Categorical actions of generalised braids

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In [32], Khovanov and Thomas constructed a categorical action of the braid group $\text{Br}_n$ on the derived category $D(T^*\text{Fl}_n)$ of coherent sheaves on the cotangent bundle of the variety $\text{Fl}_n$ of the complete flags in $\mathbb{C}^n$.

In this thesis, we define the generalised braid category $\mathcal{GBr}_n$, we define the notion of a skein-triangulated representation of $\mathcal{GBr}_3$, give a sufficient condition for the existence of a skein-triangulated representation of $\mathcal{GBr}_3$ and we construct a skein triangulated representation of $\mathcal{GBr}_3$ on $D(T^*(\text{Fl}_3(i)))$ that generalises the Khovanov and Thomas categorical braid action on $D(T^*\text{Fl}_3)$.

Summary

In [32], Khovanov and Thomas constructed a categorical action of the braid group $\text{Br}_n$ on the derived category $D(T^*\text{Fl}_n)$ of coherent sheaves on the cotangent bundle of the variety $\text{Fl}_n$ of the complete flags in $\mathbb{C}^n$.

In this thesis, we define the generalised braid category $\mathcal{GBr}_n$, we define the notion of a skein-triangulated representation of $\mathcal{GBr}_3$, give a sufficient condition for the existence of a skein-triangulated representation of $\mathcal{GBr}_3$ and we construct a skein triangulated representation of $\mathcal{GBr}_3$ on $D(T^*(\text{Fl}_3(i)))$ that generalises the Khovanov and Thomas categorical braid action on $D(T^*\text{Fl}_3)$.
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Chapter 1

Introduction

The bounded derived category of coherent sheaves can be viewed as an ultimate cohomological invariant of an algebraic variety. The notion of the derived category, together with the notion of a triangulated category which axiomatises it, was invented by Grothendieck and Verdier in 1960s in their search for the natural context for the generalisations of Serre duality and for the existence of the right adjoint of a direct image functor [46].

In the past few decades it has become increasingly more relevant in many areas of algebraic geometry. In particular, in 1994 Kontsevich formulated the Homological Mirror Symmetry conjecture at the International Congress of Mathematicians in Zürich ([33]) which was a far reaching mathematical generalisation and interpretation of a certain duality between the families of 3-dimensional Calabi-Yau varieties observed several years earlier by string theorists. The duality was on the level of Hodge numbers: there were pairs of Calabi-Yau 3-folds $X$ and $X'$ for which:

$$\dim H^p(X, \Omega^q) = \dim H^{n-p}(X', \Omega^q).$$

Kontsevich’s Homological Mirror Symmetry conjecture stated that if $X$ and $X'$ are two such dual Calabi-Yau manifolds, then the derived category of coherent sheaves $D^b(\text{Coh}(X))$ is equivalent to the derived Fukaya category of $D^b(\text{Fuk}(X'))$ and vice versa: $D^b(\text{Coh}(X)) \simeq D^b(\text{Fuk}(X'))$. The powerful intuition gained from this conjecture led to the discovery of many new mathematical structures, including spherical and $\mathbb{P}^n$-objects and their generalisations, which lie at the heart of this thesis.

Roughly, the main point of the notion of the derived category is that working with complexes is better than working with their (co)homologies. For example, there exist topological spaces $X$, $Y$ such that their homologies are isomorphic $H_*(X) \cong H_*(Y)$, but $X$ and $Y$ are not homotopy equivalent; while the Whitehead theorem states that two simplicial complexes $X$ and $Y$ have homotopy equivalent geometric realizations $|X|$ and $|Y|$ if, and only if, there exists a simplicial complex $Z$ and simplicial maps $f : Z \to X$ and $g : Z \to Y$ which are quasi-isomorphisms, that is they induce isomorphisms between the homology groups $H_i(X)$, $H_i(Z)$ and $H_i(Y)$. The notion of derived category is a realisation of the same principle: work with complexes of objects and formally invert the quasi-isomorphisms to identify any two complexes with naturally isomorphic cohomologies. We refer to [44] and [15] for an introduction to derived categories and to [34] and [27] for technical references.
Given an abelian category $\mathcal{A}$, define the category $C(\mathcal{A})$ to be the category of complexes of objects of $\mathcal{A}$. For technical purposes, we first invert all the homotopy equivalences: define the homotopy category $K(\mathcal{A})$ to be the additive category whose objects are complexes of objects in $\mathcal{A}$

$$\cdots \rightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \rightarrow \cdots \quad (n \in \mathbb{Z}, d^n \circ d^{n-1} = 0)$$

and whose morphisms spaces are the quotients of those in $C(\mathcal{A})$ by the subspaces of null-homotopic maps. The derived category $D(\mathcal{A})$ is the category whose objects are the same as those of $K(\mathcal{A})$, and whose morphisms $A^\bullet \rightarrow B^\bullet$ are certain equivalence classes $fs^{-1}$ of pairs $(s,f)$ of morphisms in $K(\mathcal{A})$ with $s$ a quasi-isomorphism.

If $X$ is a smooth quasi-projective variety we write $D^b(X)$ for the bounded derived category of the abelian category $\text{Coh}(X)$ of coherent sheaves on $X$, that is $D^b(X) := D^b(\text{Coh}(X))$. The structure of $D^b(X)$ can be studied by considering its autoequivalences and in this context categorical group actions play an important role. A categorical group action of a group $G$ on $D^b(X)$ is an assignment of an autoequivalence $F_g$ of $D^b(X)$ to every element $g \in G$ such that the group operation is compatible, up to isomorphism, with the composition of functors.

The classical result by Khovanov and Thomas in [32] states there exist $n-1$ autoequivalences $T_i$ of the derived category of the total space of the complete flag variety in $\mathbb{C}^n$ which satisfy the braid relations:

$$T_i T_j \cong T_j T_i \quad \text{for } |i - j| \geq 1.$$  
$$T_i T_j T_i \cong T_j T_i T_j \quad \text{for } |i - j| = 1.$$  

In other words, that the braid group $Br_n$ acts categorically on $D^b(T^*Fl_n)$. Here braids are configurations of $n$ disjoint pieces of string with $n$ fixed endpoints, considered up to isotopies which keep the strands disjoint.

In the work of Khovanov and Thomas, configurations of $n$ points represent the derived category of the cotangent space of complete flags in $\mathbb{C}^n$, and the cobordism between the two configurations represents an autoequivalence of this category. In this thesis, we generalise the Khovanov and Thomas result in dimension 3 to a skein-triangulated action of the category $\mathcal{GB}r_3$ of generalised braids on the derived categories of the cotangent bundles of the varieties of complete and partial flags in $\mathbb{C}^3$. Generalised braids are the braids whose strands are allowed to touch in a certain way: they can join up (two at a time), continue as a multiple strand, and split apart. Moreover, we do not distinguish any permutations of individual strands within a multiple strand — only the multiplicity matters. Therefore, technically, we define generalised braids as a certain kind of trivalent coloured graphs with fixed univalent startpoints and endpoints and satisfying flow conditions.
Due to strands having multiplicity, instead of a single endpoint configuration consisting of \( n \) disjoint points, they have multiple endpoint configurations corresponding to the ordered partitions of \( n \). This, together with the fact that such braids are no longer necessarily invertible, implies that generalised braids form a category rather than a group or a groupoid.

![Some generalised braids](image)

Figure 1.1: Some generalised braids

In Chapter 2 we describe complete and partial flag varieties as homogeneous spaces, describe their Picard group, and give a description of \( T^\ast(\text{Fl}_n) \) as Springer resolution of the nilpotent cone of \( \mathfrak{sl}_n \).

In Chapter 3 we give an overview of autoequivalences of and braid group actions on the derived categories of coherent sheaves of smooth (quasi-)projective varieties.

In Chapter 4, we define the generalised braid category \( \mathcal{GBr}_n \), we define the notion of a skein-triangulated representation of \( \mathcal{GBr}_3 \), give a sufficient condition for the existence of a skein-triangulated representation of \( \mathcal{GBr}_3 \) and we construct a skein triangulated representation of \( \mathcal{GBr}_3 \) on \( D(T^\ast(\text{Fl}_3(i))) \) that generalises the Khovanov and Thomas categorical braid action on \( D(T^\ast \text{Fl}_3) \).
Springer resolutions and flag varieties

Flag varieties are interesting geometrical objects, on one hand these Fano varieties are a natural generalisation of projective spaces and Grassmannians, on the other hand they are the model example of the notion of homogeneous spaces.

The aim of this chapter is to give a description of flag varieties as homogeneous spaces, understand their Picard group, give a convenient description of the total space of the cotangent bundle of complete and partial flag varieties. For the complete flag variety, we identify this total space with the Springer resolution of the nilpotent cone of $\mathfrak{sl}_n$.

In the first section, for a complex connected Lie group $G$, we define the nilpotent cone $\mathcal{N}_g$ of its Lie algebra $\mathfrak{g}$ and the Springer resolution $\tilde{\mathcal{N}}_g$ of $\mathcal{N}_g$. We then define the homogeneous space $G/B$, and identify the Springer resolution with the cotangent bundle $T^*G/B$. In the second section, we describe the Picard group of Grassmannians and flag varieties.

2.1 The nilpotent cone and its Springer resolution

In this section we describe the Springer resolution of the nilpotent cone of a semisimple Lie group following the Chapter 3 of [20].

Let $G$ be a complex connected Lie group and let $\mathfrak{g}$ be its Lie algebra, viewed as the tangent space at the identity $T_eG$. Assume $G$ to be semisimple, i.e. $\mathfrak{g}$ is semisimple. Let $B_0$ a Borel subgroup of $G$, let $T_0 \subset B_0$ be a maximal torus of $G$ and let $U_0$ be the unipotent radical of $B_0$. Let moreover $\mathfrak{g}, \mathfrak{b}_0, \mathfrak{h}_0$ and $\mathfrak{n}_0$ be the respective Lie algebras of $G, B, T$ and $U$.

Definition 2.1.1. Let $G$ be a Lie group and $\mathfrak{g}$ be its Lie algebra. The adjoint action $Ad_G$ of $G$ on $\mathfrak{g}$ is the differential of the adjoint action of $G$ on itself: $g \in G$ acts on $\mathfrak{g}$ by the differential $d(g(-)g^{-1}) : \mathfrak{g} \rightarrow \mathfrak{g}$. The coadjoint action $Ad^*_G$ of $G$ on $\mathfrak{g}$ is the differential of the coadjoint action of $G$ on itself: $g \mapsto d(g^{-1}(-)g)$.

Remark 2.1.2. The differential of the adjoint action $Ad_G : G \rightarrow GL(\mathfrak{g})$ is the adjoint representation of Lie Algebras $ad : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$.

Example 2.1.3. Let $G = GL_n$ so that $\mathfrak{g} = \mathfrak{gl}(\mathbb{C}^n)$. The adjoint action $Ad_G$ is the action by matrix conjugation.
Definition 2.1.4. Define $B_g$ to be the set of all Borel subalgebras of $g$.

Proposition 2.1.5. Let $G$ be a Lie group and let $g$ be its Lie algebra. The adjoint action $Ad_G$ of $G$ on $g$ defines a transitive action of $G$ on $B_g$.

Proof. Section 3.1 of [20] or [12].

Proposition 2.1.6. The normaliser $N_B(G)$ of a Borel subgroup $B$ of $G$ is $B$.


Remark 2.1.7. We have $b_0 = h_0 + n_0$ and we have $n_0 = [b_0, b_0]$. More generally, any $b \in B_g$ contains the canonical subalgebra $n_b = [b, b]$ which consists of all ad-nilpotent elements of $b$. Under the adjoint action of $G$, if $b_0$ is sent to some $b \in B_g$, then $n_0$ is sent to $n_b$, but $h_0$ can be sent to any of the Cartan subalgebras $h \subset b$.

Definition 2.1.8. An element $x \in g$ is ad-nilpotent if $ad^n_x = 0$ for some $n \in \mathbb{N}$.

Remark 2.1.9. If $g = sl_n$, then $x \in g$ is ad-nilpotent in the sense of Definition 2.1.8 if and only if its matrix is nilpotent.

Definition 2.1.10. Define the nilpotent cone $N_g$ of $g$ to be the set of all ad-nilpotent elements of $g$.

Proposition 2.1.11. The set $N_g$ has a natural structure of a quasi-projective variety: it is a closed subvariety of $g$ stable under $Ad_G$ and $\mathbb{C}^*$ actions, i.e. a cone variety singular at the origin. The set $B_g$ has the structure of a smooth projective variety: it is the closed subvariety of the Grassmannian $Gr(\dim(b), g)$ formed by all solvable Lie subalgebras of $g$.

Proof. Section 3.1.6 of [20].

Example 2.1.12. Let $g = sl_2$ generated as Lie algebra by the elements
\[
  x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\] (2.1)
subject to the relations
\[
  [x, y] = h \quad [h, x] = 2x \quad [h, y] = -2y.
\]

Then the nilpotent cone is the space
\[
  N_g = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in sl_2 \middle| -a^2 - bc = 0 \right\},
\] (2.2)
which is a quadratic cone in $\mathbb{C}^3$. Set
\[
  B_0 = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad \text{and} \quad T_0 = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\},
\]
2.1. The nilpotent cone and its Springer resolution

their respective Lie algebras \( \mathfrak{b}_0 \) and \( \mathfrak{h}_0 \) consist of the traceless matrices which are upper triangular and diagonal, respectively. Hence, in the language of Example ??:

\[
\mathfrak{b}_0 = \langle x, h \rangle, \quad \mathfrak{h}_0 = \langle h \rangle, \quad \text{and} \quad \mathfrak{n}_0 = \langle x \rangle,
\]

and we see explicitly the decomposition \( \mathfrak{b}_0 = \mathfrak{h}_0 \oplus \mathfrak{n}_0 \) of Remark 2.1.7.

Denote by \( \{e_1, e_2\} \) the standard basis of \( \mathbb{C}^2 \). We can describe \( \mathfrak{b}_0 \) as the traceless maps \( \mathbb{C}^2 \to \mathbb{C}^2 \) which preserve \( \langle e_1 \rangle \), \( \mathfrak{h}_0 \) as the traceless maps that preserve both \( \langle e_1 \rangle \) and \( \langle e_2 \rangle \), and \( \mathfrak{n}_0 \) as the maps which send the whole \( \mathbb{C}^2 \) to \( \langle e_1 \rangle \), and \( \langle e_1 \rangle \) to zero.

Since all \( \mathfrak{b} \in \mathfrak{b}_0 \) are conjugate to \( \mathfrak{b}_0 \), each \( \mathfrak{b} \in \mathfrak{b}_0 \) is the set of all traceless maps preserving a line in \( \mathbb{C}^2 \), whence \( \mathfrak{b}_0 \) is isomorphic to \( \mathbb{P}^1 \).

**Definition 2.1.13.** For any Borel subgroup \( B \subset G \), define \( G/B \) to be the set of all the left cosets \( \{xB \mid x \in G\} \).

**Remark 2.1.14.** By Propositions 2.1.5 and 2.1.6, the assignment 

\[
g \mapsto g \cdot \text{Ad}_G \mathfrak{b}_0
\]

gives a bijection 

\[
G/B_0 \cong \mathfrak{b}_0.
\]

It can be shown that the \( G/B_0 \) admits a natural structure of a smooth projective variety and that the bijection (2.3) is a \( G \)-equivariant isomorphism of varieties, see section 23.3 of [29] or [12].

**Definition 2.1.15.** Define

\[
\tilde{\mathfrak{g}} = \{(x, b) \in \mathfrak{g} \times \mathfrak{b}_0 \mid x \in \mathfrak{b}\}, \quad \text{(2.4)}
\]

\[
\tilde{\mathcal{N}}_0 := \{(x, b) \in \mathcal{N}_0 \times \mathfrak{b}_0 \mid x \in \mathfrak{b}\}. \quad \text{(2.5)}
\]

and let the two maps

\[
\begin{array}{ccc}
\tilde{\mathcal{N}}_0 & \overset{\mu}{\longrightarrow} & \mathcal{N}_0 \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathfrak{b}_0 & & \mathfrak{b}_0.
\end{array}
\]

be the corresponding natural projections.

**Definition 2.1.16.** The Springer morphism is the map 

\[
\mu : \tilde{\mathcal{N}}_0 \longrightarrow \mathcal{N}_0. \quad \text{(2.7)}
\]

**Remark 2.1.17.** Consider the trivial vector bundle over \( \mathfrak{b}_0 \) with fibers \( \mathcal{N}_0 \). We can view \( \pi : \tilde{\mathcal{N}}_0 \to \mathfrak{b}_0 \) as its subbundle whose fiber at any point \( b \in \mathfrak{b}_0 \) is \( \mathfrak{n}_b \).

**Remark 2.1.18.** \( \tilde{\mathcal{N}}_0 \) is a smooth variety since it is a vector bundle over \( \mathfrak{b}_0 \).
Example 2.1.19. If \( g = sl_2 \) as in example 2.1.12, then \( \pi : \tilde{N}_g \to B_g \) is a line bundle over \( \mathbb{P}^1 \): a point \( b \in B_g \) corresponds to the choice of 1-dimensional subspace \( l \subset \mathbb{C}^2 \) and the corresponding fiber of \( \pi \) consists of the line \( n \in N_g \) of all nilpotent operators \( \mathbb{C}^2 \to \mathbb{C}^2 \) whose image is contained in \( l \).

**Definition 2.1.20.** Define \( b^\perp \) to be the annihilator of \( b \), i.e. \( b^\perp = \{ x \in g^* \mid x(b) = 0 \} \).

**Definition 2.1.21.** Define \( G \times_{B_0} b^\perp_0 \) to be the quotient of \( G \times b^\perp_0 \) by \( B_0 \) acting on \( G \) by right multiplication and on \( b^\perp_0 \) by the coadjoint action \( Ad_{B_0}^* \).

**Proposition 2.1.22.** There is an isomorphism
\[
T^* B_g \cong G \times_{B_0} b^\perp_0.
\] (2.8)
given by the map dual to the infinitesimal \( g \)-action map \( G/B_0 \times g \to T(G/B_0) \).

**Proof.** Proposition 1.4.11 of [20].

**Lemma 2.1.23.** For any \( b \in B_g \), the identification \( g \cong g^* \) provided by the Killing form identifies \( b^\perp \) with \( n_b \).

**Proof.** Section 8.1 of [28].

We can embed \( G \times_{B_0} b^\perp_0 \) into the trivial vector bundle \( G/B_0 \times g^* \) over \( G/B_0 \):
\[
\phi : ([g, \alpha]) \mapsto ([g], g \cdot AdG \alpha).
\]
The image of \( G \times_B b^\perp_0 \) under this embedding is the subbundle
\[
\{(b, \alpha) \in G/B_0 \times g^* \mid \alpha \in b^\perp \}.
\]
We thus have the following commutative diagram
\[
\begin{array}{c}
G \times_{B} b^\perp_0 \\
\downarrow \sim \\
\{(b, \alpha) \mid \alpha \in b^\perp \} \\
\downarrow \sim \\
\tilde{N}_g \\
\end{array} \quad \begin{array}{c}
\phi \quad \quad \\
G/B \times g^* \\
\downarrow \sim \\
G/B \times g \\
\end{array}
\]

Thus, we obtain

**Proposition 2.1.24.** There exists a \( G \)-equivariant isomorphism of vector bundles over \( G/B_0 \):
\[
\tilde{N}_g \cong G \times_{B} b^\perp.
\] (2.9)
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Proof. Lemma 3.3.2 of [20] or [12].

Corollary 2.1.25. There is a natural isomorphism $\tilde{N}_g \cong T^*B_g$.

Remark 2.1.26. By Corollary 2.1.25 we can think of the Springer resolution as a map $T^*B_g \to N_g$.

2.2 Geometry of flag varieties

In this section we give a description of the Picard groups of flag varieties via Schubert calculus, following [14]. We begin with some preliminaries on Bruhat decompositions following [?]. Let $G$ be a connected complex reductive algebraic group.

Definition 2.2.1. A parabolic subgroup $P$ is a subgroup of $G$ that contains a Borel subgroup $B$.

Proposition 2.2.2. A subgroup $P \subset G$ is parabolic if and only if $G/P$ is a projective variety.

Proof. See [12] or [29].

Definition 2.2.3. Let $B \subset G$ a Borel subgroup and $T \subset B$ a maximal torus. The Weyl group of the Borel pair $(T, B)$ is the group $W = N_G(T)/T$.

Definition 2.2.4. The Bruhat decomposition of $G$ is the decomposition

$$G = \bigsqcup_{w \in W} BwB$$

(2.10)

as a disjoint union of double cosets of $B$ parameterized by the elements of $W$.

More generally, any parabolic subgroup $B \subset P_J$ defines the generalised Bruhat decomposition

$$G = \bigsqcup_{w \in W/W_J} BwP_J$$

(2.11)

where $W_J = \{[v] \in W \mid v \in P_J\}$ and $W/W_J$ is the set of right cosets of $W_J$ in $W$.

See [12] and [29] for further details.

The double cosets of the Bruhat decomposition descend to the left $B$-cosets in the right coset quotients $G/B$ and $G/P_J$. This gives the decomposition $G/B$ and $G/P_J$ into the $B$-orbits under the action of $B$ by left multiplication. These are Schubert cells, and their closures are Schubert varieties:

Definition 2.2.5. For any $w \in W$, the corresponding Schubert varieties are

$$X_w = [BwB] \subset G/B, \quad \text{and} \quad X'_w = [BwP_J] \subset G/P_J.$$
Chapter 2. Springer resolutions and flag varieties

**Definition 2.2.6.** Define the *Bruhat order* on $W$ by
\[ w \leq w' \iff X_w \subseteq X_{w'} . \tag{2.12} \]

Define the *Bruhat order* on $W_J$ by
\[ w \leq w' \iff X^J_w \subseteq X^J_{w'} . \tag{2.13} \]

**Example 2.2.7.** Let $G = GL_3(\mathbb{C})$ and let $T$ and $B$ be the subgroups of the diagonal and the upper triangular matrices, respectively:

\[
T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad \text{and} \quad B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} .
\]

The normaliser $N_G(T)$ consists of all the matrices such that the induced change of basis keeps all the diagonal matrices diagonal. Up to scaling, any such change of basis is a permutation of basis vectors. It follows that
\[ W := N_G(T)/T = S_3. \]

Moreover, we have the standard splitting $W \hookrightarrow N_G(T)$ given by the *permutation matrices*, which permute the standard basis $e_1, e_2, e_3$ of $\mathbb{C}^3$.

