty,\infty}} := \sup_{t > 0} \left\{ t^{-\alpha/2} \| P_t f \|_{L^\infty(\mathbb{R}^n)} \right\} \]
is finite, where \( P_t = e^{t\Delta}, t \geq 0 \), is the heat semigroup on \( \mathbb{R}^n \): recall that \( \{P_t\}_{t \geq 0} \) is defined by \( P_t f = f \) and
\[
P_t f(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} f(y) e^{-|x-y|^2/4t} \, dy
\]
for \( t > 0, x \in \mathbb{R}^n \). Cases of (1.5) were earlier established in [2], [3] and [4]. The inequality (1.5) is easily seen to include the classical Sobolev inequality (1.2). Ledoux’s technique requires specific information on the heat semi-group \( e^{t\Delta} \) in \( L^2(\mathbb{R}^n) \). Our first task therefore was to determine the operator semi-group associated with the inequality (1.4), namely \( e^{-tL^*L} \), where \( L = x \cdot \nabla \). This is done in section 3. We show that the analogue of (1.5) is in fact a consequence of Ledoux’s result. Corollaries of this analogue in the case \( p = 2 \), contain the following inequalities:
\[
\|rf(r\omega)\|_{L^{2^*}(\mathbb{R}^n)}^2 \leq C \left\{ \|Lf\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \|f\|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \sup_{\omega \in \mathbb{S}^{n-1}} \|f\|_{L^2(\mathbb{R}^n)}^{2(1-1/n)}
\]
(1.6)
\[
\|RF(r)\|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left\{ \|Lf\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \|f\|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \|f\|_{L^2(\mathbb{R}^n)}^{2(1-1/n)}
\]
where \( 2^* = 2n/(n-2), d\mu(r) = r^{n-1}dr, C \) is a positive constant depending only on \( n \) and, in polar co-ordinates \( x = r\omega, F(r) \) is the integral mean of \( f \) over the unit sphere \( \mathbb{S}^{n-1} \), that is,
\[
F(r) := \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} f(r\omega) \, d\omega.
\]
These have a number of consequences. One is a Hardy–Sobolev type inequality (Corollary 4) which is an analogue of the type we set out to establish of Stubbe’s inequality: that if \( f, Lf \in L^2(\mathbb{R}^n), n \geq 3 \), then, for \( \delta \in [0, n^2/4) \),
\[
\|rf\|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left[ \frac{n^2}{4} - \delta \right]^{-\frac{(n-1)}{n}} \left\{ \|Lf\|_{L^2(\mathbb{R}^n)}^2 - \delta \|f\|_{L^2(\mathbb{R}^n)}^2 \right\}.
\]
(1.7)
It also follows from (1.6) that, for \( \delta \in [0, (n-2)^2/4) \),
\[
\|F\|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left[ \frac{(n-2)^2}{4} - \delta \right]^{-\frac{(n-1)}{n}} \left\{ \|\nabla f\|_{L^2(\mathbb{R}^n)}^2 - \delta \|f\| \|\cdot\|_{L^2(\mathbb{R}^n)}^2 \right\}.
\]
Since \( \|F\|_{L^{2^*}(\mathbb{R}^n; d\mu)} \leq |\mathbb{S}^{n-1}|^{-1/2^*} \|f\|_{L^{2^*}(\mathbb{R}^n)}, \) by Hölder’s inequality, (1.7) is implied by the case \( p = 2 \) of (1.3).

We also establish the following local Hardy–Sobolev type inequalities (see Corollaries 6 and 7): if \( f \) is supported in the annulus \( A_R := \{ x \in \mathbb{R}^n : 1/R \leq |x| \leq R \} \), then
\[
\|rf(r)\|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C(\ln R)^{2(n-1)/n} \left\{ \|Lf\|_{L^2(\mathbb{R}^n)}^2 - (n^2/4) \|f\|_{L^2(\mathbb{R}^n)}^2 \right\};
\]
\[ \|F\|_{L^{2^*}(\mathbb{R}^n;\, d\mu)}^2 \leq C(\ln R)^{2(n-1)/n} \left\{ \|\nabla f\|_{L^2(\mathbb{R}^n)}^2 - \left[ \frac{n-2}{2} \right]^2 \left\| \frac{f}{|x|} \right\|_{L^2(\mathbb{R}^n)}^2 \right\} . \]  

The inequality (1.8) is reminiscent of the case \( s = 1 \) of (2.6) in [6] (proved in section 6.4); this is also proved in [1]. To be specific, it is that if \( f \in C_0^{\infty}(\Omega) \) and \( 2 \leq q < 2^* \), 
\[ \|f\|_{L^q(\mathbb{R}^n)}^2 \leq C|\Omega|^{2(1/q-1/2^*)} \left\{ \|\nabla f\|_{L^2(\mathbb{R}^n)}^2 - \left[ \frac{n-2}{2} \right]^2 \left\| \frac{f}{|x|} \right\|_{L^2(\mathbb{R}^n)}^2 \right\} , \]  

where \( |\Omega| \) denotes the volume of \( \Omega \). It is noted in [6], Remark 2.4, that, in contrast to (1.8), the \( q \) in (1.9) must be strictly less than the critical Sobolev exponent \( 2^* = 2n/(n-2) \) if \( \Omega \) includes the origin.