The upper triangular matrices are the matrices which preserve the standard flag
\[ 0 \subset E_1 \subset E_2 \subset E_3 = \mathbb{C}^3 \]
where $E_i = \langle e_1, \ldots, e_i \rangle$. Hence any right coset of $B$ consists of all the matrices which send the standard flag to a specific flag
\[ 0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3 . \]

This identifies the space $G/B$ with the flag variety $Fl_3$ of complete flags in $\mathbb{C}^3$.

There are two parabolic subgroups of $G$ containing $B$:

\[
P_1 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \quad \text{and} \quad P_2 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\} .
\]

Each $P_i$ consists of all matrices which preserve the subspace $E_i$. Its right cosets consist of the matrices which send $E_i$ to a specific $i$-dimensional subspace of $\mathbb{C}^3$. It follows that the homogeneous spaces $G/P_1$ and $G/P_2$ can be identified with the varieties $\mathbb{P}^2$ and $\mathbb{P}^{2\nu}$ of lines and planes in $\mathbb{C}^3$, respectively.

The Bruhat decomposition of a matrix $A \in GL_3$ corresponds to its reduction to column echelon form. Multiplication by an upper triangular matrix on the right is equivalent to performing a sequence of column operations where we only add to
each column a linear combination of the columns lying to the left of it. Hence in the Bruhat decomposition

\[ M = U_1 \sigma U_2 \]

the (inverse of the) matrix \( U_2 \) encodes the column operations, the (inverse of the) matrix \( \sigma \) encodes the permutation of the columns, and the matrix \( U_1 \) is the resulting column echelon form of \( M \).

Correspondingly, in \( G/B \) the Schubert cell corresponding to \( \sigma \in W \) consists of all the points which can be represented by a matrix obtained by permuting the columns of an upper triangular matrix by \( \sigma \). This translates naturally to the condition on the corresponding flag in \( \mathbb{C}^3 \). For example, the cell \( C_{123} \) which corresponds to the permutation \( (123) \) consists of the points representable by the matrix type

\[
\begin{pmatrix}
* & * & * \\
* & 0 & * \\
* & 0 & 0
\end{pmatrix}.
\]  

(2.14)

In terms of the corresponding flag \( 0 \subset V_1 \subset V_2 \subset \mathbb{C}^3 \), the shape of the first column is equivalent to the condition

\[ V_1 \not\subset E_2, \]

and then the shape of the second column is determined by the condition

\[ E_1 \subset V_2. \]

The shape of the third column is determined by the first two.

By the first condition \( V_1 \neq E_1 \), thus we can replace the second by \( V_2 = V_1 \oplus E_1 \). Thus \( C_{123} \simeq \mathbb{C}^2 \) and can be identified with \( \mathbb{P}^2 \) of lines in \( \mathbb{C}^3 \) with the line \( V_1 \subset E_2 \) removed. The Schubert variety \( X_{123} \) is the closure of \( C_{123} \). It is the subvariety \( E_1 \subset V_2 \) of \( \text{Fl}_3 \). Again, if \( V_1 \neq E_1 \), we have \( V_2 = V_1 \oplus E_1 \), but if \( V_1 = E_1 \) then we can take any \( V_2 \) containing \( E_1 \). Thus, \( X_{123} \) is the blowup of \( \mathbb{P}^2 \) at the point \( V_1 = E_1 \).

Similarly, we have:
Chapter 2. Springer resolutions and flag varieties

<table>
<thead>
<tr>
<th>$\sigma \in S_3$</th>
<th>Matrix type</th>
<th>Schubert cell $C_\sigma$</th>
<th>Schubert variety $X_\sigma$</th>
</tr>
</thead>
</table>
| Id             | \[
\begin{pmatrix}
* & * & *
0 & * & *
0 & 0 & *
\end{pmatrix}
\] | $pt: V_1 = E_1, V_2 = E_2$ | $pt: V_1 = E_1, V_2 = E_2$ |
| $(12)$         | \[
\begin{pmatrix}
* & * & *
0 & * & *
0 & 0 & *
\end{pmatrix}
\] | $C^1: V_1 \neq E_1, V_2 = E_2$ | $\mathbb{P}^1: V_2 = E_2$ |
| $(23)$         | \[
\begin{pmatrix}
* & * & *
0 & * & *
0 & 0 & *
\end{pmatrix}
\] | $C^1: V_1 = E_1, V_2 \neq E_2$ | $\mathbb{P}^1: V_1 = E_1$ |
| $(132)$        | \[
\begin{pmatrix}
* & * & *
0 & * & *
0 & 0 & 0
\end{pmatrix}
\] | $C^2: V_1 \not\subset E_2, V_2 = V_1 \oplus E_1$ | $\mathbb{P}^2$ blown up at $V_1 = E_1$: $E_1 \subset V_2$ |
| $(123)$        | \[
\begin{pmatrix}
* & * & *
* & 0 & *
0 & 0 & 0
\end{pmatrix}
\] | $C^2: V_1 = E_2 \cap V_2, E_1 \not\subset V_2$ | $\mathbb{P}^2 \vee$ blown up at $V_2 = E_2$: $V_1 \subset E_2$ |
| $(13)$         | \[
\begin{pmatrix}
* & * & *
* & 0 & *
0 & 0 & 0
\end{pmatrix}
\] | $C^3: V_1 \not\subset E_2, E_1 \not\subset V_2$ | $\text{Fl}_3$ |

Thus the Bruhat order on the Schubert subvarieties of $\text{Fl}_3$ is

\[
\begin{array}{c}
X_{13} = \text{Fl}_3 \\
X_{132} \\
X_{12} \\
X_{1} = \bullet \\
X_{123} \rightarrow X_{23} \\
X_{132} \rightarrow X_{23} \\
X_{13} \rightarrow X_{123}
\end{array}
\]

The parabolic subgroups $P_1$ and $P_2$ of $G$ intersect $W$ at subgroups $W_1 = \langle (12) \rangle$ and $W_2 = \langle (23) \rangle$. The generalised Bruhat decompositions of $G$ for $P_1$ and $P_2$ merge the Bruhat cells corresponding to the elements of $S_3$ which get identified in $W/W_1$ and $W/W_2$, respectively. Thus:

<table>
<thead>
<tr>
<th>$\sigma \in S_3/(23)$</th>
<th>Schubert cell $C^1_\sigma$</th>
<th>Schubert variety $X^1_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Id, $(23)$</td>
<td>$pt: V_1 = E_1$</td>
<td>$pt: V_1 = E_1$</td>
</tr>
<tr>
<td>$(12), (132)$</td>
<td>$C^1: V_1 \neq E_1, V_1 \subset E_2$</td>
<td>$\mathbb{P}^1: V_1 \subset E_2$</td>
</tr>
<tr>
<td>$(123), (13)$</td>
<td>$C^2: V_1 \not\subset E_2$</td>
<td>$\mathbb{P}^2$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$\sigma \in S_3/(12)$</th>
<th>Schubert cell $C^2_\sigma$</th>
<th>Schubert variety $X^2_\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Id, $(12)$</td>
<td>$pt: V_2 = E_2$</td>
<td>$pt: V_2 = E_2$</td>
</tr>
<tr>
<td>$(23), (132)$</td>
<td>$C^1: V_2 \neq E_2, E_1 \subset V_2$</td>
<td>$\mathbb{P}^1: E_1 \subset V_2$</td>
</tr>
<tr>
<td>$(123), (13)$</td>
<td>$C^2: E_1 \not\subset V_2$</td>
<td>$\mathbb{P}^2 \vee$</td>
</tr>
</tbody>
</table>
2.2. Geometry of flag varieties

Grassmanians

For this section we follow [14]. Let $G = GL_n$, $T$ and $B$ be the standard torus and the standard Borel subgroup of $GL_n$ consisting of diagonal and the upper triangular matrices, respectively. The latter are the matrices which preserve the standard flag

$$0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \mathbb{C}^n$$

obtained from the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$.

**Definition 2.2.8.** For any $0 < d < n$ define the subgroup $P_d \subset GL_n$ by

$$P_d := \{ A \in GL_n \mid A(E_d) \subseteq E_d \}.$$  

The subgroups $P_1, \ldots, P_{n-1}$ are the maximal parabolic subgroups containing $B$. Any two matrices which send $E_d$ to the same $d$-dimensional subspace $V_d \subset \mathbb{C}^n$ differ by an element of $P_d$, so the right cosets of $P_d$ are the sets of all matrices which send $E_d$ to the same $V_d$. Thus the homogeneous space $G/P_d$ can be identified with:

**Definition 2.2.9.** The Grassmanian $Gr(d, n)$ is the set of vector subspaces of $\mathbb{C}^n$ of dimension $d$:

$$Gr(d, n) := \{ V_d \subset \mathbb{C}^n \mid \text{dim}(V_d) = d \}.$$  

The following gives $Gr(d, n)$ natural structure of a smooth projective variety:

**Definition 2.2.10.** The Plücker embedding is the map

$$\text{Gr}(d, n) \rightarrow \mathbb{P}(\bigwedge^d \mathbb{C}^n)$$

defined by

$$\langle v_1, \ldots, v_n \rangle \mapsto [v_1 \wedge \cdots \wedge v_n].$$

The intersection of $P_d$ with the Weyl group $W = S_n$ is the subgroup $S_d \times S_{n-d} \subset S_n$ which consists of all the permutations in $S_n$ which preserve the subset $\{1, \ldots, d\}$. The right cosets of $S_d \times S_{n-d}$ in $S_n$ consist therefore of all the permutations which send $\{1, \ldots, d\}$ to some fixed subset of $\{1, \ldots, n\}$. We therefore identify the elements of $W/(P_d \cap W)$, the set which indexes the generalised Bruhat cells of $P_d$, with size $d$ subsets

$$I = \{i_1 < i_2 < \cdots < i_d\} \subset \{1, \ldots, n\}.$$  

Any $\sigma \in S_n$ which lies in such $I$ sends $E_d$ to the subspace

$$E_I = \langle e_{j_1}, \ldots, e_{j_d} \rangle.$$  

(2.15)

The corresponding Schubert cell $C_I$ is the orbit of $E_I$ under the left action of $B$, and hence the orbit of $E_I$ under the action of its unipotent subgroup $U \simeq \mathbb{C}^{n(n-1)}$. The
The stabiliser $\text{Stab}_U(E_I)$ are the matrices with $u_{ij} = 0$ if $j \in I$ and $i \notin I$. This gives $\sum i_k - k$ conditions in total, whence $C_I \simeq U/\text{Stab}_U(E_I) \simeq \mathbb{C}^{\sum i_k - k}$.

Any $I$ as above is uniquely determined by $\dim(E_I \cap E_j)$ for all the subspaces $E_j$ of the standard flag. Since the left action of $B$ preserves the standard flag, we have

$$C_I = \{ V_d \in Gr(d, n) \mid \dim(V_d \cap E_j) = a_j \text{ for all } 1 \leq j \leq n \},$$

where $a_j$ is the number of $i_k$ with $i_k \leq j$. The closure of $V_d \cap E_i = a_j$ in $Gr(d, n)$ is $V_d \cap E_i \geq a_j$. Since $a_{k_i} = k$, the Schubert variety $X_I$, the closure of $C_I$, is given by

$$X_I = \{ V_d \in Gr(d, n) \mid \dim(V_d \cap E_{i_k}) = k \text{ for all } i_k \in I \}. $$

Since $C_I$ is irreducible, $X_I$ is an irreducible subvariety of $Gr(d, n)$.

**Example 2.2.11.** Since the condition $\dim(V_d \cap E_d) = d$ implies $E_d = V_d$, the Schubert variety $S_{1,...,d}$ is the point $E_d \in Gr(d, n)$.

**Example 2.2.12.** Since by dimension considerations $\dim(V_d \cap E_j) \leq d + j - n$, the Schubert variety $S_{n-d+1,...,n}$ is the whole of Grassmanian $Gr(d, n)$.

**Example 2.2.13.** The Schubert variety $X_{n-d,n-d+2,...,n}$ consists of those $V_d$ whose intersection with $E_{n-d}$ is non-zero, i.e. whose projection onto $E_{n-d+1,...,n}$ is non-invertible. Thus $X_{n-d,n-d+2,...,n}$ is the hyperplane section $p_{n-d+1,...,n} = 0$ of the Plücker embedding of $Gr(d, n)$. On the other hand, $\dim(V_d \cap E_{n-d+j}) \leq j$ so if $V_d \cap E_{n-d} = 0$, we must have $\dim(V_d \cap E_{n-d+j}) = j$, i.e. $V_d \in C_{n-d+1,...,n}$. Thus

$$Gr(d, n) = C_{n-d+1,...,n} \bigsqcup X_{n-d,n-d+2,...,n}. \tag{2.16}$$

**Proposition 2.2.14.** The Picard group $\text{Pic}(Gr(d, n))$ of the Grassmannian $Gr(d, n)$ is freely generated by $\mathcal{O}(X_{n-d,n-d+2,...,n})$.

**Proof.** This follows from the decomposition (2.16) given that $C_{n-d+1,...,n} \cong \mathbb{C}^{d(n-d)}$ and $X_{n-d,n-d+2,...,n}$ is irreducible.

More generally we have

**Proposition 2.2.15.** The classes of the Schubert varieties $X_I$ give an additive basis of the cohomology ring $H^*(Gr(d, n), \mathbb{Z})$.

### Flag varieties

As before, let $G = GL_n$, let $T$ and $B$ be the subgroups of diagonal and the upper triangular matrices, and let $E_\bullet$ be the standard flag

$$0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \mathbb{C}^n$$

obtained from the standard basis $\{ e_1, \ldots, e_n \}$ of $\mathbb{C}^n$. 

2.2. Geometry of flag varieties

**Definition 2.2.16.** Let \( \lambda = (\lambda_1, \ldots, \lambda_m) \) be an ordered partition of \( n: \sum \lambda_i = n \). Define \( P_\lambda \subseteq G \) to be the subgroup of all the matrices which preserve the partial coordinate flag \( E^\lambda_\bullet \):

\[
0 \subset E_{\lambda_1} \subset E_{\lambda_1+\lambda_2} \subset \cdots \subset E_{\lambda_1+\cdots+\lambda_m} = \mathbb{C}^n.
\]  

\[\text{Equation (2.17)}\]

It is the subgroup of block upper triangular matrices with diagonal blocks of sizes \( \lambda_1, \ldots, \lambda_k \).

The groups \( P_\lambda \) are the standard parabolic groups of \( G \). Any two matrices which send \( E^\lambda_\bullet \) to the same partial flag \( V^\lambda_\bullet \) differ by an element of \( P_\lambda \), so the right cosets of \( P_\lambda \) are sets of all matrices which send \( E^\lambda_\bullet \) to some fixed \( V^\lambda_\bullet \). This identifies the homogeneous space \( G/P_\lambda \) with:

**Definition 2.2.17.** The **partial flag variety** \( \text{Fl}_n(\lambda) \) is the set of all partial flags

\[
0 \subset V_{\lambda_1} \subset V_{\lambda_1+\lambda_2} \subset \cdots \subset V_{\lambda_1+\cdots+\lambda_m} = \mathbb{C}^n
\]

\[\text{Equation (2.18)}\]

with \( \dim(V_i) = i \).

The following gives \( \text{Fl}_n(\lambda) \) the structure of a smooth projective variety of dimension \( \sum_{i<j} \lambda_i \lambda_j \):

\[
i_\lambda : \text{Fl}_n(\lambda) \hookrightarrow \prod_{i=1}^k \text{Gr}(\lambda_1 + \cdots + \lambda_i, n).
\]

\[\text{Equation (2.19)}\]

**Example 2.2.18.** \( \text{Fl}_n(1, n-1) \) is the projective space \( \mathbb{P}^{n-1} \) of lines in \( \mathbb{C}^n \).

**Example 2.2.19.** \( \text{Fl}_n(n-1, 1) \) is the dual projective space \( \mathbb{P}^{(n-1)\vee} \) of hyperplanes in \( \mathbb{C}^n \).

**Example 2.2.20.** \( \text{Fl}(d, n-d) \) is the Grassmanian \( \text{Gr}(d, n) \).

**Example 2.2.21.** \( \text{Fl}(1, n-2, 1) \) is the incidence variety of lines \( V_1 \) contained in the hyperplanes \( V_{n-1} \) of \( \mathbb{C}^n \). It is a projective subvariety of \( \mathbb{P}^{n-1} \times \mathbb{P}^{n-1\vee} \) carved out by the equation

\[
x_1y_1 + \cdots + x_ny_n = 0
\]

where \( x_i \) and \( y_i \) are the standard \( i \)-th coordinate respectively of \( \mathbb{C}^n \) and \( \mathbb{C}^{n*} \).

**Remark 2.2.22.** For every partition \( \lambda \) the quotient map \( G/B \to G/P_\lambda \) is the forgetful morphism

\[
\pi_\lambda : \text{Fl}_n \to \text{Fl}_n(\lambda)
\]

\[\text{Equation (2.20)}\]

which sends each complete flag to its corresponding partial subflag. It is therefore a flat fibration whose fibers are isomorphic to the partial flag space of the flag type complementary to \( \lambda \).
Chapter 2. Springer resolutions and flag varieties

The intersection of $P_\lambda$ with the Weyl group $W = S_n$ is the subgroup $\prod_i S_{\lambda_i}$ of all the permutations which respect the partition $\lambda$. Its right cosets can therefore be identified with partitions of the set $\{1, \ldots, n\}$ into the subsets of size $\lambda_i$. More precisely, define $S^\lambda_n$ to be the subset of $S_n$ consisting of permutations which are order preserving on each block of the partition $\lambda$. The permutations $\sigma \in S^\lambda_n$ uniquely represent all the cosets in $S_n/\prod_i S_{\lambda_i}$.

For each $\sigma \in S^\lambda_n$ the corresponding Schubert cell $C_\sigma$ is the $B$-orbit of the partial flag $\sigma(E^{\lambda}_{\bullet})$, where $\sigma(E_k) = \langle e_{\sigma(1)}, \ldots, e_{\sigma(k)} \rangle$. The Schubert variety $X_\sigma$ is its Zariski closure.

Example 2.2.23. The Schubert variety $X_{\text{Id}}$ is the single point $E_{\bullet} \in \text{Fl}_n$.

Example 2.2.24. Let $\rho$ be the order reversing permutation $(n \ n-1 \ldots 1)$. The Schubert variety $X_\rho$ is the whole of $\text{Fl}_n$.

Example 2.2.25. Let $\tau_i$ be the transposition $(i \ i+1)$. The Schubert variety $X_{\tau_i} \subset \text{Fl}_n$ consists of the flags $V_{\bullet}$ with all $V_j = E_j$ except for $V_i$. Thus it can be identified with $\mathbb{P}^1$ of choices of $i$-dimensional space $V_i$ with $E_{i-1} \subset W_i \subset E_{i+1}$.

Example 2.2.26. The Schubert variety $S_{\rho \tau_1}$ is a divisor in $\text{Fl}_n(\lambda)$.

Example 2.2.27. Let $\lambda$ be a partition of $n$ and let $\sigma \in S^\lambda_n$. The inverse image of the Schubert variety $X_\sigma \subset \text{Fl}_n(\lambda)$ under the map

$$\pi_\lambda : \text{Fl}_n \to \text{Fl}_n(\lambda)$$

is the Schubert variety $S_{\sigma \tilde{\sigma}} \subset \text{Fl}_n$, where $\tilde{\sigma}$ is the Bruhat maximal element of $\prod_i S_{\lambda_i}$, i.e. the product of the order reversing permutations. This allows to reduce some questions about partial flag varieties to the study of complete flag varieties.

Proposition 2.2.28. The Picard group $\text{Pic}(\text{Fl})$ of the complete flag variety $\text{Fl}$ is freely generated by the line bundles of the Schubert divisors $X_{\rho \tau_1}$, $X_{\rho \tau_2}$, $\ldots$, $X_{\rho \tau_n}$.

Proof. Proposition 1.4.1 of [14].

More generally:

Proposition 2.2.29. Let $\lambda$ be a partition of $n$. The classes of Schubert varieties $X_\sigma$ give an additive basis of the cohomology ring $H^*(\text{Fl}_n(\lambda), \mathbb{Z})$.

Remark 2.2.30. A nilpotent operator $\alpha$ preserving the flag $0 \subset V_1 \subset \cdots \subset \mathbb{C}^n$ has to satisfy the condition

$$\alpha(V_i) \subset V_{i-1},$$

therefore Proposition 2.1.22 and Remark 2.1.23 give us the following description of the total space of $T^* \text{Fl}_n$:

$$T^* \text{Fl}_n := \{(V_{\bullet}, \alpha) \mid \alpha : \mathbb{C}^n \to \mathbb{C}^n; \alpha(V_i) \subset V_{i-1} \}.$$
We use the following pictorial shorthand to denote such pairs $(V_\bullet, \alpha)$:

$$0 \subset V_1 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^n.$$ 

Analogously (see section 1.2 of [20]), the total space of $T^* \text{Fl}_n(\lambda)$ can be described as the space of pairs

$$T^* \text{Fl}_n(\lambda) \cong \left\{ 0 \subset V_{\lambda_1} \subset \ldots \subset V_{\lambda_k} \subset \mathbb{C}^n \mid \text{dim}(V_{\lambda_i}) = \sum_{j=1}^{i} \lambda_j \right\}.$$
— Chapter 3 —

Autoequivalences and braid group actions of derived categories

In this chapter, we focus on equivalences and autoequivalences of derived categories of coherent sheaves of reduced schemes of finite type and their relations with the braid groups.

Braid groups actions on derived categories occur in many different contexts. In this chapter, we describe the braid group action arising from an \( A_n \) configuration of spherical objects on a smooth projective variety constructed by Seidel and Thomas in [40] and the braid group action on the cotangent bundle of complete flag variety constructed by Khovanov and Thomas in [32].

In this thesis we extend the latter to a generalised braid category action on the derived categories of coherent sheaves of the cotangent bundle of partial flag varieties; in Chapter 4 we will formulate this statement precisely and we prove it in its first non-trivial instance.

The first section of this chapter is a quick introduction to the language of DG-categories, twisted complexes and DG-enhancements.

In the second section, we introduce the notion of Fourier-Mukai transforms and give some results on standard kernels and the adjunction unit and counit maps between them.

In the third section, we introduce autoequivalences of bounded derived categories of coherent sheaves over smooth projective varieties, discuss some examples and give the results of Bondal and Orlov for the case of Fano and general type varieties in [11].

The fourth section is about the spherical objects and their twists: introduced by Seidel and Thomas they were the first example of genuinely derived autoequivalences of the derived category of coherent sheaves and they could be used to contract a braid group categorical action.

In section five we discuss a generalisation of the spherical object and their twists. Anno and Logvinenko in [6] introduced the notion of spherical functors. Given a scheme \( X \), any \( \mathcal{E} \in D^b(X) \) can be considered as the functor \((-) \otimes \mathcal{E} : D^b(pt) \to D^b(X)\), spherical functors are analogues of spherical objects where the point is replaced with a scheme \( Z \). A spherical functor \( F : D^b(Z) \to D^b(X) \) induces two
autoequivalences: the twists $T_F$ of $D^b(X)$ and the cotwist $C_F$ of $D^b(X)$.

In section six, we discuss the other generalisation of spherical objects, the $\mathbb{P}^n$-objects due to Huybrechts and Thomas in [30].

Section seven covers the theory of $\mathbb{P}^n$-functors which unifies and generalises the notion of spherical functors and $\mathbb{P}^n$-objects due to Anno and Logvinenko in [7].

In section eight we focus on Mukai Flops in derived categories and on the example of the flop \(\{T^*\mathbb{P}^2 \to T^*\mathbb{P}^2\}\) which will appear in the generalised braid action we construct in the case $n = 3$.

Section nine contains some technical results on the excess bundle formula which computes the derived tensor product of two structure sheaves of two smooth subvarieties of a smooth variety.

In section ten, we define categorical group actions and we describe the Khovanov-Thomas braid group action on the total space of the cotangent bundle of a complete flag variety.

### 3.1 DG-categories and DG-enhancements

In this section we introduce DG-categories and DG-enhancements of triangulated categories, following [6].

We give the basic definitions and we will refer to [45] and [6] for further details.

Let for all this section $R$ be a commutative ring.

**Definition 3.1.1.** A DG-category is a category $\mathcal{C}$ such that for every two objects $A, B \in \text{Ob}(\mathcal{C})$ the morphism space $\text{Hom}_\mathcal{C}^\bullet(A, B)$ is a complex of $R$-modules and such that the composition map

\[
\text{Hom}_\mathcal{C}^\bullet(B, C) \otimes \text{Hom}_\mathcal{C}^\bullet(A, B) \to \text{Hom}_\mathcal{C}^\bullet(A, C)
\]  

(3.1)

is map of complexes of $R$-modules.

**Definition 3.1.2.** Let $\mathcal{C}$ be a DG-category, the homotopy category $H^0(\mathcal{C})$ is the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are given by $\text{Hom}_{H^0(\mathcal{C})}(A, B) = H^0(\text{Hom}_\mathcal{C}^\bullet(A, B))$.

**Example 3.1.3.** The DG-category $\text{Mod} - R$ is the category of complexes of $R$-modules with morphisms complex $\text{Hom}_{\text{Mod} - R}(M, N)$ defined by

\[
\text{Hom}_{\text{Mod} - R}^n(M, N) := \bigoplus_{i+j=n} \text{Hom}_R(M_i, N_j)
\]  

(3.2)

and the differential which sends $f \in \text{Hom}_{\text{Mod} - R}^n(M, N)$ to

\[
df := d_N \circ f - (-1)^n f \circ d_M.
\]  

(3.3)
3.1. DG-categories and DG-enhancements

**Definition 3.1.4.** Let $C$ be a DG-category. The opposite DG-category $C^{\text{opp}}$ is DG-category whose objects are those of $C$, whose morphisms are

$$\text{Hom}_{C^{\text{opp}}}^\bullet(A, B) := \text{Hom}_C^\bullet(B, A)$$

with the composition defined for every $\phi \in \text{Hom}_{C^{\text{opp}}}^\bullet(A, B)$ and $\psi \in \text{Hom}_{C^{\text{opp}}}^\bullet(B, C)$

$$\psi \circ_{C^{\text{opp}}} \phi := (-1)^{\deg(\phi)\deg(\psi)} \phi \circ_C \psi.$$

**Definition 3.1.5.** Let $C_1$ and $C_2$ be two DG-categories. A DG-functor $F$

$$F : C_1 \longrightarrow C_2$$

is a $R$-linear functor which commutes with the differentials of the morphism complexes and preserves the grading.

A natural transformation $t : F_1 \rightarrow F_2$ of degree $n$ between two DG-functors from $C_1$ to $C_2$ is a collection of morphisms

$$\{ t(A) \in \text{Hom}_{C_2}^n(F_1(A), F_2(A)) \}_{A \in \text{Ob}(C_1)}$$

such that for every morphism $\phi \in \text{Hom}_{C_1}^m(A, B)$ the following equivalence hold

$$t(B) \circ F_1(\phi) = (-1)^{nm} F_2(\phi) \circ t(B).$$

**Example 3.1.6.** Let $C_1$ and $C_2$ be two DG-categories. The DG-category $DG - \text{Fun}(C_1, C_2)$ is the category whose objects are the DG-functor from $C_1$ to $C_2$ and whose morphism complexes have as $n$th graded part all the natural transformations of degree $n$.

The grading is determined by the degree of the natural transformations, while the differentials and the composition are defined in $C_2$ for each $A \in \text{Ob}(C_1)$.

**Example 3.1.7.** Let $C$ be a DG-category. The category $\text{Mod} - C$ is the DG-category $DG - \text{Fun}(C^{\text{opp}}, \text{Mod} - R)$: its objects are called (right) $C$-modules.

**Definition 3.1.8.** Let $C$ be a DG-category. A twisted complex $(A^\bullet, \phi_{\bullet, \bullet})$ over $C$ is a collection

$$(A^\bullet, \phi_{\bullet, \bullet}) := \{ A_i, \phi_{i,j} : A_i \rightarrow A_j \}_{(i,j \in \mathbb{Z})}$$

where $A_i \in \text{Ob}(C)$, $A_i \neq 0$ for only a finite number of indexes and

$$\phi_{i,j} \in \text{Hom}_C^{i-j+1}(A_i, A_j)$$

satisfying

$$(-1)^j d\phi_{i,j} + \sum_k \phi_{k,j} \circ \phi_{i,k} = 0.$$
Remark 3.1.9. The twisted complexes over a DG-category $C$ have a natural structure of a DG-category with morphism complexes defined by

$$\text{Hom}^p((A^\bullet, \phi^\bullet, \psi^\bullet), (B^\bullet, \psi^\bullet, \phi^\bullet)) := \Pi_{q+m-n=p} \text{Hom}^q_C(A_n, B_m)$$

\[ (3.6) \]

with the differential which sends every $\alpha \in \text{Hom}^q_C(A_n, B_m)$ to

$$d\alpha = (-1)^m dC_\alpha + \sum_{k \in \mathbb{Z}} (\psi_{m,k} \circ \alpha - (-1)^{q+m-n} \alpha \circ \phi_{i,j}).$$

\[ (3.7) \]

Definition 3.1.10. Let $C$ be a DG-category, let $(A^\bullet, \phi^\bullet)$ be a twisted complex over $C$ and let $\oplus_i A_i[-i]$ be the $C$-module where we implicitly use the Yoneda embedding to consider $A_i$ as objects in $\text{Mod} - C$.

The convolution $\{A^\bullet, \phi^\bullet\}$ of $(A^\bullet, \phi^\bullet)$ is the $C$-module $\oplus_i A_i[-i]$ with the modified differential

$$d_{\text{conv}} := d_{\text{old}} + \sum_{n,m \in \mathbb{Z}} \phi_{n,m}.$$ 

\[ (3.8) \]

Definition 3.1.11. A DG-category $C$ is pretriangulated if its image under the Yoneda embedding is a triangulated subcategory of $H^0(\text{Mod} - C)$.

Definition 3.1.12. A functor $F : C_1 \rightarrow C_2$ between DG-categories is a quasi-equivalence if induces quasi-isomorphisms on morphism complexes and if the functor $H^0(F)$

$$H^0(F) : H^0(C_1) \rightarrow H^0(C_2)$$

is an equivalence.

Definition 3.1.13. An enhancement for a triangulated category $D$ is a couple $(C, \theta)$, where $C$ is a pretriangulated DG-category and

$$\theta : H^0(C) \xrightarrow{\cong} T$$

is an exact equivalence.

Two enhancements $(C_1, \theta_1)$ and $(C_2, \theta_2)$ are equivalent if there exists a quasi-equivalence

$$F : C_1 \rightarrow C_2.$$

Definition 3.1.14. Let $Ho(DG-Cat)$ be the localisation of the category $DG-Cat$ of small DG-categories by quasi-equivalences constructed using Tabuada’s model structure on $DG-Cat$ (See [43] for more details). Let $C_1$ and $C_2$ two DG-categories, define the set $[C_1, C_2]$ of quasi-functors to be the set of morphism from $C_1$ to $C_2$ in $Ho(DG-Cat)$.

Remark 3.1.15. The set $[C_1, C_2]$ is naturally bijective to the set of isomorphism classes in $D^{r-qr}(C_1 - C_2)$, the derived category of right quasi-representable bimodules. The standard DG-enhancements of the latter can thus be described as a DG-category of quasi-functors $C_1 \rightarrow C_2$, viewed as $R\text{Hom}(C_1, C_2)$. 

3.2. Fourier-Mukai kernels and their adjunction properties

In our work we will always work in the context of Fourier-Mukai transforms. By a fundamental result of Toën ([45]), if $C_1$ and $C_2$ are Karoubi-complete enhancements of $D^b(X)$ and $D^b(Y)$, where $X$ and $Y$ are separated schemes of finite type over a field, then $H^0(R\text{Hom}(C_1, C_2))$ is isomorphic to the subcategory of $D^b(X \times Y)$ formed by Fourier-Mukai kernels of functors $D^b(X) \to D^b(Y)$. Thus we can work with Fourier-Mukai kernels as DG-enhancements of quasi-functors.

3.2 Fourier-Mukai kernels and their adjunction properties

In this section we introduce Fourier-Mukai transforms and their behaviour under adjunction; as explained at the end of section 3.1, Fourier-Mukai trasforms are the DG-enhanceable functors between the derived categories of coherent sheaves over separated schemes of finite type.

We refer to [27] and [7] for a more general treatment.

Through this section let $k$ be an algebraically closed field and let $D_{\text{qcoh}}(-)$ the unbounded derived category of quasi-coherent sheaves.

**Definition 3.2.1.** Let $X$ and $Y$ be two separated schemes of finite type over $k$ and let $E \in D_{\text{qcoh}}(X \times Y)$. The Fourier-Mukai transform $\Phi_E$ with kernel $E$ is the functor

$$\Phi_E : D_{\text{qcoh}}(X) \longrightarrow D_{\text{qcoh}}(Y)$$

defined as

$$\Phi_E(-) = \pi_Y^* (E \otimes \pi_X^*(-))$$

where $\pi_X$ and $\pi_Y$ are the natural projections $X \times Y \to X, Y$.

Fourier-Mukai kernels admit the following composition operation:

**Definition 3.2.2.** Let $X, Y, Z$ be separated schemes of finite type over $k$. Let $E_1 \in D_{\text{qcoh}}(X \times Y)$ and let $E_2 \in D_{\text{qcoh}}(Y \times Z)$.

The composition of Fourier-Mukai kernels $E_1 \ast E_2$ is defined as

$$E_2 \ast E_1 = \pi_{13*}(\pi_{12}^*E_1 \otimes \pi_{23}^*E_2) \in D^b(X \times Z) \quad (3.10)$$

where $\pi_{12}, \pi_{23}$ and $\pi_{13}$ are the natural projections

$$
\begin{align*}
& X \times Y \times Z \\
& \pi_{12} \quad \pi_{13} \quad \pi_{23} \\
& X \times Y \quad X \times Z \quad Y \times Z
\end{align*}
$$

The composition of Fourier-Mukai kernels induces the composition of the corresponding Fourier-Mukai transforms.
Proposition 3.2.3. Let $X, Y, Z$ be separated schemes of finite type over $k$ and let $E_1 \in D_{qcoh}(X \times Y)$ and $E_2 \in D_{qcoh}(Y \times Z)$. Then we have the following isomorphism
\[ \Phi_{E_2} \circ \Phi_{E_1} \cong \Phi_{E_2 \star E_1} \quad (3.11) \]

Proof. Section 5.1 of [27] or [35].

In this thesis we always work with certain standard Fourier-Mukai kernels for direct image, inverse image, twisted inverse image and tensor product functors. The following results present some convenient expressions for the kernels involved: although we prove some of them in Chapter 4 here we refer to section 2.6 of [7], since the discussion is more systematic and more general.

Lemma 3.2.4 (Standard kernels). Let $X$ and $Y$ be separated schemes of finite type over $k$ and let $f : X \to Y$ be a map of separated schemes of finite type over $k$.

1. For any $E \in D^b(X)$, then the Fourier-Mukai kernel induces functor $E \otimes -$ \[ T_E = \pi_2^* E \otimes \Delta_* O_X \in D^b(X \times X). \quad (3.12) \]

2. The Fourier-Mukai kernel associated to the functor $f_*$ is the object \[ F_* = (\text{Id}_X \times f)_* \Delta_* O_X \in D^b(X \times Y) \quad (3.13) \]

3. The Fourier-Mukai kernel associated to the functor $f^*$ is the object \[ F^* = (\text{Id}_Y \times f)^* \Delta_* O_Y \in D^b(Y \times X) \quad (3.14) \]

4. If $f$ is perfect, the Fourier-Mukai kernel associated to the functor $f^!$ is the object \[ F^! = (\text{Id}_Y \times f)^! \Delta_* O_Y \in D^b(Y \times X) \quad (3.15) \]

5. If $f$ is perfect, the Fourier-Mukai kernel associated to the functor $f_! = f_* (\cdot \otimes f^!(O_Y))$ is the object \[ F_! = (\text{Id}_X \times f)_! \Delta_* O_X \in D^b(X \times Y) \quad (3.16) \]

Proof. Lemma 2.18 in [7].

Lemma 3.2.5. Let $f : X' \to X$ and $g : Y' \to Y$ be maps of separated schemes of finite type over $k$. Let $V \in D^b(X)$ and $W \in D^b(Y)$ and $K_1 \in D^b(X \times Y)$, $K_2 \in D^b(X' \times Y')$, then we have the following results:

1. The following isomorphism is functorial in $K_2$ \[ (f \times g)_* K_2 \cong G_* \star K_2 \star F^*. \quad (3.17) \]
2. The following isomorphism is functorial in $K_1$

$$(f \times g)^* K_1 \cong G^* K_1 \ast F_*.$$

(3.18)

3. If $f$ and $g$ are perfect maps, then the following isomorphism is functorial in $K_1$

$$(f \times g)^! K_1 \cong G^! K_1 \ast F^!.$$

(3.19)

4. If $f$ and $g$ are perfect maps, then the following isomorphism is functorial in $K_2$

$$(f \times g)_! K_1 \cong G_! K_1 \ast F_!.$$

(3.20)

5. The following isomorphism is functorial in $K_1$

$$\pi_X^* V \otimes \pi_Y^* W \otimes K_1 \cong T_V \ast K_1 \ast T_W$$

(3.21)

Proof. Lemma 2.19 in [7].

In general the functor from Fourier-Mukai kernels to Fourier-Mukai transforms is neither full neither faithful, but for the Counit of the adjunction

$$\varepsilon : \Phi_E \circ \Phi_E \to \text{Id}$$

(3.22)

it is possible to choose naturally a morphism

$$\epsilon : K \to \mathcal{O}_\Delta$$

such that it will be lifted to $\varepsilon$.

The following Lemma is a powerful tool for understanding adjunction unit and counit at level of Fourier-Mukai kernels.

**Lemma 3.2.6.** Let $f : X \to Y$ be a map of separated schemes of finite type over a field.

1. There is an isomorphism

$$F^* \ast F_* \simeq (\text{Id}_X \times f)^*(\text{Id}, f)_* \mathcal{O}_X$$

(3.23)

which identifies the adjunction counit $F^* \ast F_* \leadsto \text{Id}_X$ with the morphism

$$(\text{Id}_X \times f)^*(\text{Id}, f)_* \mathcal{O}_X \to \Delta_* \mathcal{O}_X$$

(3.24)

which is the base change map for the commutative square:

$$\begin{array}{ccc}
X & \xrightarrow{(\text{Id}, f)} & X \\
\downarrow \Delta & & \downarrow \Delta \\
X \times X & \xrightarrow{\text{Id}_X \times f} & X \times Y.
\end{array}$$

(3.25)
2. There is an isomorphism
\[ F^* \ast F_* \simeq (\text{Id}_X \times f)^*(f \times \text{Id}_Y)^* \Delta_* \mathcal{O}_Y \] (3.26)
which identifies the adjunction counit \( F^* \ast F_* \xrightarrow{\mu} \text{Id}_X \) with the composition
\[(\text{Id}_X \times f)^*(f \times \text{Id}_Y)^* \Delta_* \mathcal{O}_Y \xrightarrow{\sim} (\text{Id}_X \times f)^*(\text{Id}, f)_* \mathcal{O}_X \xrightarrow{\epsilon} i_* \mathcal{O}_{X \times Y \times X} \xrightarrow{\Delta_*} \mathcal{O}_X \] (3.27)
where the first map is the base change isomorphism for the Tor-independent fiber square at the bottom of the following commutative diagram:
\[
\begin{array}{ccc}
X & \xrightarrow{\Delta} & X \times_Y X \\
\downarrow{\pi_1} & & \downarrow{\text{Id}_X \times f} \\
X & \xrightarrow{(\text{Id}, f)} & X \times Y \\
\downarrow{f} & & \downarrow{f \times \text{Id}_Y} \\
Y & \xrightarrow{\Delta} & Y \times Y,
\end{array}
\] (3.28)
the second map is the base change map for its top fiber square, and the third map is the natural restriction of sheaves which is the image under \( i_* \) of the adjunction unit \( \mathcal{O}_{X \times Y \times X} \xrightarrow{\Delta_*} \mathcal{O}_X \times \mathcal{O}_X \times Y \).

3. We have an isomorphism
\[ F_* \ast F^* \simeq \Delta_* f_* \mathcal{O}_X \] (3.29)
which identifies the adjunction unit \( \text{Id}_Y \xrightarrow{\epsilon} F_* \ast F^* \) with the morphism
\[ \Delta_* \mathcal{O}_Y \xrightarrow{} \Delta_* f_* \mathcal{O}_X \] (3.30)
which is the image under \( \Delta_* \) of the adjunction unit for \((f^*, f_*)\).

4. If \( f \) is perfect and proper, we have an isomorphism
\[ F_* \ast F^! \simeq \Delta_* f_* f^! \mathcal{O}_Y \] (3.31)
which identifies the adjunction counit \( F_* \ast F^! \xrightarrow{\mu} \text{Id}_Y \) with the morphism
\[ \Delta_* f_* f^! \mathcal{O}_Y \xrightarrow{} \Delta_* \mathcal{O}_Y \] (3.32)
which is the image under \( \Delta_* \) of the adjunction counit for \((f_*, f^!)\).

5. If \( f \) is perfect and proper, we have an isomorphism
\[ F^! \ast F_* \simeq (\text{Id}_X \times f)^!(\text{Id}, f)_* \mathcal{O}_X \] (3.33)
which identifies the adjunction unit \( \text{Id}_X \xrightarrow{\epsilon} F^! \ast F_* \) with the morphism
\[ \Delta_* \mathcal{O}_X \xrightarrow{} (\text{Id}_X \times f)^!(\text{Id}, f)_* \mathcal{O}_X \] (3.34)
which is the twisted base change map \( \Delta_* \text{Id}^! \xrightarrow{} (\text{Id}_X \times f)^!(\text{Id}, f)_* \) for the commutative square (3.25).
6. If \( f \) is perfect and proper, we have an isomorphism
\[
F^! \star F_* \simeq (\text{Id}_X \times f)^! (f \times \text{Id}_Y)^* \Delta_* \mathcal{O}_Y
\]
(3.35)
which identifies the adjunction unit \( \text{Id}_X \lreg F^! \star F_* \) with the composition
\[
\Delta_* \mathcal{O}_X \to i_* \pi^!_1 \mathcal{O}_X \to (\text{Id}_X \times f)^! (\text{Id}_f)_* \mathcal{O}_X \xrightarrow{\sim} (\text{Id}_X \times f)^! (f \times \text{Id}_Y)^* \Delta_* \mathcal{O}_Y
\]
(3.36)
where the first map is the image under \( i_* \) of the adjunction counit \( \Delta_* \Delta^! \pi^!_1 \mathcal{O}_X \to \pi^!_1 \mathcal{O}_X \), the second map is the twisted base change map for the top fiber square in (3.28), and the third map is the base change isomorphism for the Tor-independent bottom fiber square in (3.28).

**Proof.** Proposition 2.20 in [7].

### 3.3  Autoequivalences of derived categories

In order to study a mathematical object, one can study its decomposition into subobjects or study the transformations which preserve its structure. In the context of derived categories this means either studying semiorthogonal decompositions or studying derived autoequivalences.

We focus on the latter.

The group \( \text{Aut}(D^b(X)) \) of all isomorphism classes of autoequivalences of \( D^b(X) \)

\[ D^b(X) \]
can be used to investigate the structure of \( D^b(X) \).

The following examples of autoequivalences are usually referred to as "standard" as they are either induced by autoequivalences of the abelian category \( \text{Coh}(X) \) or are the shift functors which every triangulated category is equipped with.

**Example 3.3.1.** Any automorphism \( f : X \xrightarrow{\sim} X \) induces the autoequivalence \( f_* : D^b(X) \xrightarrow{\sim} D^b(X) \) and its inverse is given by \( f^* : D^b(X) \xrightarrow{\sim} D^b(X) \).