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2. The Hardy-type inequality (1.4)

**Theorem 2.1.** Let \( n \geq 1 \) and \( 1 \leq p < \infty \). Then for all \( f \in C_0^{\infty}(\mathbb{R}^n) \)
\[ \int_{\mathbb{R}^n} |(x \cdot \nabla)f|^p \, dx \geq \left( \frac{n}{p} \right)^p \int_{\mathbb{R}^n} |f|^p \, dx . \]  

**Proof.** On integration by parts and the application of Hölder’s inequality we have
\[ n \int_{\mathbb{R}^n} |f(x)|^p \, dx = \int_{\mathbb{R}^n} \text{div}(x)|f(x)|^p \, dx \]
\[ = -p \text{Re} \int_{\mathbb{R}^n} (x \cdot \nabla)f(x)|f(x)|^{p-2}\overline{f}(x) \, dx \]
\[ \leq p \left( \int_{\mathbb{R}^n} |(x \cdot \nabla)f(x)|^p \, dx \right)^{1/p} \left( \int_{\mathbb{R}^n} |f(x)|^p \, dx \right)^{(p-1)/p} \]
which yields (2.1). \[ \square \]

**Remark 2.2.** The inequality (2.1) implies (1.1) for \( 1 \leq p \leq n \). For we have from
\[ \nabla(|x|f) = \frac{x}{|x|}f + |x|\nabla f \]
that
\[ \|\nabla(|x|f)\|_{L^p(\mathbb{R}^n)} \geq \|\nabla f\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)} \]
\[ \geq \|(x \cdot \nabla)f\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)} \]
\[ \geq \left( \frac{n-p}{p} \right) \|f\|_{L^p(\mathbb{R}^n)} \]
whence (1.1) on replacing \( f(x) \) by \( f(x)/|x| \).
3. Calculation of the Semigroup \( e^{-tL^*L} \)

**Theorem 3.1.** Let \( L = x \cdot \nabla, x = r\omega, r = |x| \). Then the semigroup \( e^{-tL^*L} \) is given by

\[
(e^{-tL^*L}\psi)(x) = \frac{e^{-tn^2/4}}{\sqrt{4\pi t}} r^{-n/2} \int_0^\infty e^{-\frac{(\ln r - \ln s)^2}{4t}} s^{-n/2} \psi(s\omega) s^{n-1} ds.
\] (3.1)

**Proof.** Before embarking on the proof, some preliminary remarks and results might be helpful. The gist of the proof is that after a change of co-ordinates, \( L^*L \) is seen to be related to the Laplacian in \( \mathbb{R} \), and this then yields the result.

The co-ordinate change is determined by the map \( \Phi : L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times S^{n-1}) \) defined by

\[
(\Phi \psi)(s, \omega) := e^{tn/2} \psi(e^s \omega)
\] (3.2)

for \( \omega \in S^{n-1} \) and \( s \in \mathbb{R} \). Note that we equip \( \mathbb{R} \times S^{n-1} \) with the usual one dimensional Lebesgue measure on \( \mathbb{R} \) and the usual surface measure on \( S^{n-1} \). Thus \( \Phi \) preserves the \( L^2 \) norm. The inverse of \( \Phi \) satisfies \( \Phi^{-1} : L^2(\mathbb{R} \times S^{n-1}) \to L^2(\mathbb{R}^n) \) and is given by

\[
(\Phi^{-1} \varphi)(x) = r^{-n/2} \varphi(\ln r, \omega).
\] (3.3)

The dilations \( U(t) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \) given by

\[
U(t)\psi(x) := e^{tn/2} \psi(e^t x)
\]

form a group of unitary operators with generator \( U(t) = e^{itA} \), where \( A \) is given by

\[
iA\psi = \frac{\partial}{\partial t} U(t)\psi \big|_{t=0} = (x \cdot \nabla + \frac{n}{2})\psi = \frac{1}{2}(x \cdot \nabla + \nabla \cdot x)\psi.
\]