**Example 3.3.2.** For every line bundle \( \mathcal{L} \) on \( X \) the functor
\[
\mathcal{L} \otimes (-) : D^b(X) \to D^b(X)
\]
is an autoequivalence with inverse functor \( \mathcal{L}^{-1} \otimes (-) \).

**Example 3.3.3.** The shift functor \([n] \) is an autoequivalence of \( D^b(X) \) for every integer number; its inverse is the shift functor \([-n] \).

Do there exist any other autoequivalences of the bounded derived category of coherent sheaves on a smooth projective variety?

These would be genuinely derived in the sense of interacting non-trivially with the triangulated structure on \( D^b(X) \).

The following Theorem gives an answer when \( X \) is a Fano variety or it is a variety of general type.
Proposition 3.3.4 (Bondal, Orlov). Let $X$ be a smooth projective variety with ample (anti)-canonical bundle. The group of autoequivalences $\text{Aut}(D^b(X))$ of $D^b(X)$ is generated by

1. Derived pushforwards $f_*$ of automorphisms $f$ of $X$.
2. Shift functors $[n](-)$, for $n \in \mathbb{Z}$.
3. Twists $\mathcal{L} \otimes -$ by line bundles, $\mathcal{L} \in \text{Pic}(X)$.

Thus
\[
\text{Aut}(D^b(X)) \cong \mathbb{Z} \times (\text{Aut}(X) \ltimes \text{Pic}(X)).
\] (3.37)

Proof. Section 4.2 of [27] or [11].

When the canonical bundle is neither ample or anti-ample, for example in the Calabi-Yau case, the group $\text{Aut}(D^b(X))$ has a richer structure.

3.4 Spherical twists

Consider a compact symplectic manifold $(M, \beta)$ and a Lagrangian sphere $S$ inside $M$. In [39] Seidel associated to such $S$ a symplectic automorphism called the generalised Dehn twist along $S$.

In [40], Seidel and Thomas introduced the spherical twists, which are the analogues of the generalised Dehn twist under Homological Mirror Symmetry; these provided an early example of genuinely derived autoequivalences and were used to construct an example of braid group action.

For this section we refer to [40] and [27].

Let $X$ be smooth projective variety and let $D^b(X)$ be the full subcategory of the derived category of $\mathcal{O}_X$-modules consisting of complexes with bounded and coherent cohomology.

Definition 3.4.1. An object $\mathcal{E}$ in $D^b(X)$ is called spherical if:

1. $\text{Hom}_{D(X)}^r(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{C}, & \text{if } r = 0, \text{dim}(X), \\ 0, & \text{otherwise}. \end{cases}$

2. $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$, where $\omega_X$ is the canonical bundle.

Definition 3.4.2. Given two objects $\mathcal{F} \in D(X)$ and $\mathcal{G} \in D(Y)$ we define
\[
\mathcal{F} \boxtimes \mathcal{G} = \pi_1^*\mathcal{F} \otimes \pi_2^*\mathcal{G}.
\] (3.38)

Notice that the a bifunctor
\[
\pi_1^*(-) \otimes \pi_2^*(-) : \text{Coh}(X) \times \text{Coh}(Y) \to \text{Coh}(X \times Y)
\]
is exact in each of its two arguments.
3.4. Spherical twists

**Definition 3.4.3.** Let $E \in D^b(X)$ over the scheme $X$ and let $\Delta : X \to X \times X$ be the diagonal embedding. The twist functor of $T_E$ is the Fourier-Mukai transform $\Phi_P$ where

$$\mathcal{P} := \text{Cone}(E^\vee \boxtimes E \xrightarrow{\eta} O_\Delta).$$

(3.39)

where $\eta$ is the canonical pairing which sends

$$\bigoplus_{i-j=k} (\pi_1^*E_{[j]})^\vee \otimes \pi_2^*E_{[i]} \to 0 \quad \forall k \in \mathbb{Z}, k \neq 0$$

and

$$\bigoplus_{i-j=0} (\pi_1^*E_{[j]})^\vee \otimes \pi_2^*E_{[i]} \to O_X$$

by the usual canonical pairing of sheaves.

**Definition 3.4.4.** An $(A_m)$-configuration, $m \geq 1$, in $D^b(X)$ is a collection of $m$ spherical objects $E_1, \ldots, E_m$ such that

$$\dim \text{Hom}_{D^b(X)}^*(E_i, E_j) = \begin{cases} 1 & |i-j| = 1, \\ 0 & |i-j| \geq 2. \end{cases}$$

**Theorem 3.4.5 (Seidel, Thomas).** The twist $T_E$ along any spherical object $E$ is an autoequivalence of $D^b(X)$. Moreover, if $E_1, \ldots, E_m$ is an $(A_m)$-configuration, the twists $T_{E_i}$ satisfy the braid relations:

$$T_{E_i}T_{E_{i+1}}T_{E_i} \cong T_{E_{i+1}}T_{E_i}T_{E_{i+1}} \quad \text{for } i = 1, \ldots, m - 1,$$

$$T_{E_i}T_{E_j} \cong T_{E_j}T_{E_i} \quad \text{for } |i-j| \geq 2.$$

**Proof.** See [40] or Propositions 8.6 and 8.22 of [27].

**Remark 3.4.6.** The second part of Theorem 3.4.5 is an example of a braid group action on the derived category $D^b(X)$. See section 3.10 for details on categorical group actions.

**Example 3.4.7.** Let $C$ be a smooth projective curve and let $x \in C$ be a point on $C$. The skyscraper sheaf $O_x$ is a spherical object and

$$T_{O_x}(-) \cong O_C(x) \otimes (-).$$

**Example 3.4.8.** If $C$ is a smooth rational curve with $C^2 = -2$ in a smooth projective surface, then the structure sheaf $O_C$ is as spherical object.

**Example 3.4.9.** If $X$ be a Calabi-Yau variety, then any line bundle $L \in \text{Pic}(X)$ is a spherical object.
Example 3.4.10. If $C$ is a smooth projective curve contained in a Calabi-Yau threefold with normal bundle isomorphic to

$$
N_{C/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1),
$$

(3.40)

Then $\mathcal{O}_C$ is a spherical object.

Example 3.4.11. If $Y$ is a smooth projective subvariety of a Calabi-Yau variety $X$ of dimension $2n+1$, such that $Y \cong \mathbb{P}^n$ and the normal bundle of $Y$ is isomorphic to

$$
N_{Y/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \cdots \oplus \mathcal{O}(-1),
$$

(3.41)

then $\mathcal{O}_Y$ is a spherical object.

Remark 3.4.12. If $X$ is Fano or of general type variety then $\omega_X$ is ample or anti-ample. The condition

$$
\mathcal{E} \otimes \omega_X \cong \mathcal{E}
$$

(3.42)

forces the support of the spherical object to be 0-dimensional.

3.5 Spherical functors

Anno and Logvinenko in [6] introduced the notion of spherical functors. Given a scheme $X$, any $\mathcal{E} \in D^b(X)$ can be considered as the functor

$$
(-) \otimes \mathcal{E} : D^b(pt) \to D^b(X).
$$

Spherical functors are analogues of spherical objects where the point is replaced with a scheme $Z$.

A spherical functor $F : D^b(Z) \to D^b(X)$ induces two autoequivalences: the twist $T_F$ of $D^b(X)$ and the cotwist $C_F$ of $D^b(X)$.

Definition 3.5.1. Let $C_1$ and $C_2$ be enhanced triangulated categories and let

$$
F : D(C_1) \to D(C_2)
$$

be an enhanceable functor with enhanceable left and right adjoints

$$
L, R : D(C_2) \to D(C_1).
$$

The spherical twist $T : D(C_2) \to D(C_2)$ of $F$ is the enhanceable functor that fits in the natural exact triangle

$$
FR \xrightarrow{\text{tr}} Id \to T \xrightarrow{\epsilon} FR[1].
$$

(3.43)

The spherical cotwist $C : D(C_1) \to D(C_1)$ of $F$ is the enhanceable functor that fits in the natural exact triangle

$$
C \to Id \xrightarrow{\text{act}} RF \xrightarrow{\zeta} C[1].
$$

(3.44)

The functor $F$ is spherical if the following conditions hold:
3.5. Spherical functors

1. The twist $T$ is an autoequivalence of $D(C_2)$.

2. The cotwist $C$ is an autoequivalence of $D(C_1)$.

3. The following composition in an isomorphism:

$$LT \xrightarrow{Lt} LFR[1] \xrightarrow{trR} R[1]$$

4. The following composition is an isomorphism:

$$R \xrightarrow{Ract} RFL[1] \xrightarrow{cL} CL[1]$$

**Theorem 3.5.2.** Any two of the conditions 1-4 in the definition above imply all four.

*Proof.* Theorem 5.1 of [6].

**Remark 3.5.3.** By the argument in Lemma 5.16 in [7] if condition (2) of definition 3.5 holds, it is enough to show

$$R \simeq CL[1]$$

for condition (4) to hold.

**Example 3.5.4.** Let $Z = Spec(\mathbb{C})$, let $X$ be a smooth projective variety, let $E$ be an object in $D^b(X)$ and set $F$ to be the functor

$$F : D(Z) \xrightarrow{-\otimes E} D(X).$$

Then $E$ is a spherical object if and only if $F$ satisfies conditions 2 and 4 of Theorem 3.5.2, so if and only if $F$ is a spherical functor. (See example 3.5 of [5].)

**Example 3.5.5.** If $D$ is a divisor of an algebraic variety $X$ with inclusion map

$$i : D \hookrightarrow X$$

(3.45)

Then $F := i_*$ is a spherical functor.

Indeed, we have the adjunctions

$$L = i^* \dashv F \dashv R = i^!$$

(3.46)

The Fourier-Mukai kernels are:

- $i_* : \mathcal{O}_D \in D(D \times X)$;
- $i^* : \mathcal{O}_D \in D(X \times D)$;
- $i^! \simeq i^* \otimes \mathcal{O}(D)[-1] : \mathcal{O}_D(D)[-1] \in D(X \times D)$. 
By Proposition 3.2.3 the composition $RF$ has Fourier-Mukai kernel $O_D \oplus O_D(D)[-1].$

Therefore the twist $T = \text{Cone}(\text{Id} \to RF)$ has Fourier-Mukai kernel $O_D(D)[-1] \in D(D \times D)$ and 

$$CL \cong R.$$

**Example 3.5.6.** Let $X$ be a variety of dimension $n$, let $i : D \to X$ be a divisorial inclusion and let $\pi : D \to Y$ be a projective bundle of rank $k$ over the variety $Y$ of dimension $n - 1 - k.$

\[
\begin{array}{c}
D \\
\downarrow \pi \\
Y
\end{array} 
\quad \xrightarrow{i} \quad 
\begin{array}{c}
X
\end{array}
\]

Define the functor $F$

$$F := i_* \pi^* : D(Y) \to D(X).$$ (3.47)

The right adjoint $R$ of $F$ is the functor

$$R = \pi_* i^! : D(X) \to D(Y)$$ (3.48)

while the left adjoint is the functor $L$

$$L = \pi! i_* : D(X) \to D(Y).$$ (3.49)

The functor $F$ is spherical if and only if

$$N_{D/X} \cong \omega_{D/Y} \otimes \pi^* \mathcal{L}$$

for some $\mathcal{L} \in \text{Pic}(Y)$ (See [7]).

**3.6 $\mathbb{P}^n$ twists**

The notion of $\mathbb{P}^n$ twists was introduced by Huybrechts and Thomas in [30] as mirror Dehn twists of Lagrangian $\mathbb{C}\mathbb{P}^n$s.

When $n = 2$ the notion of a $\mathbb{P}^n$ twist coincides with the notion of the square of a spherical twist, but in higher dimensions, the two notions are different.

Let $\mathcal{C}$ a triangulated category and let

$$A \xrightarrow{f} B \xrightarrow{g} C$$ (3.50)

be a complex of objects in $\mathcal{C}$, so that $g \circ f = 0.$
Definition 3.6.1. A right Postnikov system of the complex 3.50 is a diagram

\[
\begin{array}{c}
A \\
\downarrow f \\
B \\
\downarrow j \\
\downarrow i \\
X \\
\downarrow h \\
C
\end{array}
\]

where \( B \to C \to X \) is a distinguished triangle and \( f = i \circ j \).

Definition 3.6.2. The convolution of the right Postnikov system (3.51) is the cone \( \text{Cone}(A[1] \to X) \).

Definition 3.6.3. A left Postnikov system of the complex (3.50) is a diagram

\[
\begin{array}{c}
A \\
\downarrow f \\
B \\
\downarrow l \\
\downarrow m \\
Y \\
\downarrow k \\
C
\end{array}
\]

where \( A \to B \to Y \) is a distinguished triangle and \( g = m \circ k \).

Definition 3.6.4. The convolution of the left Postnikov system (3.52) is the cone \( \text{Cone}(Y \to C) \).

Definition 3.6.5. An object \( E \in \mathcal{C} \) is a convolution of the complex (3.50) if it is a convolution of a right or a left Postnikov system associated to it.

Definition 3.6.6. Let \( D^b(X) \) be the bounded derived category of coherent sheaves on a smooth projective variety. An object \( E \in D^b(X) \) is a \( \mathbb{P}^n \)-object if:

1. \( \text{Hom}^r_{D^b(X)}(E, E) = \begin{cases} \mathbb{C}, & \text{if } r = 2i, i \in \{0, \ldots, n\} \\ 0, & \text{otherwise.} \end{cases} \)

2. \( E \otimes \omega_X \cong E \), where \( \omega_X \) is the canonical bundle.

Remark 3.6.7. Let \( E \) be a \( \mathbb{P}^n \) object in \( D^b(X) \) with \( \dim(X) = m \), then by Serre duality

\[
\text{Ext}^i(E, E) = \text{Ext}^{m-i}(E, E \otimes \omega_X)^* = \text{Ext}^{m-i}(E, E)^*
\]

that forces \( m = n \).

Example 3.6.8. Let \( X \) be an hyperkähler manifold of dimension \( 2n \), and take \( \mathbb{P}^n \hookrightarrow X \).

In this case \( N_{\mathbb{P}^n/X} \cong \Omega_{\mathbb{P}^n} \), hence \( \text{Ext}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) = \Omega^2_{\mathbb{P}^n} \).

Thus the local to global spectral sequence
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\[ E_2^{p,q} = H^p(X, \text{Ext}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})) \Rightarrow \text{Ext}_X^{p+q}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \]

gives the following isomorphism of rings

\[ \text{Ext}_X^*(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \cong H^*(\mathbb{P}^n, \Omega^*_{\mathbb{P}^n}) \cong H^*(\mathbb{P}^n, \mathcal{C}). \]

Thus \( \mathcal{O}_{\mathbb{P}^n} \) is a \( \mathbb{P}^n \)-object.

**Example 3.6.9.** Let \( X \) be a K3 surface and let \( C \) be a curve on it; we have that \( C \cong \mathbb{P}^1 \subset X \); so by example 3.6.8 \( \mathcal{O}_C \in D^b(X) \) is a \( \mathbb{P}^1 \) object.

Moreover, by Example 3.4.7 \( \mathcal{O}_C \) is also a spherical object, which agrees with \( S^2 \cong \mathbb{P}^1 \).

**Remark 3.6.10.** Suppose that \( \mathcal{E} \) is a \( \mathbb{P}^n \) object, then \( \text{Ext}^2(\mathcal{E}, \mathcal{E}) \) is one dimensional vector space let

\[ \phi : \mathcal{E}[-2] \rightarrow \mathcal{E} \]

We have \( \text{Ext}^2(\mathcal{E}, \mathcal{E}) \cong \text{Ext}^2(\mathcal{E}', \mathcal{E}') \), so just define \( \phi' \) to be the image of \( \phi \) under the group isomorphisms of the Exts; \( \phi' \) is a generator for \( \text{Ext}^2(\mathcal{E}', \mathcal{E}') \) and can be represented by a morphism

\[ \phi : \mathcal{E}'[-2] \rightarrow \mathcal{E}'. \]

Let \( \tilde{\phi} = \phi' \otimes \text{Id} - \text{Id} \otimes \phi \),

\[ \tilde{\phi} : \mathcal{E}' \otimes \mathcal{E}[-2] \rightarrow \mathcal{E}' \otimes \mathcal{E}. \]

The trace map \( tr : \mathcal{E}' \otimes \mathcal{E} \rightarrow \mathcal{O}_{\Delta} \) factorise throughout the cone \( \text{Cone}(\tilde{\phi}) \) of \( \tilde{\phi} \)

\[ \mathcal{E}' \otimes \mathcal{E} \xrightarrow{\text{tr}} \Delta \xrightarrow{\psi} \mathcal{O}_{\Delta} \]

**Definition 3.6.11.** Let \( \mathcal{E} \in D^b(X) \) be a \( \mathbb{P}^n \)-object and let \( P_\mathcal{E} = \text{Cone}(\psi) \). The \( \mathbb{P}^n \) twist \( T^\mathbb{P}_\mathcal{E} \) associated to \( \mathcal{E} \) is the Fourier-Mukai transform \( \Phi_{P_\mathcal{E}} \) with kernel \( P_\mathcal{E} \)

\[ T^\mathbb{P}_{\mathcal{E}} := \Phi_{P_\mathcal{E}} : D^b(X) \rightarrow D^b(X) \]

**Theorem 3.6.12.** If \( P \in D^b(X) \) is a \( \mathbb{P}^n \) object, then the \( \mathbb{P}^n \) twist \( T^\mathbb{P}_P \) is an autoequivalence of \( D^b(X) \).

**Proof.** See Proposition 2.6 of [30] or Proposition 8.19 of [27].
### 3.7 \( \mathbb{P}^n \) functors

The notion of \( \mathbb{P}^n \)-functor unifies and generalises those of spherical functors and of \( \mathbb{P}^n \)-objects. In its general form, it is due to Anno and Logvinenko, we refer the reader to [7] for all technical details.

**Definition 3.7.1.** Let \( C_1 \) and \( C_2 \) be enhanced triangulated categories and let

\[
F : C_1 \to C_2
\]

be an enhanceable functor with enhanceable left and right adjoints

\[
L, R : C_2 \to C_1.
\]

\( F \) is a \( \mathbb{P}^n \)-functor if it can be equipped with a triple \((H, Q_n, \gamma)\):

- \( H \) is an enhanced autoequivalence of \( C_1 \) such that \( H(Ker(F)) = Ker(F) \).
- \( Q_n \) is cyclic degree \( n \) coextension of \( Id \) by \( H \) of the form:

\[
\begin{array}{cccccccc}
\text{Id} & \overset{i_1}{\to} & Q_1 & \overset{i_2}{\to} & Q_2 & \cdots & Q_{n-2} & \overset{i_{n-1}}{\to} & Q_{n-1} & \overset{i_n}{\to} & Q_n \\
\text{H} & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots \\
\text{H}^2 & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots \\
\vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
\text{H}^n & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots & \overset{\ast}{\to} & \cdots \\
\end{array}
\]

Here all starred triangles are exact and all the remaining triangles are commutative. Let \( \iota = \iota_n \circ \cdots \circ \iota_1 \).

- The map \( \gamma \) is an isomorphism

\[
\gamma : Q_n \xrightarrow{\sim} RF
\]

that interweaves the adjunction unit \( \text{Id} \xrightarrow{\varepsilon} RF \) with the map \( \text{Id} \xrightarrow{\iota} Q_n \);

Note that as \( F \xrightarrow{F\varepsilon} FRF \) is a retract, so is \( F\iota \). Hence the exact triangle \( FR \xrightarrow{F\iota} FQ_1 R \to FHR \) is also split. Choose any splitting \( FHR \xrightarrow{\phi} FQ_1 R \) and denote by \( \phi \) the composition

\[
FHR \xrightarrow{\phi} FQ_1 R \xrightarrow{\iota_n \circ \cdots \circ \iota_2} FQ_n R \xrightarrow{F\gamma R} FRFR.
\]

(3.56)

Define the map \( FHR \xrightarrow{\psi} FR \) to be the composition

\[
FHR \xrightarrow{\phi} FRFR \xrightarrow{FR tr - tr FR} FR.
\]

Note that any choice of the splitting \( FHR \xrightarrow{\phi} FQ_1 R \) in the definition of \( \phi \) will produce the same map \( \psi \), since the composition \( (FR tr - tr FR) \circ F\varepsilon R \) is zero.

satisfying the following conditions:
1. The monad condition. The map
\[ v : FHQ_{n-1} \xrightarrow{FH_{n}} FHR \xrightarrow{\psi_F} FRF \xrightarrow{F\kappa} FC[1] \] (3.57)
is an isomorphism, where \( C \) is the spherical cotwist of \( F \) defined by the exact triangle \( \text{Id} \xrightarrow{\sim} RF \xrightarrow{\sim} C[1] \).

2. The adjoints condition. The map
\[ w : FR \xrightarrow{FR_{\varepsilon}} FRL \xrightarrow{F\mu} FH^nL \] (3.58)
is an isomorphism.

3. The highest degree term condition. There exists an isomorphism
\[ u : FH^nL \xrightarrow{\sim} FHH^nH'L \]
that makes the following diagram commutate:

\[
\begin{array}{ccc}
FHQ_{n-1}L & \xrightarrow{FH_{n}L} & FHRFL \\
\text{Id} & & \downarrow^{\sim} \\
FHQ_{n-1}L & \xrightarrow{FH_{n}L} & FHRFL \xrightarrow{FHR\psi'} RHRFH'L \xrightarrow{FHH^nH'L} FHH^nH'L
\end{array}
\] (3.59)

where \( H' \) is the inverse of \( H \) the map \( \psi' : FL \to FH' \) is the left dual of \( \psi : FHR \to FR \).

**Theorem 3.7.2.** Let \( F : C_1 \to C_2 \) equipped with the triple \( (H, Q_n, \gamma) \), as above, be a \( \mathbb{P}^n \)-functor.

Let the \( \mathbb{P} \)-twist \( P_F \) of \( F \) to be the unique convolution of the two-step complex
\[ FHR \xrightarrow{\psi} FR \xrightarrow{\text{tr}} \text{Id}, \] (3.60)
see [4] for the uniqueness of the convolution.

Then \( P_F \) is an autoequivalence of the category \( C_2 \).

**Proof.** See Theorem 4.1 in [7]. \( \square \)

**Remark 3.7.3.** Definition 3.7.1 is the more general version of the definition of a split \( \mathbb{P}^n \)-functor, an enhanced functor \( F \) equipped with an isomorphism
\[ RF \simeq \text{Id} \oplus H \ldots H^n \]
for which the following conditions hold:
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- Strong monad condition: The matrix \( A_l \) of the left multiplication by \( H \) in \( RF \) has the form

\[
\begin{pmatrix}
* & * & \ldots & * & * \\
1 & * & \ldots & * & * \\
0 & 1 & \ldots & * & * \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & * \\
\end{pmatrix},
\]

i.e., \( a_{kj} = 0 \) for \( k > j + 1 \) and \( a_{j+1,j} \) are identities for \( 0 \leq j < n \).

- Weak adjoints condition: There exists some isomorphism \( R \cong H^n L \).

**Example 3.7.4.** Let \( C_1 \) and \( C_2 \) be enhanced triangulated categories and let

\[
F : D(C_1) \to D(C_2)
\]

be a enhanceable functor with enhanceable left and right adjoints

\[
L, R : D(C_2) \to D(C_1).
\]

\( F \) is a spherical functor with twists \( T \) and cotwist \( C \) if and only if is a \( \mathbb{P}^1 \)-functor with

\[
H \cong C[1]
\]

and the degree 1 coextension of \( \text{Id} \) by \( H \) structure on \( RF \) defined by the exact triangle

\[
C \to \text{Id} \xrightarrow{act} RF \xrightarrow{c} C[1].
\]

(See Proposition 7.1 [7]).

**Example 3.7.5.** Let \( X \) be a smooth projective variety and let \( P \in D^b(X) \) be a \( \mathbb{P}^n \) object, let \( Z = \text{Spec}(\mathbb{C}) \) and consider the functor \( F \)

\[
F : D^b(Z) \xrightarrow{\otimes \mathbb{P}} D^b(X)
\]

with \( \mathbb{P} \in D^b(X) \cong D^b(Z \times X) \).

We have

\[
RF \cong \text{Hom}(\mathbb{P}, \mathbb{P}) \cong \mathbb{C} \oplus \mathbb{C}[-2] \oplus \cdots \oplus \mathbb{C}[-2n]
\]

which decomposes \( RF \) as

\[
RF \cong \text{Id} \oplus H \oplus \cdots \oplus H^n
\]

(3.64)

where \( H = [-2] \) and (3.64) gives \( F \) a structure of split \( \mathbb{P}^n \)-functor.

Moreover, from the weak adjoint condition we have that

\[
R \cong H^n L.
\]

(See [1] for details.)
Example 3.7.6. Let $X$ be the Hilbert scheme of $n$ points on a projective K3 surface $Z$ so that

$$X = Z^{[n]}$$

and define $F$ to be the functor

$$F : D^b(Z) \to D^b(X)$$

realised by the Fourier-Mukai transform with Fourier-Mukai kernel the universal ideal sheaf $\mathcal{I} \in D^b(Z \times X)$.

There exists an isomorphism

$$RF \cong \text{Id} \oplus H \oplus H^2 \oplus \cdots \oplus H^{n-1}$$

(3.65)

where $H = \text{Id}[-2]$ and (3.65) gives $F$ the structure of a split $\mathbb{P}^{n-1}$ functor.

(See Theorem 2 of [1] for details).

Example 3.7.7. Let $\mathcal{V}$ be a vector bundle on a smooth projective variety $Z$, with $\mathbb{P}^n$ fibration

$$\pi : \mathbb{P} \mathcal{V} \to Z$$

and embedding in a smooth projective variety $X$

$$i : \mathbb{P} \mathcal{V} \hookrightarrow X$$

with normal bundle of rank $n$ isomorphic to

$$\mathcal{N}_{\mathcal{P} \mathcal{V}/X} \cong \Omega^1_{\mathbb{P} \mathcal{V}/X}.$$

Define the functors $f_k$ to be

$$f_k := i_* \circ (\mathcal{O}_{\mathbb{P} \mathcal{V}}(k) \otimes \pi^*(-)) : D^b(Z) \to D^b(X).$$

(3.66)

let $r_k$ be their right adjoints and define

$$h := \text{Id}[-2] : D^b(Z) \to D^b(Z).$$

(3.67)

Let $F_k$, $R_k$, $L_K$ and $H$ be their standard enhancements.

There exist a structure of cyclic extension of degree $n$ of $\text{Id}$ by $H$ on the adjunction monad $R_k F_k$ that makes $F_k$ a $\mathbb{P}^n$-functor (see Theorem 7.2 in [7]).

3.8 Mukai flops in derived categories

For the first part of this section we refer to section 11.4 of [27], for the second part we refer to section 5 of [3].

Let $X$ be a smooth projective variety of even dimension and let $P$ be a smooth subvariety of $X$ with dimension half the dimension of $X$. 
Let moreover $P$ be isomorphic to $\mathbb{P}^n$, with

$$\mathcal{N}_{P/X} \cong \Omega_P. \quad (3.68)$$

Let $\tilde{X} = Bl_P(X)$ be the blow up of $X$ along $P$ with projective morphism

$$q : \tilde{X} \rightarrow X$$

and exceptional divisor $E \cong \mathbb{P}(\mathcal{N}_{P/X}) \cong \mathbb{P}(\Omega_P)$.

If we consider $P \cong \mathbb{P}(V)$, with $V$ an $n+1$ dimensional complex vector space, then from the Euler sequence

$$0 \rightarrow \Omega_P \rightarrow V^* \otimes \mathcal{O}(-1) \rightarrow \mathcal{O}_P \rightarrow 0 \quad (3.69)$$

we have that

$$\mathbb{P}(\Omega_P) \hookrightarrow \mathbb{P}(V^* \otimes \mathcal{O}(-1)) \cong \mathbb{P}(V^*) \times \mathbb{P}(V)$$

The exceptional divisor $E \cong \mathbb{P}$ inside $P \times P^\vee \cong \mathbb{P}(V^*) \times \mathbb{P}(V)$ is the incidence variety of pairs $(l, H)$ of lines $l \subset V$ and planes $H \subset V$ such that $l \subset H$.

Indeed, the fiber $\Omega_P(l)$ over a line $[l] \subset P$ is naturally isomorphic to the space of linear functions $\alpha : V \rightarrow \mathbb{C}$ which vanish on $l$.

Therefore, because $E \in |\mathcal{O}(-1,1)|$, from the adjunction formula we deduce that the canonical bundle of the exceptional divisor

$$\omega_E \cong \mathcal{O}(-n,-n)|_E \quad (3.70)$$

Moreover, using the adjunction formula for $E \subset \tilde{X}$, we have that the same canonical bundle is isomorphic to

$$\omega_E \cong (\omega_{\tilde{X}} \otimes \mathcal{O}(E))|_E \cong (q^* \omega_X \otimes \mathcal{O}((n-1)E))|_E \otimes \mathcal{O}_E(E) \cong \pi^*(\omega_X|_P) \otimes \mathcal{O}_E(nE)$$

where $\pi$ is the natural map $\pi : E \rightarrow P$.

Since by assumption $\omega_X|_P$ is the trivial bundle, we have that $\omega_E \cong \mathcal{O}(-1,-1)$ and conclude that

$$\mathcal{O}_E(E) \cong \mathcal{O}(-1,-1) \quad (3.71)$$

The previous isomorphism, ensure that exists a birational morphism

$$p : \tilde{X} \rightarrow X'$$

which restricted to $E$ is the second projection $E \subset P \times P^\vee \rightarrow P^\vee$ and away from $E$ is an isomorphism (by the Fujiki-Nakano criterion).

Moreover,

$$\mathcal{N}_{P^\vee/X'} \cong \Omega_{P^\vee} \text{ and } \omega_{X'}|_{P^\vee} \cong \mathcal{O}_{P^\vee}$$
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Proposition 3.8.1. The Fourier-Mukai transform induced by the composition of the functors

\[ p_* \circ q^* : D^b(X) \to D^b(X') \quad (3.72) \]

is not fully faithful.

Proof. Proposition 11.28 of [27]. \qed

Theorem 3.8.2. Given the reduced subvariety \( \tilde{X} \cup (P \times P^\vee) \) inside \( X \times X' \), and the object \( \mathcal{O}_{\tilde{X} \cup (P \times P^\vee)} \in D^b(X \times X') \), the Fourier-Mukai transform with kernel \( \mathcal{O}_{\tilde{X} \cup (P \times P^\vee)} \)

\[ \Phi_{\mathcal{O}_{\tilde{X} \cup (P \times P^\vee)}} : D^b(X) \to D^b(X') \quad (3.73) \]

is an equivalence.

Proof. Proposition 11.28 of [27] or [31] and [36]. \qed

Consider the three dimensional complex vector space \( \mathbb{C}^3 \) and the three dimensional flag variety \( Fl_3 \), with the natural projection maps to \( \mathbb{P}^2 \) and \( \mathbb{P}^2 \star \)

\[ Fl_3 \]

\[ p_1 \quad \downarrow \quad p_2 \]

\[ \mathbb{P}^2 \quad \mathbb{P}^2 \star \]

Let \( C \) be the total space of the cotangent bundle of \( Fl_3 \)

\[ C = T^* Fl_3. \]

Let \( A \) and \( E \) be the quasi-projective varieties defined as the total space of the cotangent bundle of \( \mathbb{P}^2 \) and \( \mathbb{P}^2 \star \):

\[ A = T^* \mathbb{P}^2 \quad E = T^* \mathbb{P}^2 \star. \]

Let \( B \) and \( D \) be the quasi-projective varieties defined as the total space of the pullback via \( p_1 \) and \( p_2 \) on \( Fl_3 \) of the cotangent bundles of \( \mathbb{P}^2 \) and \( \mathbb{P}^2 \star \):

\[ B = p_1^* T^* \mathbb{P}^2 \quad E = p_2^* T^* \mathbb{P}^2 \star. \]
Then we have the following diagram

\[
\begin{array}{c}
B & \xleftarrow{i_B} & C & \xleftarrow{i_D} & D \\
\downarrow{\pi_A} & & \downarrow{\pi_E} & & \\
A & & E & & \\
\end{array}
\]

(3.74)

where \(i_B, i_D\) are divisorial inclusions and \(\pi_A, \pi_E\) are \(\mathbb{P}^1\)-bundles respectively on \(A\) and \(E\).

From remark 2.2.30, we have the following descriptions of the quasi-projective varieties

\[
A = \left\{ \begin{array}{c}
\alpha \\
0 \subset V_1 \subset \mathbb{C}^3; \dim(V_1) = 1
\end{array} \right\},
\]

and

\[
E = \left\{ \begin{array}{c}
\alpha \\
0 \subset V_2 \subset \mathbb{C}^3; \dim(V_2) = 2
\end{array} \right\}.
\]

Moreover the four dimensional subvariety of \(C\) defined as the (transverse) intersection of \(B\) and \(D\) can be described as

\[
B \cap D = \left\{ \begin{array}{c}
\alpha \\
0 \subset V_1 \subset V_2 \subset \mathbb{C}^3
\end{array} \right\}.
\]

equipped with the two natural forgetful maps

\[
\begin{array}{c}
B \cap D \\
\downarrow{q_1} \downarrow{q_2} \\
A & & E
\end{array}
\]

where \(q_i\) is the map that forget the choice of the \(n-i\)-th space.

Both \(q_1\) and \(q_2\) are isomorphic to the blow up of the zero section carved out by \(\{\alpha = 0\}\) in respectively \(A\) and \(E\).

Both the blow-ups have the same exceptional divisor \(Fl_3\) carved out by \(\{\alpha = 0\}\) and the resulting birational transformations

\[
q_2 \circ q_1^{-1}: A \dashrightarrow E
\]
are a local model of a four dimensional Mukai flop.

Therefore, from Theorem 3.8.2 we have the following

**Corollary 3.8.3.** Given the reduced subvariety $(B \cap D) \cup (\mathbb{P}^2 \times \mathbb{P}^2)$ inside $E \times A$, and the object $O_{(B \cap D) \cup (\mathbb{P}^2 \times \mathbb{P}^2)} \in D^b(E \times A)$, the Fourier-Mukai transform with kernel $O_{(B \cap D) \cup (\mathbb{P}^2 \times \mathbb{P}^2)}$:

$$
\Phi_{O_{(B \cap D) \cup (\mathbb{P}^2 \times \mathbb{P}^2)}} : D^b(E) \longrightarrow D^b(A)
$$

is an equivalence.

### 3.9 The excess bundle formula

This section is on the excess bundle formula which computes the derived tensor product of two structure sheaves of two smooth subvarieties of a smooth variety: we refer to [38] and [7] for more details.

Let $Z_1, Z_2$ be two locally complete intersection subvarieties of a smooth algebraic variety $Z$ with their intersection $W = Z_1 \cap Z_2$ being a locally complete subvariety of $Z_1$ and $Z_2$.

Let $i_1, i_2, j_1, j_2$ be the inclusion of the following fiber square

$$
\begin{array}{c}
Z_1 \cap Z_2 = W \\
| \\
| \\
\downarrow j_1 \\
Z_1
\end{array}
\quad
\begin{array}{c}
\downarrow j_2 \\
Z_2
\end{array}
\quad
\begin{array}{c}
\uparrow i_1 \\
Z
\end{array}
\quad
\begin{array}{c}
\uparrow i_2 \\
Z
\end{array}
$$

**Definition 3.9.1.** The excess bundle $\mathcal{E}_W$ of the intersection $W = Z_1 \cap Z_2$, is the locally free sheaf which fits in the short exact sequence of sheaves on $W$

$$
0 \longrightarrow \mathcal{N}_{W/Z} \longrightarrow j_1^*\mathcal{N}_{Z_1/Z} \oplus j_2^*\mathcal{N}_{Z_2/Z} \longrightarrow \mathcal{E}_W \longrightarrow 0
$$

(3.76)

**Theorem 3.9.2.** Under the previous hypothesis, the cohomology sheaves of

$$
i_2^*i_1^*O_{Z_1} \in D^b(Z_2)
$$

are

$$
H^{-q}(i_2^*i_1^*O_{Z_1}) = j_1^*(\bigwedge^q \mathcal{E}_W^*)
$$

(3.77)

**Proof.** See [38].

Definition 3.9.3. The intersection $W = Z_1 \cap Z_2$ of two locally complete intersection subvarieties $Z_1, Z_2$ of a smooth algebraic variety $Z$ is called transverse if both the following conditions are satisfied:

1. $W$ is a smooth subvariety of $Z$.
2. $\text{codim}_Z(W) = \text{codim}_Z(Z_1) + \text{codim}_Z(Z_2)$.

Corollary 3.9.4. If the intersection $W = Z_1 \cap Z_2$ is transverse then we have the following isomorphism

$$O_{Z_1} \otimes O_{Z_2} \cong O_W. \quad (3.78)$$

3.10 A braid group action on $D^b(T^*F_{l_n})$

In this section, we present the Khovanov and Thomas braid group action on the derived category of coherent sheaves of the cotangent bundle of complete flag varieties $D^b(T^*F_{l_n})$: we refer to [32] for more details.

Definition 3.10.1. An action of a group $G$ on a category $\mathcal{C}$ is an assignment of an invertible functor $F_g : \mathcal{C} \rightarrow \mathcal{C}$ to each $g \in G$ such that $F_g \circ F_h \cong F_{g \cdot h}$. Moreover, the following diagram has to be commutative

$$
\begin{array}{ccc}
F_f \circ F_g \circ F_h & \cong & F_f \circ F_{g \cdot h} \\
\downarrow & & \downarrow \\
F_{f \cdot g} \circ F_h & \cong & F_{f \cdot g \cdot h}
\end{array}
$$

Remark 3.10.2. The $n$-braid group $\text{Br}_n$ is generated by the elements $\{t_1, \ldots, t_n\}$ subjects to the relations

$$
\begin{cases}
t_i t_j = t_j t_i & \text{if } |i - j| > 1 \quad \text{"commutation"} \\
t_i t_j t_i = t_j t_i t_j & \text{if } |i - j| = 1 \quad \text{"braiding".}
\end{cases}
$$

Recall that the the complete flag variety is defined

$$F_{l_n} = \left\{ 0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V \mid \dim(V_i) = i \right\}$$

and the total space of its cotangent bundle can be described as the space
Chapter 3. Autoequivalences and braid group actions of derived categories

\[\mathbb{T}^*\mathrm{Fl}_n \cong \left\{ \begin{array}{c}
0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V ; \quad \dim(V_i) = i
\end{array} \right\}.\]

Denote by \(C = \mathbb{T}^*\mathrm{Fl}_n(1, \ldots, 1)\) the total space of the cotangent bundle on the complete flag and by \(A_i = \mathbb{T}^*\mathrm{Fl}_n(\hat{i})\) the total space of the cotangent bundle on the flag where the choice of \(V_i\) is skipped.

Let the map \(p_i : \mathrm{Fl}_n \to \mathrm{Fl}_n(\hat{i})\) be the projection which just skip the choice of \(V_i\) containing a \(V_{i-1}\) and contained in \(V_{i+1}\), therefore a \(\mathbb{P}^1\) bundle over \(\mathrm{Fl}_n(\hat{i})\).

Denote \(B_i = p_i^*\mathbb{T}^*\mathrm{Fl}_n(\hat{i})\) the total space of the pullback via the morphism \(p_i\) on \(\mathrm{Fl}_n\) of the cotangent bundle of \(\mathrm{Fl}_n(\hat{i})\).

As before, \(B_i\) can be described as the space

\[\left\{ \begin{array}{c}
0 \subset V_1 \subset \ldots \subset V_{i-1} \subset V_i \subset V_{i+1} \subset \ldots \subset V ; \quad \dim(V_i) = i
\end{array} \right\}.\]

For every integer \(i \in \{1, \ldots, n\}\) we have a couple of natural maps \(j_i\) and \(\pi_i\)

\[B_i \xrightarrow{j_i} C \quad \pi_i \downarrow \quad \begin{array}{c} \downarrow \pi_i \end{array} \quad A_i\]

which are respectively the divisorial inclusion of \(B_i\) in \(C\) and the canonical projection of \(B_i\) onto \(A_i\).

**Lemma 3.10.3.** Let \(\mathcal{V}_i\) be the pullback on \(\mathrm{Fl}_n\) of the tautological bundle of \(\mathrm{Gr}(i,n)\). The normal bundle \(\mathcal{O}_{B_i}\) of the divisor \(B_i\) inside \(C\) is isomorphic to

\[\mathcal{O}(B_i) \cong (\Lambda^i V_i^*)^{-2} \otimes \Lambda^{i-1} V_{i-1}^* \otimes \Lambda^{i+1} V_{i+1}^*\]

**Proof.** Section 4.1 of [32].

For every \(i \in \{1, \ldots, n-1\}\), define the functors

\[F_i = j_i^* \circ p_i^* : D^b(A_i) \to D^b(C)\]
3.10. A braid group action on $D^b(T^*Fl_n)$

and denote their right adjoints by

$$R_i = p_{i*} \circ j_i^! : D^b(C) \to D^b(A_i)$$

Since we are in the hypothesis of example 3.5.6 $F_i$ are spherical functors and their spherical twists are autoequivalences.

Moreover, since $B_i$ is a divisor inside $C$ the right adjoint of the functor $j_i^!$ is the functor $j_{i*}(-) \otimes \mathcal{O}(-B_i)$, so by Lemma 3.10.3 we have the isomorphism

$$j_{i*}(p_i^*(-) \otimes \omega_{p_i})) \otimes \mathcal{O}(-B_i)[2] \cong j_{i*}p_i^*[2]$$

thus, the following adjuctions hold

$$F_i \dashv R_i \dashv F_i[2]$$

(3.80)

with the respective counit and unit maps

$$\varepsilon_i : F_iR_i \to Id \quad \epsilon_i : Id[-2] \to F_iR_i.$$  

and, therefore, the cotwist around $F_i$ is $[2]$.

Define the functors

$$T_i = \text{Cone}(F_iR_i \xrightarrow{\varepsilon_i} Id) \quad T'_i = \text{Cone}(Id \xrightarrow{\epsilon_i} F_iR_i).$$

(3.81)

**Theorem 3.10.4.** The functors $T_i$ defined in 3.81 are autoequivalences of $D^b(T^*Fl_n)$ with respective inverses $T'_i$, i.e.

$$T'_i \circ T_i \cong Id \cong T_i \circ T'_i$$

Moreover the $n - 1$ autoequivalences $T_i$ satisfy the braid relations:

$$T_iT_j \cong T_jT_i \quad \text{for } |i - j| \geq 1.$$  

$$T_iT_jT_i \cong T_jT_iT_j \quad \text{for } |i - j| = 1.$$  

Thus, there is a categorical action of the braid group $B_n$ on $D^b(T^*Fl_n)$.

**Proof.** Theorem 4.1 in [32].

To conclude this section we prove two results on the canonical bundle of the quasi-projective varieties $C$, $B_i$ and $A_i$.

**Proposition 3.10.5.** Let $X$ be a smooth projective variety and let $p : E \to X$ be the cotangent vector bundle $\Omega_X$. Let $Y = \text{Spec}(\text{Sym}(E^*))$ be the total space of $E$, then the canonical bundle of $\omega_Y$ is trivial.
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Proof. Consider the projection $\pi$

$$\pi : Y \longrightarrow X,$$

then the relative tangent bundle $T_{Y/X}$ of $Y$ over $X$ is isomorphic to

$$T_{Y/X} \cong \pi^* E$$

thus the relative canonical bundle $\omega_{Y_X}$ is isomorphic to

$$\omega_{Y_X} \cong \pi^* \det(E).$$

Therefore the canonical bundle $\omega_Y$ of the total space of $E$ is isomorphic to

$$\omega_Y \cong \pi^* \omega_X \otimes \omega_{Y/X} \cong \pi^*(\omega_X \otimes \det(T_X^*)) \cong \pi^* \mathcal{O}_X \cong \mathcal{O}_Y$$

since the dual commutes with the determinant. 

As a corollary of Proposition 3.10.5, the canonical bundle of $C$ and $A_i$ is trivial for every $i$.