Thus

\[
A = \frac{1}{i}(x \cdot \nabla + \frac{n}{2}) = -iL - \frac{n}{2}.
\]

and so

\[
L = iA - \frac{n}{2},
\]

where \( A \) is the self-adjoint generator of dilations in \( L^2(\mathbb{R}^n) \). In particular,

\[
L^*L = (-iA - \frac{n}{2})(iA - \frac{n}{2}) = A^2 + \frac{n^2}{4}.
\]

Since

\[
(\Phi \psi)(s, \omega) = (U(s)\psi)(\omega)
\]

for \( \omega \in S^{n-1} \) and \( s \in \mathbb{R} \), it follows from the group property of the dilations \( U(\cdot) \) that

\[
(\Phi(U(t)\psi))(s, \omega) = (U(s)(U(t)\psi))(\omega) = (U(s + t)\psi)(\omega) = (\Phi \psi)(s + t, \omega).
\]

In particular, in the new co-ordinates given by \( \Phi \), the dilations \( U(t) \) act simply as shifts by \( t \) and should be diagonalizable with the help of a Fourier transform! We now proceed to confirm this prediction.
Define $M : L^2(\mathbb{R}^n) \to L^2(\mathbb{R} \times S^{n-1})$ by
\[
(M \psi)(\tau, \omega) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} (\Phi \psi)(s, \omega) ds,
\] (3.4)
so that $M = \mathcal{F} \circ \Phi$, where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}$. Then
\[
(MU(t) \psi)(\tau, \omega) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ist} (\Phi \psi)(s + t, \omega) ds = e^{it\tau} (M \psi)(\tau, \omega).
\] (3.5)
The map $M = \mathcal{F} \circ \Phi$ is the Mellin transformation and has an explicit representation using the group structure of $\mathbb{R}^+$ under multiplication: it is the Fourier transform on this group.

The next step is to show that
\[
(MA \psi)(\tau, \omega) = \tau (M \psi)(\tau, \omega)
\] (3.6) for $\psi$ in the domain $\mathcal{D}(A)$: it follows that $\psi \in \mathcal{D}(A)$ if and only if $(\tau, \omega) \mapsto \tau (M \psi)(\tau, \omega) \in L^2(\mathbb{R} \times S^{n-1})$. To see (3.6) we note that $iAe^{itA} = \partial_t U(t)$ and so, from (3.5)
\[
(MiAe^{itA} \psi)(\tau, \omega) = (M\partial_t U(t) \psi)(\tau, \omega) = \partial_t (MU(t) \psi)(\tau, \omega)
\]
\[
= \partial_t e^{it\tau} (M \psi)(\tau, \omega) = i\tau e^{it\tau} (M \psi)(\tau, \omega).
\]
Setting $t = 0$ yields (3.6).

We are now in a position to complete the proof of the theorem. We have $e^{-tL^*L} = e^{-tn^2/4}e^{-tA^2}$ and by (3.4)
\[
(Me^{-tA^2} \psi)(\tau, \omega) = e^{-t\tau^2} (M \psi)(\tau, \omega).
\]
So
\[
e^{-tA^2} = M^{-1}e^{-t\tau^2} M.
\]
Since $M = \mathcal{F} \circ \Phi$, we see that
\[
e^{-tA^2} = \Phi^{-1} \circ \mathcal{F}^{-1} (e^{-t\tau^2} \mathcal{F} \circ \Phi).
\]
Of course,
\[
\mathcal{F}^{-1} (e^{-t\tau^2} M \psi)(\lambda, \omega) = \mathcal{F}^{-1} (e^{-t\tau^2} \mathcal{F} \circ \Phi)(\lambda, \omega)
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\lambda \tau} e^{-t\tau^2} e^{-ist} (\Phi \psi)(s, \omega) ds d\tau
\]
\[
= \frac{1}{2\pi} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-\tau^2 + i(\lambda-s)\tau} d\tau \right) (\Phi \psi)(s, \omega) ds
\]
The integral in big parentheses is a Gaussian integral which gives
\[
\int_{\mathbb{R}} e^{-\tau^2 + i(\lambda-s)\tau} d\tau = \sqrt{\frac{\pi}{i}} e^{-\frac{(\lambda-s)^2}{4t}}.
\]
Thus
\[ F^{-1}(e^{-t\sigma^2}M\psi)(\lambda, \omega) = \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(\lambda-s)^2}{4t}}(\Phi\psi)(s, \omega) \, ds = \varphi_t(\lambda, \omega) \]
and, with \( x = r\omega \),
\[ (e^{-tA^2}\psi)(r\omega) = (\Phi^{-1}\varphi_t)(r\omega) = \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(\ln r-s)^2}{4t}}(\Phi\psi)(s, \omega) \, ds. \]
Since \( (\Phi\psi)(s, \omega) = e^{sn/2}\psi(e^s\omega) \), we get from the change of variables \( z = e^s \),
\[ (e^{-tA^2}\psi)(r\omega) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(\ln r-ln z)^2}{4t}}z^{n/2}^{-1}\psi(z\omega) \, dz. \]
So
\[ (e^{-tL^*L}\psi)(r\omega) = e^{-tn^2/4}(e^{-tA^2}\psi)(r\omega) = \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(\ln r-ln z)^2}{4t}}z^{n/2}^{-1}\psi(z\omega) \, dz \]
which is (3.1).