**Proposition 3.10.6.** The canonical bundle of $B_i$ is

$$\omega_{B_i} \cong \mathcal{O}_{B_i}(-B_i) \cong (\Lambda^i \mathcal{V}_i^*)^2 \otimes (\Lambda^{i-1} \mathcal{V}_{i-1}^*)^{-1} \otimes (\Lambda^{i+1} \mathcal{V}_{i+1}^*)^{-1} \quad (3.82)$$

Proof. Using adjunction formula and Proposition 3.10.5 we have that

$$\omega_{B_i} \cong \mathcal{O}_{B_i}(-B_i)$$

and from Lemma 3.10.3 follows that

$$\omega_{B_i} \cong (\Lambda^i \mathcal{V}_i^*)^2 \otimes (\Lambda^{i-1} \mathcal{V}_{i-1}^*)^{-1} \otimes (\Lambda^{i+1} \mathcal{V}_{i+1}^*)^{-1}.$$
Chapter 4

Categorical action of generalised braids

Braids are topological configurations of \( n \) disjoint pieces of string with \( n \) fixed endpoints, considered up to isotopies which keep the strands disjoint. In [32] Khovanov and Thomas constructed a categorical action of the braid group \( \text{Br}_n \) on the derived category \( D(T^* \text{Fl}_n) \) of coherent sheaves on the cotangent bundle of the variety \( \text{Fl}_n \) of the complete flags in \( \mathbb{C}^n \). The configuration of \( n \) distinct fixed endpoints is represented by \( D(T^* \text{Fl}_n) \) and a braid starting and ending at such configuration is represented by an auto-equivalence of this category.

In this chapter we describe a more general structure: \( \mathcal{GB}_{r_n} \), the \textit{generalised braid category} on \( n \)-strands. Our goal is to study its categorical representations. The definition of \( \mathcal{GB}_{r_n} \) resembles that of the category \( \text{Web}_n \) of \( \mathfrak{s}\mathfrak{l}_n \)-webs ([19], [37]), but unlike the latter category, \( \mathcal{GB}_{r_n} \) is not additive. Like in \( \text{Web}_n \), the objects of \( \mathcal{GB}_{r_n} \) are ordered partitions \( \vec{i} = (i_1, \ldots, i_k) \) of \( n \) and the morphisms are generated by certain elementary diagrams modulo relations. The key difference is that \( \text{Web}_n \) is enriched over \( \mathbb{Z}[q, q^{-1}] \), and some relations are additive. The category \( \mathcal{GB}_{r_n} \) is topological in nature, and its relations stem from isotopies. Our main interest, however, lies in \textit{skein triangulated representations} of \( \mathcal{GB}_{r_n} \) where we impose triangulated relations on functors that conjecturally categorify those of \( \text{Web}_n \).

In particular, it is expected that there is a certain skein-triangulated action of \( \mathcal{GB}_{r_n} \) on the derived categories of coherent sheaves of the cotangent bundles of \( \text{Fl}_n(\vec{i}) \), the varieties of complete and partial flags in \( \mathbb{C}^n \). This action consists of a network of functors between these derived categories, some of which are well-known in geometric representation theory: the Khovanov-Thomas braid group action [32] comprises a limited subset of the endofunctors of the full flag variety, a single node in the network. The Cautis-Kamnitzer-Licata categorical \( \mathfrak{s}\mathfrak{l}_2(\mathbb{C}) \)-action [18] comprises a limited subset of our functors between the Grassmanians, a few of the nodes in the network. To obtain the Cautis-Kamnitzer tangle calculus [17], on the other hand, we restrict a part of our network to a small slice of each flag variety, the resolution of \((n,n)\)-Slodowy slice. On these slices, the generalised braid relations simplify and become the tangle calculus.

In Section 1, we define the generalised braid category \( \mathcal{GB}_{r_n} \). In Section 2, we define the notion of a skein-triangulated representation of \( \mathcal{GB}_{r_3} \) and give the conditions on fork generator functors which allow us to construct such representation out of them. In Section 3, we introduce the setup for a generalised braid action.
on $D^b(T^* \text{Fl}_3)$, we define the merge and the fork functors and compute their compositions at level of Fourier-Mukai kernels. In Section 4, we describe a conjectural program which allows us to inductively compute the multiple crossing functors and verify it for $\mathcal{GB}_3$. In Section 5, we construct the conjectured skein triangulated representation of $\mathcal{GB}_3$ on $D(T^*(\text{Fl}_3(\bar{i})))$.

4.1 Generalised braid category

Intuitively, generalised braids should be thought of as braids where we remove the restriction that the strands are not allowed to touch each other. They can now come together, continue as a strand with multiplicity, and then possibly split apart:

Any two strands with multiplicities $p$ and $q$ can join up and continue as a strand with multiplicity $p + q$. Any strand with multiplicity $p + q$ can split up into two strands with multiplicities $p$ and $q$. Instead of a single configuration of $n$ disjoint endpoints, we have multiple configurations indexed by the ordered partitions of $n$. Finally, we want to consider the generalised braids up to isotopies which preserve the intervals on which strands come together. An isotopy can make such interval shorter or longer, but can’t make it vanish completely or join two such intervals into one.

This intuitive idea needs to be coarsened: we do not want to distinguish individual strands within a strand with multiplicity. One approach would be to take the definition above, and factorise by the action of the permutation group $S_p$ on each multiplicity $p$ strand which permutes the individual strands within it.

We take another approach, that of embedded trivalent graphs:

Definition 4.1.1. The generalised braid category $\mathcal{GB}_n$ is the category with:

- Objects: ordered partitions of $n$:

$$\left\{ \vec{i} = (i_1 \ldots i_k) \mid \sum_{s=1}^{k} i_s = n \right\}.$$

- Morphisms: The morphisms

$$\vec{i} = (i_1 \ldots i_k) \rightarrow \vec{j} = (j_1 \ldots j_l)$$
4.1. Generalised braid category

are the generalised braids with startpoint/endpoint configurations \( \bar{i} \) and \( \bar{j} \). Such braid is an oriented graph with edges colored by integers from 1 to \( n \). We refer to the colors as the multiplicities. The graph must have:

- \( k \) ordered 1-valent startpoint vertices with the emerging edges of multiplicities \( i_1, \ldots, i_k \),
- \( l \) ordered 1-valent endpoint vertices with the terminating edges of multiplicities \( j_1, \ldots, j_l \),
- The remaining vertices are trivalent with the flow condition respected: the total multiplicity of the terminating edges equals that of the emerging edges.

We consider this oriented graph together with an embedding into \( \mathbb{R}^2 \times [0,1] \), which satisfies:

- The startpoint vertices are \((1,0,0), (2,0,0), \ldots, (k,0,0)\).
- The endpoint vertices are \((1,0,1), (2,0,1), \ldots, (l,0,1)\).
- The orientation at any point must project positively onto \([0,1]\).

We considered these generalised braids up to equivalence generated by two types of relations. One is the isotopy of embedded trivalent graphs. The other is the multifork and multimerge relations, where we identify the graphs which can be obtained one from another by the modification:

\[
\begin{align*}
\begin{array}{c}
\begin{tikzpicture}
\node (p) at (-2,0) {$p$};
\node (q) at (-1,1) {$q$};
\node (r) at (0,1) {$r$};
\node (pq) at (1,.5) {$p+q$};
\node (pqr) at (1,0) {$p+q+r$};
\draw[purple,thick,->] (p) -- (q);
\draw[blue,thick,->] (q) -- (pq);
\draw[blue,thick,->] (pq) -- (pqr);
\end{tikzpicture}
\end{array}
& \Rightarrow \\
\begin{array}{c}
\begin{tikzpicture}
\node (p) at (-2,0) {$p$};
\node (q) at (-1,1) {$q$};
\node (r) at (0,1) {$r$};
\node (pq) at (1,.5) {$p+q$};
\node (pqr) at (1,0) {$p+q+r$};
\draw[purple,thick,->] (p) -- (q);
\draw[blue,thick,->] (q) -- (pq);
\draw[blue,thick,->] (pq) -- (pqr);
\end{tikzpicture}
\end{array}
\end{align*}
\]

and its obvious analogue for merge vertices.

- **The composition:** The composition is given by concatenation of graphs.
- **Identity morphisms:** The identity morphism from \( \bar{i} = (i_1, \ldots, i_k) \) to itself is the graph which consists of \( k \) vertical edges: from the \( j \)-th startpoint vertex to the \( j \)-th endpoint vertex for all \( l \leq j \leq k \).
4.2 Skein-triangulated representations of $\mathcal{GB}r_3$

General notation

Let $\vec{i} = (i_1, \ldots, i_k)$ denote a partition of $n$. The general principle is that the source partition and the target partition are denoted by the subscript and the superscript, respectively. Specifically, if $\vec{i}$ and $\vec{j}$ allow a fork between them, let $f_{\vec{i}}^{\vec{j}}$ denote this unique fork. Similarly, let $g_{\vec{i}}^{\vec{j}}$ denote the unique merge and when $\vec{i} \neq \vec{j}$, let $t_{\vec{i}}^{\vec{j}}$ and $d_{\vec{i}}^{\vec{j}}$ denote the unique positive and the negative crossings. When $\vec{i} = \vec{j}$, there is an ambiguity, so we overscore the subscript indices which correspond to the strands being crossed, e.g. $t_{1\overline{1}11}^{1\overline{1}1}$ or $t_{1\overline{1}11}^{1\overline{1}1}$.

The generators of $\mathcal{GB}r_3$

The generalised braid category $\mathcal{GB}r_3$ has the following generators:

1. Four forks:

   \begin{align*}
   f_{3}^{12} & \\
   f_{3}^{21} & \\
   f_{12}^{111} & \\
   f_{21}^{111} &
   \end{align*}

   Figure 4.1: Forks

2. Four merges:

   \begin{align*}
   g_{12}^{3} & \\
   g_{21}^{3} & \\
   g_{111}^{12} & \\
   g_{111}^{21} &
   \end{align*}

   Figure 4.2: Merges

3. Two positive and two negative $(1,1)$-crossings:

4. Two positive and two negative $(2,1)$- and $(1,2)$-crossings:
Inspired by the results in [8], we formulate the following conjecture.

**Conjecture 4.2.1.** All the relations between these generators in $GBr_3$ can be obtained from the following four basic relations via three operations: vertical reflection (swap the source and the target partitions, change all forks into merges and vice versa, reverse the parity of the crossings); horizontal reflection (swap the partitions $12$ and $21$, reverse the parity of the crossings); and blackboard reflection (reverse the parity of the crossings).

1. Multifork relation: $f_{21}^{111} f_{3}^{111} = f_{12}^{111} f_{3}^{12}$.  

   ![Figure 4.5: The Multifork relation](image)

2. The braid relation: $t_{111}^{111} t_{111}^{111} t_{111}^{111} = t_{111}^{111} t_{111}^{111} t_{111}^{111}$.  

3. Inverses relations: $t_{21}^{12} d_{12}^{21} = id_{12}$, $d_{12}^{21} t_{21}^{12} = id_{21}$, $t_{111}^{111} d_{111}^{111} = t_{111}^{111} d_{111}^{111} = id_{111}$.  

4. The pitchfork relation: $f_{12}^{111} t_{21}^{12} = t_{111}^{111} t_{111}^{111} f_{21}^{111}$.  

Remark 4.2.2. The category $\mathcal{GBr}_2$ has a simpler structure; indeed, the generators of $\mathcal{GBr}_2$ are $f^{11}_2$, $g^{22}_{11}$, $t^{11}_{11}$, $d^{11}_{11}$ and the only relation between them is
\[ t^{11}_{11}d^{11}_{11} = \text{Id} = d^{11}_{11}t^{11}_{11}. \] (4.2)
4.2. Skein-triangulated representations of $\mathcal{GB}_r$

In this section, we describe a special type of categorical representations of $\mathcal{GB}_r$ which we call *skein-triangulated*. In these representations, $\mathcal{GB}_r$ acts on enhanced triangulated categories and certain additional relations are satisfied which make use of the triangulated structure of the target categories.

Let $\mathcal{C}_3$, $\mathcal{C}_{12}$, $\mathcal{C}_{21}$, and $\mathcal{C}_{111}$ be enhanced triangulated categories. Let

\[
F^{111}_{21}: D(\mathcal{C}_{21}) \to D(\mathcal{C}_{111}),
\]

\[
F^{111}_{12}: D(\mathcal{C}_{12}) \to D(\mathcal{C}_{111}),
\]

\[
F^{21}_{3}: D(\mathcal{C}_3) \to D(\mathcal{C}_{21}),
\]

\[
F^{12}_{3}: D(\mathcal{C}_3) \to D(\mathcal{C}_{12}),
\]

be enhanced functors with enhanced 2-categorical left and right adjoints. Denote the left and right adjoints of each $F^j_i$ by $L^i_j$ and $R^i_j$.

Assume that Conjecture 4.2.1 holds and assume moreover that

1. $(1,1)$-forks $F^{111}_{12}$ and $F^{111}_{21}$ are split spherical functors with cotwist $[-2]$.
2. $(1,2)$-fork $F^{12}_{3}$ and $(2,1)$-fork $F^{21}_{3}$ are split $\mathbb{P}^2$-functors with $H = [-2]$.
3. There exists a multifork isomorphism $\alpha: F^{111}_{21}F^{21}_{3} \cong F^{111}_{12}F^{12}_{3}$.
4. There exist isomorphisms

\[
R^{3}_{12}R^{12}_{111}F^{111}_{12}F^{12}_{3} \simeq \text{Id}_3 \oplus [-2] \oplus [-2] \oplus [-4] \oplus [-4] \oplus [-6] \simeq R^{3}_{21}R^{21}_{111}F^{111}_{21}F^{21}_{3},
\]

which together with $\mathbb{P}^2$-functor structures on $F^{12}_{3}$ and $F^{21}_{3}$ identify the maps

\[
R^{3}_{12}F^{12}_{3} \xrightarrow{R^{3}_{12}\text{ act } F^{12}_{3}} R^{3}_{12}R^{12}_{111}F^{111}_{12}F^{12}_{3},
\]

\[
R^{3}_{21}F^{21}_{3} \xrightarrow{R^{3}_{21}\text{ act } F^{21}_{3}} R^{3}_{21}R^{21}_{111}F^{111}_{21}F^{21}_{3},
\]

with the maps

\[
\text{Id}_3 \oplus [-2] \oplus [-4] \to \text{Id}_3 \oplus [-2] \oplus [-2] \oplus [-4] \oplus [-4] \oplus [-6]
\]

which are the direct summand embeddings whose images are $\text{Id}_3 \oplus$ the first $[-2] \oplus$ the first$[-4]$, and $\text{Id}_3 \oplus$ the second $[-2] \oplus$ the second $[-4]$, respectively.
5. The following diagram can be completed to an exact triangle in $D(C_{12}-C_{12})$:

$$F_3^{12} R_{12}^3 \rightarrow R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} \rightarrow \text{Id}_{12}[-2]. \quad (4.10)$$

Here the first map is the composition

$$F_3^{12} R_{12}^3 \rightarrow R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111},$$

and the second map is the composition

$$R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} \simeq L_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} [-2] \xrightarrow{\text{tr}[2]} \text{Id}_{12}[-2]. \quad (4.11)$$

6. The following diagram can be completed to an exact triangle in $D(C_{21}-C_{21})$:

$$F_3^{21} R_{21}^3 \rightarrow R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111} \rightarrow \text{Id}_{21}[-2]. \quad (4.13)$$
4.2. Skein-triangulated representations of $\mathcal{GBr}_3$

Its two maps are defined analogously to (4.11) and (4.12):

\[
F_3^{21} R_{21}^{3} \\
\act F_3^{21} R_{21}^{3} \act
\]
\[
R_{111}^{21} F_{21}^{111} F_3^{21} R_{21}^{3} R_{111}^{21} F_3^{111} \\
\sim\text{multifork}
\]
\[
R_{111}^{21} F_{12}^{111} F_3^{12} R_{12}^{12} R_{111}^{21} F_2^{111} \\
\flat R_{111}^{21} F_{12}^{111} \flat R_{111}^{21} F_2^{111}
\]
\[
R_{111}^{21} F_{12}^{111} R_{111}^{12} F_2^{111}
\]

and

\[
R_{111}^{21} F_{12}^{111} R_{111}^{12} F_2^{111} \simeq R_{111}^{21} F_{12}^{111} R_{111}^{12} F_2^{111} \simeq R_{111}^{21} F_{12}^{111} \simeq R_{111}^{21} F_{12}^{111} \simeq R_{111}^{21} F_{21}^{111} \simeq L_{111}^{21} F_{12}^{111} R_{111}^{12} F_2^{111} \rightarrow \text{Id}_{21}[-2].
\]

\[
R_{111}^{21} F_{12}^{111} R_{111}^{12} F_2^{111} \simeq L_{111}^{21} F_{12}^{111} R_{111}^{12} F_2^{111} \rightarrow \text{Id}_{21}[-2].
\]

\[\text{Theorem 4.2.3.} \quad \text{Under the assumptions above, the following assignments define a categorical action of } \mathcal{GBr}_3:\]

1. Each partition $\bar{i}$ of 3 is represented by the enhanced triangulated category $\mathcal{C}_{\bar{i}}$.

2. Each fork $\tilde{f}_i^2$ is represented by the functor $F_{i}^\tilde{f}$.

3. (1,1)-merges $g_{i}^2$ are represented by the functors $G_{i}^\tilde{f}$ defined by

\[
L_{i}^{\tilde{f}}[-1] \simeq G_{i}^{\tilde{f}} \simeq R_{i}^{\tilde{f}}[1].
\]

4. (1,2)- and (2,1)-merges $g_{i}^\tilde{f}$ are represented by the functors $G_{i}^\tilde{f}$ defined by

\[
L_{i}^{\tilde{f}}[-2] \simeq G_{i}^{\tilde{f}} \simeq R_{i}^{\tilde{f}}[2].
\]
5. \((1, 1)\)-crossings are represented by the spherical twists of \((1, 1)\)-forks:

\[
T_{111}^{11} = \text{Cone} \left( F_{21}^{11} G_{111}^{21} [-1] \xrightarrow{\text{tr}} \text{Id}_{111} \right),
\]
\[
T_{111}^{11} = \text{Cone} \left( F_{12}^{11} G_{111}^{12} [-1] \xrightarrow{\text{tr}} \text{Id}_{111} \right),
\]
\[
D_{111}^{11} = \text{Cone} \left( \text{Id}_{111} [-1] \xrightarrow{\text{act}} F_{21}^{11} G_{111}^{21} \right),
\]
\[
D_{111}^{11} = \text{Cone} \left( \text{Id}_{111} [-1] \xrightarrow{\text{act}} F_{12}^{11} G_{111}^{12} \right),
\]

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{crossing1111}} \\
= \text{Cone} \left( \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{fork1111}} \\
[-1] \xrightarrow{\text{tr}} \\
\text{\includegraphics[width=0.1\textwidth]{id1111}}
\end{array} \right)
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{crossing1112}} \\
= \text{Cone} \left( \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{fork1112}} \\
[-1] \xrightarrow{\text{tr}} \\
\text{\includegraphics[width=0.1\textwidth]{id1112}}
\end{array} \right)
\end{array}
\end{align*}
\]

6. \((1, 2)\)- and \((2, 1)\)-crossings are represented by the cones:

\[
T_{12}^{21} = \text{Cone} \left( F_{3}^{21} G_{12}^{3} [-1] \xrightarrow{\lambda} G_{111}^{21} F_{12}^{11} \right),
\]
\[
T_{21}^{12} = \text{Cone} \left( F_{3}^{12} G_{21}^{3} [-1] \xrightarrow{\mu} G_{111}^{12} F_{12}^{11} \right),
\]
\[
D_{12}^{21} = \text{Cone} \left( G_{111}^{21} F_{12}^{11} [-1] \xrightarrow{\lambda'} F_{3}^{21} G_{12}^{3} \right),
\]
\[
D_{21}^{12} = \text{Cone} \left( G_{111}^{12} F_{21}^{11} [-1] \xrightarrow{\mu'} F_{3}^{12} G_{21}^{3} \right),
\]

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{crossing1211}} \\
= \text{Cone} \left( \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{fork1211}} \\
[-1] \xrightarrow{\text{tr}} \\
\text{\includegraphics[width=0.1\textwidth]{id1211}}
\end{array} \right)
\end{array}
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{crossing1212}} \\
= \text{Cone} \left( \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{fork1212}} \\
[-1] \xrightarrow{\text{tr}} \\
\text{\includegraphics[width=0.1\textwidth]{id1212}}
\end{array} \right)
\end{align*}
\]

\[
\begin{align*}
\begin{array}{c}
\text{\includegraphics[width=0.2\textwidth]{crossing2112}} \\
= \text{Cone} \left( \begin{array}{c}
\text{\includegraphics[width=0.1\textwidth]{fork2112}} \\
[-1] \xrightarrow{\text{tr}} \\
\text{\includegraphics[width=0.1\textwidth]{id2112}}
\end{array} \right)
\end{align*}
\]
4.2. Skein-triangulated representations of $\mathcal{GB}_3$

where $\lambda$ is defined by either of the two equal compositions

\[ \begin{align*}
F_{12}^3 R_{12}^3 & \quad F_{12}^3 R_{12}^3 \\
\downarrow F_{12}^3 R_{12}^3 \text{act} & \quad \downarrow \text{act } F_{12}^3 R_{12}^3 \\
F_{12}^3 R_{12}^3 R_{111}^3 F_{12}^3 & \quad R_{111}^3 F_{12}^3 L_{12}^3 R_{111}^3 \\
\sim \text{multifork} & \quad \sim \text{multifork} , \\
F_{12}^3 R_{12}^3 L_{111}^3 F_{12}^3 & \quad R_{12}^3 F_{12}^3 L_{12}^3 R_{111}^3 \\
\downarrow \text{tr } R_{111}^3 F_{12}^3 & \quad \downarrow R_{111}^3 F_{12}^3 \text{tr } \\
L_{111}^3 F_{12}^3 & \quad R_{111}^3 F_{12}^3 \\
\lambda' \text{ is defined by either of the two equal compositions}
\end{align*} \]

\[ \begin{align*}
L_{111}^3 F_{12}^3 & \quad L_{111}^3 F_{12}^3 \\
\downarrow L_{111}^3 F_{12}^3 \text{act} & \quad \downarrow \text{act } L_{111}^3 F_{12}^3 \\
L_{111}^3 F_{12}^3 L_{12}^3 & \quad F_{12}^3 L_{12}^3 L_{111}^3 F_{12}^3 \\
\sim \text{multifork} & \quad \sim \text{multifork} , \\
L_{111}^3 F_{12}^3 & \quad F_{12}^3 F_{12}^3 L_{111}^3 \\
\downarrow \text{tr } F_{12}^3 L_{12}^3 & \quad \downarrow F_{12}^3 L_{12}^3 \text{tr } \\
F_{12}^3 L_{12}^3 & \quad F_{12}^3 L_{12}^3 \\
\text{and } \mu \text{ and } \mu' \text{ are defined similarly.}
\end{align*} \]

Moreover, this categorical action also satisfies the following condition:

- $\text{Flop + Flop = Twist}$:
  \[ \begin{align*}
1. & \quad T_{12}^3 T_{12}^3 \text{ is isomorphic in } D(C_{12} - C_{12}) \text{ to the } P\text{-twist of } F_{12}^3, \\
2. & \quad T_{12}^3 T_{12}^3 \text{ is isomorphic in } D(C_{21} - C_{21}) \text{ to the } P\text{-twist of } F_{12}^3.
\end{align*} \]

Motivated by the previous result, we give the following definition.