Once it is realised that \( A \) is simply multiplication by \( \tau \) in the sense of (3.6), it is clear that \( A \) is the momentum operator on \( \mathbb{R} \), that is, \( \Phi A \Phi^{-1} \) is given by
\[ \Phi A \Phi^{-1} = -i\partial_s \otimes 1_{6n-1}. \]
On using this and the functional calculus we get
\[ \Phi L^*L\Phi^{-1} = (\Phi A \Phi^{-1})^2 + \frac{n^2}{4} = -\partial_s^2 \otimes 1_{6n-1} + \frac{n^2}{4}. \]
Thus, \( L^*L = -\Phi^{-1}\partial_s^2 \otimes 1_{6n-1} \Phi + \frac{n^2}{4} \) and
\[ e^{-tL^*L} = e^{-tn^2/4}e^{-t\Phi^{-1}\partial_s^2\otimes 1_{6n-1}\Phi} = e^{-tn^2/4}\Phi^{-1}e^{-t\partial_s^2\otimes 1_{6n-1}\Phi} \]
which is a convenient way of expressing (3.1). \( \square \)

On substituting (3.2) and (3.3) and making an obvious change of variables, we obtain from (3.1) the following representation for \( e^{-tA^2} \); see also (3.7).

**Corollary 3.2.** Let \( P_t \) denote \( e^{-tA^2} \). Then
\[ \Phi P_t\Phi^{-1}\varphi(r, \omega) = \frac{1}{\sqrt{4\pi t}} \int_\mathbb{R} \exp\{-\frac{1}{4t}(r-s)^2\}\varphi(s\omega) \, ds. \]
4. The main inequalities

The fact that \( \Phi e^{-A^2} \Phi^{-1} \) in (4.8) is essentially radial means that the analogue of (1.5) derived by Ledoux’s technique is a consequence of the one-dimensional case of (1.5). Defining \( B^\alpha \) to be the space of all tempered distributions \( g \) on \( \mathbb{R} \times \mathbb{S}^{n-1} \) for which the norm

\[
\|g\|_{B^\alpha} := \sup_{t>0} \left\{ t^{-\alpha/2} \|\Phi e^{-A^2} \Phi^{-1} g\|_{L^\infty(\mathbb{R} \times \mathbb{S}^{n-1})} \right\} < \infty,
\]

one obtains from the \( n = 1 \) case of (1.5), that for any \( \omega \in \mathbb{S}^{n-1} \),

\[
\int_{\mathbb{R}} |g(r, \omega)|^q dr \leq C_q \int_{\mathbb{R}} \left| \frac{\partial g(r, \omega)}{\partial r} \right|^p dr \times \left( \sup_{t>0, r \in \mathbb{R}} t^{\theta/2(1-\theta)} \left| \frac{1}{\sqrt{4\pi t}} \int e^{-\frac{(r-s)^2}{4t}} g(s, \omega) ds \right| \right)^{(1-\theta)}
\]

\[= C_q \int_{\mathbb{R}} \left| \frac{\partial g(r, \omega)}{\partial r} \right|^p dr \times \left( \sup_{t>0, r \in \mathbb{R}} t^{\theta/2(1-\theta)} \left| \Phi e^{-A^2} \Phi^{-1} g(r, \omega) \right| \right)^{(1-\theta)}
\]

\[\leq C_q \int_{\mathbb{R}} \left| \frac{\partial g(r, \omega)}{\partial r} \right|^p dr \|g\|_{B^{\theta/(\theta-1)}}^{(1-\theta)}.
\]

On integrating with respect to \( \omega \) over \( \mathbb{S}^{n-1} \) we obtain

**Theorem 4.1.** Let \( 1 \leq p < q < \infty \) and suppose that \( g \) is such that \( \Phi A \Phi^{-1} g \equiv -i(\partial/\partial r)g, L^p(\mathbb{R} \times \mathbb{S}^{n-1}) \) and \( g \in B^{\theta/(\theta-1)}, \theta = p/q \). Then there exists a positive constant \( C \), depending on \( p \) and \( q \), such that

\[
\|g\|_{L^q(\mathbb{R} \times \mathbb{S}^{n-1})} \leq C \|\Phi A \Phi^{-1} g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}^{1-\theta} \|g\|_{B^{\theta/(\theta-1)}}^\theta.
\]

(4.2)

The theorem has two natural corollaries featuring the Hardy-type inequality (2.1), the first an inequality of Sobolev type, and the second of Gagliardo-Nirenberg type.

**Corollary 4.2.** (i) Let \( p^* := np/(n-p) \), \( 1 \leq p \leq n-1 \), and suppose \( (\partial/\partial r)g \in L^p(\mathbb{R} \times \mathbb{S}^{n-1}) \) and \( \sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^p(\mathbb{R})} < \infty \). Then

\[
\|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \leq C \|\Phi A \Phi^{-1} g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}^{1/n} \sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^p(\mathbb{R})}^{(n-1)/n}.
\]

(4.3)

(ii) If \( G = \mathcal{M}(g) \) denotes the integral mean of \( g \), namely,

\[
G(r) = \mathcal{M}(g)(r) := \frac{1}{|\mathbb{S}^{n-1}|} \int_{\mathbb{S}^{n-1}} g(r, \omega) d\omega,
\]

then if \( g, (\partial/\partial r)g \in L^p(\mathbb{R} \times \mathbb{S}^{n-1}) \),

\[
\|G\|_{L^p(\mathbb{R})} \leq C \|\Phi A \Phi^{-1} g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}^{1/n} \|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}^{(n-1)/n}.
\]

(4.4)
If $g$ is supported in $[-\Lambda, \Lambda] \times \mathbb{S}^{n-1}$, then
\[
\|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \leq CA^{(n-1)/n^2} \|\partial/\partial r g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}^{1/n} \sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^p(\mathbb{R})}^{(n-1)/n} ; \tag{4.5}
\]
also
\[
\|G\|_{L^p(\mathbb{R})} \leq CA^{(n-1)/n} \|\partial/\partial r g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}. \tag{4.6}
\]

Proof. From (3.8), it follows that, for any $s \in [1, \infty)$,
\[
t^{\theta/2(\theta-1)} \|\Phi P^1 \Phi^{-1} g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \leq Ct^{-\theta/2(\theta-1)-1/2s} \sup_{\omega \in \mathbb{S}^{n-1}} \|g\|_{L^p(\mathbb{R})}.
\]
If $1 \leq p < n - 1$ set $\theta = p/q$, $q = p(p+1)$ and $s = p$. Then, from Theorem 4.1
\[
\|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \leq C\|\partial/\partial r g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}^{1/(p+1)} \sup_{\omega \in \mathbb{S}^{n-1}} \|g\|_{L^p(\mathbb{R})}^{(p+1)/p}. \tag{4.7}
\]
Thus $g \in L^{p(p+1)}(\mathbb{R} \times \mathbb{S}^{n-1}) \cap L^p(\mathbb{R} \times \mathbb{S}^{n-1})$, and since
\[
\frac{np}{n-p} = \frac{p(p+1)}{n-p} + \frac{p(n-p-1)}{n-p}
\]
we have by Hölder’s inequality,
\[
\int_{\mathbb{R} \times \mathbb{S}^{n-1}} |g(p)|^n d\lambda \leq \left( \int_{\mathbb{R} \times \mathbb{S}^{n-1}} |g(p)^{p+1} d\lambda \right)^{1/(n-p)} \left( \int_{\mathbb{R} \times \mathbb{S}^{n-1}} |g|^p d\lambda \right)^{(n-p-1)/(n-p)}.
\]
Hence, from (4.7),
\[
\|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \leq \|g\|_{L^{p(p+1)}(\mathbb{R} \times \mathbb{S}^{n-1})} \|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}^{(n-p-1)/n}
\leq C\|\partial/\partial r g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}^{1/n} \sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^p(\mathbb{R})}^{(n-1)/n}.
\]
If $p = n - 1$, we choose $s = n - 1, q = p^* = n(n-1)$ and $\theta = 1/n$. Then
Theorem 3 gives (4.3) immediately. The inequality (4.5) follows on applying Hölder’s inequality to $\|g(\cdot, \omega)\|_{L^p(\mathbb{R})}$. The inequalities (4.4) and (4.6) follow from (4.3) and (4.5) respectively, on substituting $G$ for $g$ and noting that
\[
\|G\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \leq \|\partial/\partial r g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}
\|G\|_{L^p(\mathbb{R})} \leq \|\mathbf{S}^{n-1}\|^{-1/p} \|g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})}.
\]