**Definition 4.2.4.** A skein-triangulated representation of $\mathcal{GB}_3$ is a system of categories and functors satisfying the assignments properties of Theorem 4.2.3.

**Proof of Theorem 4.2.3.** It suffices to prove that the four basic relations between the generators of $\mathcal{GB}_3$ listed in (4.2.1) hold under the assumptions of the theorem. The proofs for the relations obtained from these four by vertical, horizontal and blackboard reflections are identical.

$(T_{111}^{111}, D_{111}^{111})$ and $(T_{111}^{111}, D_{111}^{111})$ are pairs of mutually inverse equivalences.

This follows from $F_{12}^{111}$ and $F_{12}^{111}$ being spherical functors, see [6], Theorem 5.1.
The pitchfork relation

We need to show the existence of an isomorphism

\[ F_{12}^{111} \cong T_{111}^{111} T_{111}^{111} F_{21}^{111}. \]

By definition we have

\[ T_{111}^{111} = \text{Cone} \left( F_{21}^{111} G_{111}^{21} [-1] \xrightarrow{t} \text{Id}_{111} \right), \]
\[ T_{111}^{111} = \text{Cone} \left( F_{12}^{111} G_{111}^{12} [-1] \xrightarrow{t} \text{Id}_{111} \right), \]

therefore \( T_{111}^{111} T_{111}^{111} F_{21}^{111} \) is isomorphic in \( D(C_{111} - C_{111}) \) to the convolution of the twisted complex

\[ F_{21}^{111} R_{111,21}^{12} R_{12,21}^{111} R_{111,21}^{111} F_{21}^{111} \xrightarrow{\left( \begin{array}{ccc} trF_{111}^{111} R_{111,21}^{12} F_{111}^{111} & 0 \\ -F_{21}^{111} R_{111,21}^{12} trF_{21}^{111} F_{111}^{111} \end{array} \right)} F_{12}^{111} R_{111,21}^{12} R_{111,21}^{111} F_{21}^{111} \oplus F_{21}^{111} R_{111,21}^{12} R_{111,21}^{111} F_{21}^{111} \xrightarrow{(trF_{111}^{111} trF_{21}^{111})} F_{21}^{111} \]

By Lemma 5.10 of [6], we have a homotopy equivalence of twisted complexes between \( F_{21}^{111} R_{111,21}^{12} F_{111}^{111} \xrightarrow{trF_{111}^{111}} F_{21}^{111} \) and \( F_{21}^{111} \xrightarrow{F_{21}^{111} \text{act}} F_{21}^{111} R_{111,21}^{12} F_{111}^{111} \).

It follows by the Replacement Lemma that \( T_{111}^{111} T_{111}^{111} F_{21}^{111} \) is further isomorphic to the convolution of

\[ F_{21}^{111} R_{111,21}^{12} R_{111,21}^{111} F_{21}^{111} \oplus F_{21}^{111} \xrightarrow{\left( \begin{array}{ccc} trF_{111}^{111} R_{111,21}^{12} F_{111}^{111} & 0 \\ -F_{21}^{111} R_{111,21}^{12} trF_{21}^{111} F_{111}^{111} \end{array} \right)} F_{12}^{111} R_{111,21}^{12} R_{111,21}^{111} F_{21}^{111} \]

Moreover, there exists an homotopy equivalence from \( F_{21}^{111} F_{3}^{21} R_{21}^{3} \) to

\[ F_{21}^{111} R_{111,21}^{12} R_{111,21}^{111} F_{21}^{111} \oplus F_{21}^{111} \xrightarrow{\left( \begin{array}{ccc} -F_{21}^{111} R_{111,21}^{12} trF_{21}^{111} F_{111}^{111} \end{array} \right)} F_{21}^{111} R_{111,21}^{12} R_{111,21}^{111} F_{21}^{111} \]

whose \( F_{21}^{111} F_{3}^{21} R_{21}^{3} \xrightarrow{F_{21}^{111} R_{111,21}^{12} F_{111}^{111} R_{12,21}^{111} F_{21}^{111}} \) component is the map \( F_{21}^{111} (4.11) \).

It follows by the Replacement Lemma, that \( T_{111}^{111} T_{111}^{111} F_{21}^{111} \) is therefore isomorphic to the convolution of

\[ F_{21}^{111} F_{3}^{21} R_{21}^{3} \xrightarrow{trF_{12}^{111} R_{111,21}^{12} F_{21}^{111} (4.11)} F_{12}^{111} R_{111,21}^{12} F_{21}^{111} \]

and hence by the multifork isomorphism to the convolution of

\[ F_{12}^{111} F_{3}^{21} R_{21}^{3} \xrightarrow{trF_{12}^{111} R_{111,21}^{12} F_{21}^{111} (4.11) \text{multifork}} F_{12}^{111} R_{111,21}^{12} F_{21}^{111}. \]

On the other hand, by the definition of \( T_{21}^{12} \) the object \( F_{12}^{111} T_{21}^{12} \) is isomorphic in \( D(C_{111} - C_{111}) \) to the convolution if the twisted complex
4.2. Skein-triangulated representations of $\mathcal{G}Br_3$

\[
F_{12}^{111} F_3^{112} R_{21}^3 \xrightarrow{F_{12}^{111} R_{111}^{12}} F_{12}^{111} R_{111}^{11} F_{21}^{111} .
\]

And since

\[
tr F_{12}^{111} R_{111}^{11} F_{21}^{111} \circ F_{12}^{111} (4.11) \circ \text{multifork} = F_{12}^{111} \mu
\]

we have

\[
F_{12}^{111} T_{21}^{12} \simeq T_{111}^{111} T_{111}^{111} F_{21}^{111} .
\]

The braid relation

Consider the following twisted complexes of enhanced functors:

\[
F_{21}^{111} R_{111}^{12} R_{21}^{11} F_{21}^{111} R_{111}^{11} + F_{21}^{111} R_{111}^{12} R_{21}^{11} - F_{12}^{111} R_{111}^{11} F_{21}^{111} .
\]

\[
F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{11} + F_{12}^{111} R_{111}^{12} F_{21}^{111} .
\]

There are natural maps from both to $F_{21}^{111} R_{111}^{12} F_{12}^{111} R_{111}^{11} \oplus F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{11}

\]

induced by the maps

\[
F_{21}^{111} R_{111}^{12} F_{12}^{111} R_{111}^{11} + F_{21}^{111} R_{111}^{12} R_{21}^{11} - F_{12}^{111} R_{111}^{11} F_{21}^{111} .
\]

\[
F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{11} + F_{12}^{111} R_{111}^{12} F_{21}^{111} .
\]

By [6], Theorem 6.2, if there exists a $D(C_{111}-C_{111})$ isomorphism between the convolutions of the twisted complexes (4.28) and (4.29) which intertwines the maps induced by (4.30) and (4.31), then the braid relation holds for $T_{111}^{111}$ and $T_{111}^{111}$.

Claim: There exist homotopy equivalences from the objects $F_{21}^{111} F_3^{21} R_{21}^3 R_{111}^{11}$ and $F_{12}^{111} F_3^{12} R_{12}^{12} R_{111}^{11}$ to the twisted complexes (4.28) and (4.29) whose

\[
F_{21}^{111} F_3^{21} R_{21}^3 R_{111}^{11} \to F_{21}^{111} R_{111}^{11} F_2^{111} R_{111}^{12} F_{21}^{111} R_{111}^{12}.
\]

\[
F_{12}^{111} F_3^{12} R_{12}^{12} R_{111}^{11} \to F_{12}^{111} R_{111}^{11} F_2^{111} R_{111}^{12} F_{12}^{111} R_{111}^{12}.
\]

components are the maps $F_{21}^{111} (4.14) R_{111}^{21}$ and $F_{12}^{111} (4.11) R_{111}^{12}$.
Indeed, since (4.5) fits into an exact triangle in $D(C_{21}-C_{21})$, there exist a homotopy equivalence of twisted complexes of the form

\[
\begin{align*}
F_{21}^{111} F_{21}^{211} R_{21}^{211} R_{21}^{111} & \\
\downarrow F_{21}^{111} (4.14) R_{21}^{211} & \\
F_{21}^{111} R_{111}^{211} F_{12}^{111} R_{111}^{111} & \rightarrow F_{21}^{111} R_{111}^{211} [-2].
\end{align*}
\]

(4.32)

Since $F_{21}^{111}$ is split-spherical with cotwist $[-3]$, there is a split exact triangle

\[
F_{21}^{111} R_{111}^{211} \rightarrow F_{21}^{111} R_{111}^{211} R_{21}^{111} \rightarrow F_{21}^{111} R_{111}^{211} [-2].
\]

Therefore, the object $F_{21}^{111} R_{111}^{211} [-2]$ is homotopy equivalent to the twisted complex $F_{21}^{111} R_{111}^{211} R_{21}^{111}$. It follows by the Replacement Lemma ([7], Lemma 2.1) that there exists a homotopy equivalence of form

\[
\begin{align*}
F_{21}^{111} R_{111}^{211} F_{21}^{111} R_{111}^{211} & \\
\downarrow F_{21}^{111} (4.15) R_{21}^{211} & \\
F_{21}^{111} R_{111}^{211} F_{12}^{111} R_{111}^{111} & \rightarrow F_{21}^{111} R_{111}^{211} [-2].
\end{align*}
\]

(4.33)

The composition of (4.32) and (4.33) is the desired homotopy equivalence from $F_{21}^{111} F_{21}^{211} R_{21}^{211} R_{21}^{111}$ to the twisted complex (4.28).

Therefore, we have proven the claim for the twisted complex (4.28); the proof for (4.29) are similar.

Now, recall that we have the multifork isomorphism $\alpha: F_{21}^{111} F_{21}^{211} \sim F_{12}^{111} F_{3}^{12}$.

Let $\beta$ denote its inverse $F_{12}^{111} F_{3}^{12} \sim F_{21}^{111} F_{3}^{12}$, and let $\beta^R: R_{21}^{111} R_{111}^{211} \sim R_{12}^{111} R_{12}^{111}$.

We now claim that the $D(C_{111}-C_{111})$ isomorphism between the convolutions of (4.28) and (4.29) which is induced via the isomorphisms provided by the Claim from the isomorphisms

\[
F_{21}^{111} F_{3}^{211} R_{21}^{211} R_{111}^{211} \overset{\alpha^R}{\longrightarrow} F_{12}^{111} F_{3}^{12} R_{12}^{111}
\]

is the requisite isomorphism intertwining the maps (4.30) and (4.31).

The target of the maps is the direct sum

\[
F_{21}^{111} R_{111}^{211} F_{12}^{111} R_{111}^{111} \oplus F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{211}.
\]

We prove that the isomorphism intertwines the components of (4.30) and (4.31) which go into the second direct summand. The proof that it intertwines the first direct summand components is similar.
4.2. Skein-triangulated representations of $\mathcal{GBr}_3$

It suffices to show that the following diagram commutes in $D(\mathcal{C}_{111})$:

$$\begin{array}{cccc}
F_{12}^{111}F_3^{21}R_2^1R_{111}^{11} & \xrightarrow{\alpha \beta R} & F_{12}^{111}F_3^{12}R_2^1R_{111}^{11} \\
F_{21}^{111}F_2^{11}R_1^{11}F_2^{11}R_{21}^{11} & \xrightarrow{\beta G} & F_{12}^{111}F_2^{11}R_1^{11}F_2^{11}R_{21}^{11} \\
F_{12}^{111}F_2^{11}R_1^{11}R_{21}^{11}F_2^{11}R_{21}^{11} & \xrightarrow{\alpha \beta R} & F_{12}^{111}F_2^{11}R_1^{11}R_{21}^{11}F_2^{11}R_{21}^{11} \\
\end{array}$$

This can be simplified to:

$$\begin{array}{cccc}
F_{21}^{111}F_3^{12}R_2^1R_{111}^{11} & \xrightarrow{\alpha \beta R} & F_{12}^{111}F_3^{12}R_2^1R_{111}^{11} \\
F_{21}^{111}F_3^{12}R_2^1R_{111}^{11} & \xrightarrow{\alpha \beta R} & F_{12}^{111}F_3^{12}R_2^1R_{111}^{11} \\
F_{21}^{111}F_3^{12}R_2^1R_{111}^{11} & \xrightarrow{\alpha \beta R} & F_{12}^{111}F_3^{12}R_2^1R_{111}^{11} \\
\end{array}$$

The commutativity of this diagram reduces to the commutativity of the diagram

$$\begin{array}{cccc}
F_{12}^{122}F_3^{21}R_2^1R_{111}^{11} & \xrightarrow{\alpha \beta R} & F_{12}^{122}F_3^{21}R_2^1R_{111}^{11} \\
F_{12}^{122}F_3^{21}R_2^1R_{111}^{11} & \xrightarrow{\alpha \beta R} & F_{12}^{122}F_3^{21}R_2^1R_{111}^{11} \\
F_{12}^{122}F_3^{21}R_2^1R_{111}^{11} & \xrightarrow{\alpha \beta R} & F_{12}^{122}F_3^{21}R_2^1R_{111}^{11} \\
\end{array}$$

and then further to the commutativity of

$$\begin{array}{cccc}
\end{array}$$

which commutes since $\beta R$ is the right dual of $\beta$. 

Chapter 4. Categorical action of generalised braids

Flop+flop=twist

We only prove the first of the “flop-flop = twist” relations, the second is proved identically. We need to show that

\[
\{ F_3^1 R_{21}^3 \to R_{11}^3 F_{11}^{11} \} \{ F_3^1 R_{12}^3 \to R_{11}^{21} F_{12}^{11} \} [2]
\]

is isomorphic to

\[
F_3^1 R_{12}^3 [-2] \psi F_3^1 R_{12}^3 \to \text{Id}_{C_{12}}.
\]

The tensor of product of convolutions in (4.34) is isomorphic to the convolution of the twisted complex

\[
\begin{pmatrix}
F_3^1 R_{21}^3 F_3^1 R_{12}^3 \\
\begin{pmatrix}
\mu F_3^1 R_{12}^3 \\
-F_3^2 R_{12}^3
\end{pmatrix}
\end{pmatrix}
\]

(4.35)

Since \( F_{21}^{11} \) and \( F_{12}^{11} \) are split spherical with cotwist \([-3]\), we have

\[
R_{11}^{12} F_{21}^{11} F_{3}^3 R_{12}^3 \simeq R_{11}^{12} F_{12}^{11} F_{3}^3 R_{12}^3 \simeq F_3^1 R_{12}^3 \oplus F_3^1 R_{12}^3 [-2],
\]

\[
F_3^1 R_{21}^3 R_{11}^{11} F_{12}^{11} \simeq F_3^1 R_{12}^3 R_{11}^{12} F_{12}^{11} \simeq F_3^1 R_{12}^3 \oplus F_3^1 R_{12}^3 [-2],
\]

Since \( F_3^{21} \) is split \( \mathbb{P}^2 \)-functor with \( H = [-2] \) we also have

\[
F_3^1 R_{21}^3 F_3^1 R_{12}^3 \simeq F_3^1 R_{12}^3 \oplus F_3^1 R_{12}^3 [-2] \oplus F_3^1 R_{12}^3 [-4].
\]

Thus three out of four objects in the twisted complex (4.35) are isomorphic to direct sums of shifted copies of \( F_2 R_2 \). Under these identifications (4.35) becomes:

\[
\begin{pmatrix}
F_3^1 R_{12}^3 [2] \oplus F_3^1 R_{12}^3 \oplus F_3^1 R_{12}^3 [-2] \\
\begin{pmatrix}
\text{Id} & 0 & 0 \\
0 & \text{Id} & \psi_1 \\
-\text{Id} & 0 & 0 \\
0 & -\text{Id} & -\psi_2
\end{pmatrix}
\end{pmatrix}
\]

(4.36)

\[
\begin{pmatrix}
F_3^1 R_{12}^3 [2] \\
\text{Deg.0}
\end{pmatrix}
\]

where \( \psi_1, \psi_2 \) are the compositions

\[
F_3^1 R_{12}^3 [-2] \to F_3^1 R_{12}^3 F_3^1 R_{12}^3 \to F_3^1 R_{12}^3,
\]

\[
F_3^1 R_{12}^3 \to F_3^1 R_{12}^3.
\]
and \( \phi_1 \) and \( \phi_2 \) are the compositions

\[
\begin{align*}
F_3^{12} R_{12}^3 & \quad F_3^{12} R_{12}^3 \\
R_{111}^{12} F_{21}^{111} R_{12}^{11} F_{3}^{21} R_{12}^{3} & \quad R_{111}^{12} F_{21}^{111} R_{12}^{11} F_{3}^{21} R_{12}^{3} \\
R_{111}^{12} F_{21}^{111} R_{12}^{11} F_{3}^{21} R_{12}^{3} & \quad R_{111}^{12} F_{21}^{111} R_{12}^{11} F_{3}^{21} R_{12}^{3}
\end{align*}
\]

Note that since the map (4.12) is the composition

\[
R_{111}^{12} F_{21}^{111} R_{12}^{11} F_{3}^{21} R_{12}^{3} \xrightarrow{R_{111}^{12} \operatorname{tr} F_{12}^{11}} R_{111}^{12} F_{12}^{111} \xrightarrow{\operatorname{Id}_{111}[-2]}.
\]

we have the following equality of maps \( F_3^{12} R_{12}^3 \to \operatorname{Id}_3 \):

\[
\phi_1 \circ (4.12) = \operatorname{tr} = \phi_2 \circ (4.12).
\]

The map \( \psi \) in the definition of the \( \mathbb{P} \)-twist of \( F_3^{12} \) is \( \psi_1 - \psi_2 \). Changing the basis of the middle term of (4.36) to the diagonal and the antidiagonal we obtain:

\[
\begin{align*}
& F_3^{12} R_{12}^3 [2] \oplus F_3^{12} R_{12}^3 \oplus F_3^{12} R_{12}^3 [-2] \\
& \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \psi \\
2 \psi & 0 & 0
\end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
& (F_3^{12} R_{12}^3 [2] \oplus F_3^{12} R_{12}^3) \oplus (F_3^{12} R_{12}^3 [2] \oplus F_3^{12} R_{12}^3) \\
& \begin{pmatrix}
2(4.11) & \phi_1 + \phi_2 & 0 & \phi_1 - \phi_2
\end{pmatrix}
\end{align*}
\]

(4.37)

We can remove the following acyclic subcomplex of (4.37)

\[
\begin{align*}
& F_3^{12} R_{12}^3 [2] \oplus F_3^{12} R_{12}^3 \xrightarrow{\begin{pmatrix}
2 \psi & 0 \\
0 & 2 \psi
\end{pmatrix}} F_3^{12} R_{12}^3 [2] \oplus F_3^{12} R_{12}^3
\end{align*}
\]
using the Replacement Lemma. Since the subcomplex has no external arrows emerging from its degree $-2$ part, no other differentials are affected. We obtain:

\[
F_3^{12} R_{12}^3 [-2] \xrightarrow{\psi} F_3^{12} R_{12}^3 [2] \oplus F_3^{12} R_{12}^{21} \xrightarrow{2(4.11) \phi_1 + \phi_2} R_{12}^{11} F_{21}^{11} R_{12}^{21} F_{12}^{11} [2].
\] (4.38)

Since (4.5) fits into an exact triangle, we have a homotopy equivalence of form

\[
\begin{array}{ccc}
F_3^{12} R_{12}^3 [2] & \xrightarrow{2(4.11)} & R_{12}^{11} F_{21}^{11} R_{12}^{21} F_{12}^{11} [2] \\
\downarrow & \searrow & \downarrow \\
& \Id_{12} \deg 0 & \uparrow \frac{1}{2}(4.12)
\end{array}
\] (4.39)

By the Replacement Lemma, we finally obtain:

\[
F_3^{12} R_{12}^3 [-2] \xrightarrow{\psi} F_3^{12} R_{12}^3 [2] \xrightarrow{2(4.11) \phi_1 + \phi_2} \Id_{12} \deg 0.
\] (4.40)

Since $4.12 \circ \phi_i = \tr$, this is homotopy equivalent to the $P$-twist of $F_3^{12}$:

\[
F_3^{12} R_{12}^3 [-2] \xrightarrow{\psi} F_3^{12} R_{12}^3 [2] \xrightarrow{\tr} \Id_{12} \deg 0.
\] (4.41)

$(T_{21}^{12}, D_{12}^{21})$ and $(T_{12}^{12}, D_{12}^{21})$ are pairs of mutually inverse equivalences.