Corollary 4.3. (i) Let $1 \leq p < q < \infty, m = (q/p) - 1$, and suppose that $(\partial/\partial r) g \in L^p(\mathbb{R} \times \mathbb{S}^{n-1})$ and \( \sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^m(\mathbb{R})} < \infty \). Then
\[
\|g\|_{L^q(\mathbb{R} \times \mathbb{S}^{n-1})} \leq C\|\partial/\partial r g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \sup_{\omega \in \mathbb{S}^{n-1}} \|g(\cdot, \omega)\|_{L^m(\mathbb{R})}^{1-p/q} \tag{4.8}
\]
(ii) If $(\partial/\partial r) g \in L^p(\mathbb{R} \times \mathbb{S}^{n-1})$ and $g \in L^m(\mathbb{R} \times \mathbb{S}^{n-1})$, then, with $G = \mathcal{M}(g)$,
\[
\|G\|_{L^q(\mathbb{R})} \leq C\|\partial/\partial r g\|_{L^p(\mathbb{R} \times \mathbb{S}^{n-1})} \|g\|_{L^m(\mathbb{R} \times \mathbb{S}^{n-1})}^{1-p/q}. \tag{4.9}
\]
Proof. From (3.8), with \( \theta = p/q \) and \( m = q/p - 1 \), we deduce that
\[
t^{-\theta/2(\theta-1)} \| \Phi P_t \Phi^{-1} g \|_{L^\infty(\mathbb{R} \times S^{n-1})} \leq C t^{-\theta/2(\theta-1)-1/2m} \sup_{\omega \in S^{n-1}} \| g(\cdot, \omega) \|_{L^m(\mathbb{R})}
\]
\[
\leq C \sup_{\omega \in S^{n-1}} \| g(\cdot, \omega) \|_{L^m(\mathbb{R})}
\]
and this yields (4.8). The inequality (4.9) follows from (4.8) on substituting \( G \) for \( g \).
\( \square \)

The case \( p = 2 \) of Corollary 4.2 is of special interest.

**Corollary 4.4.** (i) Let \( f \) be such that \( Lf \in L^2(\mathbb{R}^n), L = x \cdot \nabla, \) and
\[
\sup_{\omega \in S^{n-1}} \| f(\cdot, \omega) \|_{L^2(\mathbb{R}^n; d\mu)} < \infty.
\]
Then, for \( n \geq 3 \),
\[
\| rf(r\omega) \|_{L^{2^*}(\mathbb{R}^n)}^2 \leq C \left\{ \| Lf \|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \| f \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \sup_{\omega \in S^{n-1}} \| f(\cdot, \omega) \|_{L^{2(1-1/n)}(\mathbb{R}^n)}^{2(1-1/n)}.
\]
where \( 2^* = 2n/(n-2) \) and \( d\mu = r^{n-1} dr \).

(ii) If \( f, Lf \in L^2(\mathbb{R}^n) \), then, with \( F := \mathcal{M}(f) \),
\[
\| rF(r) \|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left\{ \| Lf \|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \| f \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \| f \|_{L^{2(1-1/n)}(\mathbb{R}^n)}^{2(1-1/n)}.
\]
For \( 0 \leq \delta < n^2/4 \), we have
\[
\| rF(r) \|_{L^{2^*}(\mathbb{R}^n; d\mu)}^2 \leq C \left( \frac{n^2}{4} - \delta \right)^{(n-1)/n} \left\{ \| Lf \|_{L^2(\mathbb{R}^n)}^2 - \delta \| f \|_{L^2(\mathbb{R}^n)}^2 \right\}.
\]

**Proof.** On using the facts that \( \Phi : L^2(\mathbb{R}^n) \rightarrow L^2(\mathbb{R} \times S^{n-1}) \) is an isometry and, with \( g := \Phi f \),
\[
\| (\partial/\partial r) g \|_{L^2(\mathbb{R} \times S^{n-1})}^2 = \| \Phi A \Phi^{-1} g \|_{L^2(\mathbb{R} \times S^{n-1})}^2 = \| Af \|_{L^2(\mathbb{R}^n)}^2
\]
\[
= \| Lf \|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \| f \|_{L^2(\mathbb{R}^n)}^2
\]
since \( A^2 = L^* L - (n^2/4) \) from (3.6), it follows from (4.3) that
\[
\| \Phi f \|_{L^{2^*}(\mathbb{R} \times S^{n-1})}^2 \leq C \left\{ \| Lf \|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4} \| f \|_{L^2(\mathbb{R}^n)}^2 \right\}^{1/n} \times \sup_{\omega \in S^{n-1}} \| f(\cdot, \omega) \|_{L^{2(1-1/n)}(\mathbb{R}^n)}^{2(1-1/n)}.
\]
Then (4.10) follows since
\[
\| \Phi f \|_{L^{2^*}(\mathbb{R} \times S^{n-1})} = \| rf(r, \omega) \|_{L^{2^*}(\mathbb{R}^n)}.
\]
The inequality (4.11) follows in a similar way from (4.4) since
\[ \|M(\Phi f)\|_{L^2(\mathbb{R})} = \|rF(r)\|_{L^2(\mathbb{R}^+; d\mu)} \]

From Young’s inequality we have for any \( \varepsilon > 0 \) that
\[ n\left[ \varepsilon/(n - 1) \right]^{1 - 1/n} a^b \leq a^n + \varepsilon b^{n/(n-1)}. \]

On applying this to (4.11) we get
\[ \varepsilon^{1-1/n} \|rF(r)\|^2_{L^2(\mathbb{R}^+; d\mu)} \leq C\left\{ \|L f\|^2_{L^2(\mathbb{R}^n)} - \left[ \left(\frac{N}{2}\right)^2 - \varepsilon \right] \|f\|^2_{L^2(\mathbb{R}^n)} \right\}. \]

This yields (4.12) on setting \( \varepsilon = n^2/4 - \delta \).

\[ \square \]

**Corollary 4.5.** (i) Let \( \nabla h \in L^2(\mathbb{R}^n), n \geq 3, \) and
\[ \sup_{\omega \in \mathbb{S}^{n-1}} \|h(\cdot, \omega)/|\cdot||^2_{L^2(\mathbb{R}^n; d\mu)} < \infty. \]

Then
\[ \|h\|^2_{L^2(\mathbb{R}^n)} \leq C\left\{ \|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \left(\frac{n - 2}{2}\right)^2 \|h/|\cdot||^2_{L^2(\mathbb{R}^n)} \right\}^{1/n} \times \sup_{\omega \in \mathbb{S}^{n-1}} \left\{ \|h(\cdot, \omega)/|\cdot||^2_{L^2(\mathbb{R}^n; d\mu)} \right\}^{1-1/n}. \] (4.13)

(ii) If \( h, \nabla h \in L^2(\mathbb{R}^n) \) then, with \( H := M(h), \)
\[ \|H\|^2_{L^2(\mathbb{R}^n; d\mu)} \leq C\left\{ \|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \left(\frac{n - 2}{2}\right)^2 \|h/|\cdot||^2_{L^2(\mathbb{R}^n)} \right\}^{1/n} \times \left\{ \|h/|\cdot||^2_{L^2(\mathbb{R}^n)} \right\}^{1-1/n}. \] (4.14)

For \( 0 \leq \delta < (n - 2)^2/4, \) we have
\[ \|H\|^2_{L^2(\mathbb{R}^n; d\mu)} \leq C \left( (n - 2)^2/4 - \delta \right)^{(n-1)/n} \left\{ \|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \delta \|h/|\cdot||^2_{L^2(\mathbb{R}^n)} \right\}. \] (4.15)

**Proof.** Since \( n \geq 3, \) we have that \( f := h/|\cdot| \in L^2(\mathbb{R}^n). \) We claim that \( Lf \in L^2(\mathbb{R}^n). \) For
\[ |\nabla(|x|f)|^2 = \left| \frac{x}{|x|} f + |x| \nabla f \right|^2 = |f|^2 + (|x||\nabla f|)^2 + 2 \text{Re} [\overline{f}(x \cdot \nabla)f] \]
and, on integration by parts, initially for \( f \in C_0^\infty(\mathbb{R}^n) \) and then by the usual continuity argument,
\[ \int_{\mathbb{R}^n} \overline{f}(x \cdot \nabla)f \, dx = \sum_{j=1}^n \int_{\mathbb{R}^n} x_j \overline{f} \frac{\partial f}{\partial x_j} \, dx \]
\[ = -\sum_{j=1}^n \int_{\mathbb{R}^n} f \left\{ \overline{f} + x_j \frac{\partial f}{\partial x_j} \right\} \, dx \]
\[ = -\int_{\mathbb{R}^n} \left\{ n|f|^2 + f(\nabla f)^2 \right\} \, dx. \]
This gives
\[ 2\text{Re} \int_{\mathbb{R}^n} [\overline{f}(\mathbf{x} \cdot \nabla)f] \, d\mathbf{x} = -n \int_{\mathbb{R}^n} |f|^2 \, d\mathbf{x} \]
and hence
\[ \int_{\mathbb{R}^n} |\nabla(|f|)|^2 \, d\mathbf{x} = \int_{\mathbb{R}^n} (|\nabla f|^2 \mathbf{x} - (n - 1) \int_{\mathbb{R}^n} |f|^2 \, d\mathbf{x} \]
\[ \geq \int_{\mathbb{R}^n} |Lf|^2 \, d\mathbf{x} - (n - 1) \int_{\mathbb{R}^n} |f|^2 \, d\mathbf{x} \quad (4.16) \]
which confirms our claim. On substituting (4.16) and \( f = h/|\cdot| \) in (4.10), we get
\[ \|h\|^2_{L^2}(\mathbb{R}^n) \leq C \left\{ \|\nabla h\|^2_{L^2(\mathbb{R}^n)} + (n - 1)\|h/|\cdot||^2_{L^2(\mathbb{R}^n)} \right\} \]
\[ - \left( \frac{n^2}{4} \right) \|h/|\cdot||^2_{L^2(\mathbb{R}^n)} \right\} \sup_{\omega \in \mathbb{S}^{n-1}} \|h/|\cdot||^{2(1-1/n)}_{L^2(\mathbb{R}^+;d\mu)} \]
which yields (4.13); (4.14) follows similarly from (4.11) and (4.14) yields (4.15).

If in (4.6) \( g = \Phi f \), where \( f \) is supported in the annulus \( A_R := \{\mathbf{x} \in \mathbb{R}^n : 1/R \leq |\mathbf{x}| \leq R\} \), then \( G \) is supported in the interval \([-\ln R, \ln R]\) and we have as in the proof of Corollary 4

**Corollary 4.6.** Let \( f \in C_0^\infty(A_R) \). Then, with \( F := \mathcal{M}(f) \),

\[ \|rF(r)\|^2_{L^2(\mathbb{R}^+;d\mu)} \leq C (\ln R)^{2(\frac{n-1}{n})} \left\{ \|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4} \|f\|^2_{L^2(\mathbb{R}^n)} \right\}. \quad (4.17) \]

On putting \( f = h/|\cdot| \) in (4.17) and using (4.16), we have

**Corollary 4.7.** Let \( h \in C_0^\infty(A_R) \). Then, with \( H := \mathcal{M}(h) \),

\[ \|H\|^2_{L^2(\mathbb{R}^+;d\mu)} \leq C (\ln R)^{2(\frac{n-1}{n})} \left\{ \|\nabla h\|^2_{L^2(\mathbb{R}^n)} - \frac{(n - 2)^2}{4} \|\cdot\|^2_{L^2(\mathbb{R}^n)} \right\}. \]

Finally we have the following \( p = 2 \) case of Corollary 3(ii).

**Corollary 4.8.** Let \( 2 < q < \infty \) and \( m = q/2 - 1 \). Then, if \( f \) is such that \( f, Lf \in L^2(\mathbb{R}^n) \) and \( \int_{\mathbb{R}^+} \int_{\mathbb{S}^{n-1}} |f(s,\omega)|^m s^{\frac{mn}{2}-1} \, ds \, d\omega < \infty \), we have that \( \int_{\mathbb{R}^+} |F(s)|^q s^{\frac{qr}{2}-1} \, ds < \infty \) and

\[ \int_{\mathbb{R}^+} |F(s)|^q s^{\frac{qr}{2}-1} \, ds \leq C \left\{ \|Lf\|^2_{L^2(\mathbb{R}^n)} - \frac{n^2}{4} \|f\|^2_{L^2(\mathbb{R}^n)} \right\}^2 \times \left\{ \int_{\mathbb{R}^+} \int_{\mathbb{S}^{n-1}} |f(s,\omega)|^m s^{\frac{mn}{2}-1} \, ds \, d\omega \right\}^2 \]
Proof. Corollary 4.3(ii) with $p = 2$ yields

$$
\|M(\Phi f)\|_{L^q(\mathbb{R})} \leq C \left\{ \|Lf\|_{L^2(\mathbb{R}^n)}^2 - \frac{n^2}{4}\|f\|_{L^2(\mathbb{R}^n)}^2 \right\}^{2/q} \times \|\Phi f\|_{L^m(\mathbb{R} \times S^{n-1})}^{1-2/q}.
$$

Since

$$\|M(\Phi f)\|_{L^q(\mathbb{R})}^q = \int_{\mathbb{R}^+} |F(s)|^q s^{(\frac{nq}{2} - 1)} ds$$

and

$$\|\Phi f\|_{L^m(\mathbb{R} \times S^{n-1})}^m = \int_{\mathbb{R}^+} \int_{S^{n-1}} |f(s, \omega)|^m s^{\left(\frac{2n}{2} - 1\right)} ds d\omega$$

the corollary follows. \qed

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