This follows from the “twist-twist=flop” relations. Indeed, $T_{21}^{12} \circ T_{12}^{21}$ and $T_{12}^{21} \circ T_{21}^{12}$ are isomorphic to $P$-twists of $F_3^{12}$ and $F_3^{21}$ and hence are both autoequivalences. Therefore $T_{12}^{21}$ and $T_{21}^{12}$ are also autoequivalences. On the other hand, the maps $\lambda'$ and $\mu'$ which define the functors $D_{12}^{21}$ and $D_{21}^{12}$ are the left duals of the maps $\mu$ and $\lambda$ which define the functors $T_{12}^{21}$ and $T_{21}^{12}$. Hence $D_{12}^{21}$ and $D_{21}^{12}$ are the left adjoints of $T_{12}^{21}$ and $T_{21}^{12}$. The claim now follows. \(\blacksquare\)
4.3 A skein triangulated action of $\mathcal{GB}_r$ on $T^* \text{Fl}_3(\bar{i})$

The aim of the rest of this chapter is to define a network of categories and functors that satisfy the assumption of Theorem 4.2.3.

In particular, we construct such a network on $T^* \text{Fl}_3(\bar{i})$ and we prove that all the hypothesis for a skein triangulated action of $\mathcal{GB}_r$ are verified.

In the following, let $D^b(X)$ be the derived category of coherent sheaves on a smooth quasi-projective variety $X$ and assume all the functors derived, i.e. we omit $R$ and $L$.

We omit the pushforward $i_*$ applied to structure sheaves when $i$ is an embedding and the pullbacks applied to line bundles; we also write, when $D_1$ and $D_2$ are divisors respectively in $X$ and $Y$, $\mathcal{O}_{X \times Y}(D_1, D_2)$ for the line bundle $\mathcal{O}_X(D_1) \boxtimes \mathcal{O}_Y(D_2)$ on the fiber product $X \times_Z Y$.

We write the total space $T^* \text{Fl}_n(\bar{i})$ of the cotangent bundle of the flag $\text{Fl}_n(\bar{i})$ with the little abuse of notation, as in remark 2.2.30

$$T^* \text{Fl}_n(\bar{i}) = \left\{ 0 \subset V_{i_1} \subset \ldots \subset V_{i_{k-1}} \subset \mathbb{C}^n \right\},$$

and we also write

$$\left\{ 0 \subset V_{\lambda_1} \subset \ldots \subset V_{\lambda_{k-1}} \subset \mathbb{C}^n \right\} \times \left\{ 0 \subset V_{j_1} \subset \ldots \subset V_{j_{h-1}} \subset \mathbb{C}^n \right\}$$

for the subspace of $T^* \text{Fl}_n(\bar{i}) \times T^* \text{Fl}_n(\bar{j})$ where the maps $\alpha$ need to satisfy $\alpha(V_{i_k}) \subset V_{i_{k-1}}$ and $\alpha(V_{j_k}) \subset V_{j_{k-1}}$.

We write $\mathcal{V}_i$ for the pullback on $\text{Fl}_n$ of the tautological bundle of $\text{Gr}(i, n)$.

Recall that if $X$ and $Y$ are quasi-projective subvarieties of $Z$ such that their intersection $X \cap Y$ is a Cartier divisor in $Y$ than we have the short exact sequence of coherent sheaves of $Z$

$$0 \to \mathcal{O}_Y(-X \cap Y) \to \mathcal{O}_{X \cup Y} \to \mathcal{O}_X \to 0. \tag{4.42}$$

Recall moreover that if $f : X \to Y$ is proper, then we have the following adjunction of derived functors

$$f^* \dashv f_* \dashv f^! \tag{4.43}$$

where $f^! = f^*(-) \otimes \omega_{X/Y}[\text{dim} Y - \text{dim} X]$.

If $f$ is a divisorial embedding we write $\mathcal{O}_{X \times X Y} \in D^b(X \times Y)$ and $\mathcal{O}_{Y \times X X} \in D^b(Y \times X)(X, 0)[-1]$ for the Fourier-Mukai kernels representing respectively $f_*$ and $f^!$. 

If \( f \) is a fibration we write \( \mathcal{O}_{X \times X} \in D^b(X \times Y) \) and \( \mathcal{O}_{Y \times X} \in D^b(Y \times X) \) for the Fourier-Mukai kernels representing respectively \( f_* \) and \( f^* \).

We write moreover \( \mathcal{O}_{X \times X} \in D^b(X \times X) \) for the Fourier-Mukai kernel representing the identity on \( X \).

As in section 3.8, let \( C \) be our ambient variety the total space of the contangent bundle of \( \text{Fl}_3 \)

\[
C = T^*\text{Fl}_3.
\]

Let \( A \) and \( E \) be the quasi-projective varieties defined as the total space of the contangent bundle of \( \mathbb{P}^2 \) and \( \mathbb{P}^{2\vee} \):

\[
A = T^*\mathbb{P}^2 \quad E = T^*\mathbb{P}^{2\vee}.
\]

Let \( B \) and \( D \) be the quasi-projective varieties defined as the total space of the pullback via \( p_1 \) and \( p_2 \) on \( \text{Fl}_3 \) of the contangent bundle of \( \mathbb{P}^2 \) and \( \mathbb{P}^{2\vee} \):

\[
B = p_1^*T^*\mathbb{P}^2 \quad D = p_2^*T^*\mathbb{P}^{2\vee}.
\]

We have therefore the following diagram

\[
\begin{array}{ccc}
B & \xleftarrow{i_B} & C \\
\downarrow{\pi_A} & & \downarrow{\pi_E} \\
A & \xleftarrow{i_{\mathbb{P}^2}} & \mathbb{P}^2 \\
\end{array}
\begin{array}{ccc}
D & \xleftarrow{i_D} & E \\
\downarrow{\pi_{\text{pt}}} & & \downarrow{\pi_{\text{pt}1}} \\
\mathbb{P}^{2\vee} & \xleftarrow{i_{\mathbb{P}^{2\vee}}} & \text{pt} \\
\end{array}
\]

(4.44)

where \( i_B, i_D, i_{\mathbb{P}^{2\vee}} \) and \( i_{\mathbb{P}^2} \) are divisorial inclusions and \( B \xrightarrow{\pi_A} A \) and \( D \xrightarrow{\pi_E} E \) are \( \mathbb{P}^1 \) bundles.

**Definition 4.3.1.** We define the categories \( \mathcal{C}_{(3)} = D^b(pt) \), \( \mathcal{C}_{(1,2)} = D^b(A) \), \( \mathcal{C}_{(2,1)} = D^b(E) \), and \( \mathcal{C}_{(1,1,1)} = D^b(C) \).

### 4.4 A skein triangulated action of \( \mathcal{G}\text{Br}_3 \): generators

In this section we define the generators of skein triangulated action of \( \mathcal{G}\text{Br}_3 \), the forks and the merges.

The functors \( F_{12}^{11} \), \( F_{21}^{11} \) and their respective right adjoints \( R_{11}^{12} \), \( R_{11}^{21} \) are the spherical functors \( F_i \) and their right adjoints \( R_i \) of section 3.10.
4.4. A skein triangulated action of $\mathcal{GBr}_3$: generators

After giving a description of them as Fourier-Mukai transforms, we will compute the first compositions of them.

In the following sections, we use $D^b(pt)$, $D^b(A)$, $D^b(E)$, $D^b(C)$ instead of the notation of Definition 4.3.1.

**Definition 4.4.1.** We define the first fork functor

$$F_{21}^{111} = i_{D*} \circ \pi_E^* : D^b(E) \to D^b(C)$$

and define the second fork functor

$$F_{12}^{111} = i_{B*} \circ \pi_A^* : D^b(A) \to D^b(C).$$

![First and second forks](image)

**Figure 4.10: First and second forks**

**Proposition 4.4.2.** The Fourier-Mukai kernel of $F_{21}^{111}$ and $F_{12}^{111}$ are respectively the sheaves $\mathcal{O}_{E \times E \times D} \in D^b(E \times C)$ and $\mathcal{O}_{A \times A \times B} \in D^b(A \times C)$.

**Proof.** Let $\pi_{12}, \pi_{23}, \pi_{13}$ be the natural projections

$$\begin{array}{ccc}
E \times D \times C & \xrightarrow{\pi_{12}} & E \\
\downarrow{\pi_{13}} & & \downarrow{\pi_{23}} \\
E \times D & \to & D \times C & \to & D \times E.
\end{array}$$

By Proposition 3.9.4, the composition $F_{21}^{111}$ of $i_{D*} \circ \pi_E^*$ is represented by the convolution of their Fourier-Mukai kernels, thus

$$\pi_{13*}(\pi_{12}^* \mathcal{O}_{E \times E} \otimes \pi_{23}^* \mathcal{O}_{D \times D}) = \pi_{13*}(\mathcal{O}_{E \times E \times C} \otimes \mathcal{O}_{E \times D \times D})$$

By Corollary 3.9.4 the latter derived tensor product of the structure sheaves is isomorphic to the structure sheaf of the intersection

$$\mathcal{O}_{E \times E \times C} \otimes \mathcal{O}_{E \times D \times D} \simeq \mathcal{O}_{E \times E \times E}$$

Indeed the subvariety $E \times E \times C$ is of codimension 4 inside $E \times D \times C$ while the condimension of the subvariety $E \times D \times D$ is 6 inside $E \times D \times C$.

Their intersection $(E \times E \times D \times C) \cap (E \times D \times D)$ is smooth and of codimension 9, therefore they intersect transversally.
Chapter 4. Categorical action of generalised braids

Notice that $E \times_E E \times E D$ is isomorphic to a copy of $D$ in the third component, therefore $\pi_{13} : E \times E E \times E D \to E \times C$ is an embedding, so the derived pushforward of $O_{E \times_E E \times E D}$ is just the structure sheaf of the image of the support.

So in conclusion the Fourier-Mukai kernel of $F_{21}^{11}$ is isomorphic to

$$O_{E \times_E D}.$$  

The same argument applies to $i_B \circ \pi_A^*$ for showing that the kernel of $F_{12}^{111}$ is isomorphic to

$$O_{A \times A B}.$$  

Remark 4.4.3. The first merge

$$R_{111}^{21} = \pi^*_{E} \circ i_D^! : D^b(C) \to D^b(E)$$

and the second merge

$$R_{111}^{12} = \pi^*_{A} \circ i_B^! : D^b(C) \to D^b(A)$$

are respectively the right adjoints of $F_{21}^{111}$ and $F_{12}^{111}$.

![Figure 4.11: First and second merges](image)

Proposition 4.4.4. The Fourier-Mukai kernel to $R_{111}^{21}$ and $R_{111}^{12}$ are respectively the objects $O_{D \times_E E}(D, 0)[-1] \in D^b(C \times E)$ and $O_{B \times_A A}(B, 0)[-1] \in D^b(C \times A)$.

Proof. Let $\pi_{12}, \pi_{23}, \pi_{13}$ be the natural projections as in diagram (4.59)

By Proposition 3.2.3, the composition $R_{111}^{21}$ of $p_E^* \circ i_D^!$ is represented by the convolution of their Fourier-Mukai kernels, thus

$$\pi_{13}^*(\pi_{23}^* O_{D \times_D D}(D, 0) \otimes \pi_{12}^* O_{E \times E E}) \approx \pi_{13}^*(O_{E \times D \times_D D}(D, 0, 0) \otimes O_{E \times E E \times C})$$

As in the proof of Proposition (4.59), by transversality of the intersection of $E \times D \times_D D$ with $E \times_E E \times C$, we can apply Corollary 3.9.4 and obtaining the isomorphism

$$O_{E \times D \times_D D}(D, 0, 0) \otimes O_{E \times E E \times C} \approx O_{E \times E E \times E D}(D, 0, 0).$$

Notice that $E \times_E E \times E D$ is isomorphic to a copy of $D$ in the third component, therefore $\pi_{13} : E \times_E E \times E D \to E \times C$ is an embedding, so the derived pushforward of $O_{E \times_E E \times E D}$ is just the structure sheaf of the image of the support.
4.4. A skein triangulated action of $\mathcal{GBr}_3$: generators

Since $\pi_{13}$ is an embedding restricted to $E \times_E E \times_E D$, we conclude that the Fourier-Mukai kernel associated to $R_{111}^{21}$ is isomorphic to

$$\mathcal{O}_{D \times E}(D, 0)[-1].$$

The same argument applies to $i_{B*} \circ \pi_A^*$ for showing that the Fourier-Mukai associated to $R_{111}^{12}$ is isomorphic to

$$\mathcal{O}_{B \times A}(B, 0)[-1].$$

\[\Box\]

**Definition 4.4.5.** We define the third and fourth forks

$$F_{12}^3 := i_{P^2*} \circ \pi_{pt1}^*: D^b(pt) \to D^b(A), \quad F_{21}^3 := i_{P^2*} \circ \pi_{pt2}^*: D^b(pt) \to D^b(E)$$

and the third and fourth merges

$$R_{12}^3 = \pi_{pt1*} \circ i_{P^2*}^*: D^b(A) \to D^b(pt), \quad R_{21}^3 = \pi_{pt2*} \circ i_{P^2*}^*: D^b(E) \to D^b(pt).$$

![Figure 4.12: Third and fourth forks](image)

![Figure 4.13: Third and fourth merges](image)

**Remark 4.4.6.** The functor $F_{12}^3$ (and similarly for the functor $F_{21}^3$) can be also described as the functor that maps the 1-dimensional vector space to $\mathcal{O}_{P^2}$ and its right adjoint $R_{12}^3$ is the functor $RHom(\mathcal{O}_{P^2}, -)$.

$R_{111}^{12}$ is represented by the Fourier-Mukai kernel $\mathcal{O}_{B \times A}(B, 0)[-1]$.

$F_{21}^{111}$ is represented by the Fourier-Mukai kernel $\mathcal{O}_{E \times E D}$. 
Proposition 4.4.7. The Fourier-Mukai kernel associated to $R^{12}_{111}F^{111}_{21}$ is $\mathcal{O}_{Z_1} \otimes V_1^* \otimes (\Lambda^2 V_2)^2[-1]$, where

$$Z_1 := \left\{ \begin{array}{c} \alpha \vphantom{0} \\ 0 \subset V_2 \subset \mathbb{C}^3 \end{array} \right\} \times \left\{ \begin{array}{c} \alpha \vphantom{0} \\ 0 \subset V_1 \subset \mathbb{C}^3 \end{array} \right\}$$

is a line bundle over $\text{Fl}_3$ and it is also the blow up of $A$ or $E$ along their zero sections.

Proof. Let $\pi_{12}, \pi_{23}, \pi_{13}$ be the natural projections

$$\begin{array}{c c c}
E \times C \times A & \overset{\pi_{12}}{\longrightarrow} & E \times C \\
\downarrow \pi_{13} & & \downarrow \pi_{23} \\
E \times A & \overset{\pi_{13}}{\longrightarrow} & C \times A \\
\end{array}$$

By Proposition 3.2.3, the composition is represented by the convolution of their Fourier-Mukai kernels, thus the kernel of $R^{12}_{111}F^{111}_{21}$ is isomorphic to

$$\pi_{13*}(\mathcal{O}_{E \times E D} \otimes \mathcal{O}_{E \times B \times A}(B, 0)[-1]) = \pi_{13*}(\mathcal{O}_{E \times E D} \otimes \mathcal{O}_{E \times B \times A}(0, B, 0)[-1])$$

The subvariety $E \times E D \times A$ is of codimension 5 inside $E \times C \times A$; the same codimension is the one of $E \times B \times A$ in $E \times C \times A$.

Their intersection $(E \times E D \times A) \cap (E \times B \times A A) = E \times E (D \cap B) \times A A$ is smooth and of codimension 10, therefore $E \times E D \times A$ and $E \times B \times A A$ intersect transversally in $E \times C \times A$.

By Corollary 3.9.4,

$$\mathcal{O}_{E \times E D \times A} \otimes \mathcal{O}_{E \times B \times A} \simeq \mathcal{O}_{E \times E (D \cap B) \times A A}$$

The subvariety $E \times E (D \cap B) \times A A$ can be described as the space

$$\left\{ \begin{array}{c} \alpha \vphantom{0} \\ 0 \subset V_2 \subset \mathbb{C}^3 \end{array} \right\} \times \left\{ \begin{array}{c} \alpha \vphantom{0} \\ 0 \subset V_1 \subset \mathbb{C}^3 \end{array} \right\} \times \left\{ \begin{array}{c} \alpha \vphantom{0} \\ 0 \subset V_2 \subset \mathbb{C}^3 \end{array} \right\} \times \left\{ \begin{array}{c} \alpha \vphantom{0} \\ 0 \subset V_1 \subset \mathbb{C}^3 \end{array} \right\}.$$
4.5. A skein triangulated action of $\mathcal{GBr}_3$: main theorem

Notice that by 3.10.3 $\mathcal{O}_Z(0, B, 0) \simeq \mathcal{V}^*_1 \otimes (\Lambda^2 \mathcal{V}_2)^2$, so the Fourier-Mukai kernel of $R_{111}^{12} F_{21}^{111}$ is isomorphic to

$$\mathcal{O}_Z \otimes \mathcal{V}^*_1 \otimes (\Lambda^2 \mathcal{V}_2)^2[-1].$$

$\square$

The single crossings are the autoequivalences $T_i$ of section 3.10 that induces the Khovanov-Thomas braid group action on $D^b(C)$ of Theorem 3.10.4.

**Remark 4.4.8.** The simple crossing functors $T_{111}^{111}$ $T_{111}^{111}$ are

$$T_{111}^{111} = \text{Cone} \left( F_{21}^{111} R_{111}^{21}[-1] \xrightarrow{\text{tr}} \text{Id}_{111} \right),$$

$$T_{111}^{111} = \text{Cone} \left( F_{12}^{111} R_{111}^{12}[-1] \xrightarrow{\text{tr}} \text{Id}_{111} \right),$$

where $\text{tr}$ is the counit of the adjunction.

**Proposition 4.4.9.** The Fourier-Mukai kernel associated to $T_{111}^{111}$ is the sheaf

$$\mathcal{O}_{(C \times C) \cup (D \times D)}(D, 0).$$

The Fourier-Mukai kernel associated to $T_{111}^{111}$ is the sheaf $\mathcal{O}_{(C \times C) \cup (B \times B)}(B, 0)$.

**Proof.** Proposition 4.4 in [32].

$\square$

4.5 A skein triangulated action of $\mathcal{GBr}_3$: main theorem

In this section, we prove the main theorem of the thesis verify assumptions for a skein triangulated action of $\mathcal{GBr}_3$ on $D^b(T^* \text{Fl}_3(\tilde{i}))$.

**Lemma 4.5.1.** The following isomorphism holds

$$F_{21}^{111} F_{3}^{21} \simeq F_{12}^{111} F_{3}^{12}$$

\[\begin{array}{c}
\text{Diagram}
\end{array}\]
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Proof. The Fourier-Mukai kernel representing $F_{111}^{11} F_{21}^{21} F_{3}^{12}$ is isomorphic to

$$\pi_{13*}(\pi_{12}^* (O_{pt \times pt A}) \otimes \pi_{23}^* (O_{A \times A B})) \simeq \pi_{13*}(O_{pt \times pt \mathbb{P}^2 \times C} \otimes O_{pt \times A \times B}) \simeq \pi_{13*}(O_{pt \times pt \mathbb{P}^2 \times \mathbb{P}^2 F_3}) \simeq O_{pt \times F_3}.$$ (4.50)

Similarly the Fourier-Mukai kernel associated to $F_{111}^{11} F_{21}^{21} F_{3}^{12}$ is isomorphic to

$$O_{pt \times F_3}.$$

Remark 4.5.2. From Remark 4.4.6, Lemma 4.5.1 can be also proved showing that functors $F_{111}^{11} F_{21}^{21} F_{3}^{12}$ both map the 1-dimensional vector space to $O_{F_3}$. 

Lemma 4.5.3. The mapping cone of the morphism

$$R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} \xrightarrow{\text{tr}[-2]} \text{Id}[-2]$$

is isomorphic to

$$\text{Cone}(\text{tr}[-2]) \simeq F_{3}^{12} R_{12}^{3}.$$ (4.51)

Proof. The Fourier-Mukai kernel representing $F_{3}^{12} R_{12}^{3}$ is isomorphic to

$$\pi_{13*}(\pi_{12}^* (O_{A \times pt A}) \otimes \pi_{23}^* (O_{pt \times pt A}(0, \mathbb{P}^2))[1]) \simeq \pi_{12}^* (O_{\mathbb{P}^2 \times pt \times A} \otimes O_{A \times pt \times \mathbb{P}^2 (0, \mathbb{P}^2)}[-1]) \simeq \pi_{13*}(O_{\mathbb{P}^2 \times pt \times \mathbb{P}^2 (0, \mathbb{P}^2)} [-1]) \simeq O_{\mathbb{P}^2 \times \mathbb{P}^2 (0, \mathbb{P}^2)} [-1].$$

The Fourier-Mukai kernel representing $R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111}$ is isomorphic, by the proof of Lemma 4.4.7, to

$$\pi_{15*}(O_{A \times A (B \cap D) \times E \times C \times A (0, D, 0, 0, 0) \otimes O_{A \times C \times E \times E (B \cap D) \times A (0, 0, 0, B, 0)} [-2].$$

The subvarieties $A \times A (B \cap D) \times E \times C \times A$ and $A \times C \times E \times E (B \cap D) \times A$ are both of codimension 10 in $A \times C \times E \times C \times A$. Their intersection $A \times A (B \cap D) \times E$
4.5. A skein triangulated action of $\mathcal{GBr}_3$: main theorem

$E \times (B \cap D) \times_A A$ is of codimesion 20 in $A \times C \times E \times C \times A$ therefore by Corollary 3.9.4 the Fourier-Mukai kernel representing $R_{12}^{11} F_{111}^{111} R_{111}^{21} F_{12}^{111}$ is isomorphic to

$$\pi_{15*}(O_{A \times A(B \cap D) \times E \times E(B \cap D) \times A}(0, D, 0, B, 0))[-2]$$

whose support is isomorphic to $(B \cap D) \times_E (B \cap D)$.

The variety $(B \cap D) \times E (B \cap D)$ has two irreducible components, one of them $X_1$ is isomorphic to $Z_1$ of Proposition 4.4.7, and the other one $X_2$ is the blow up of $\mathbb{P}^2 \times \mathbb{P}^2$ along the diagonal.

The intersection $X_1 \cap X_2$ of the two components is isomorphic to $Fl_3$ which is the exceptional divisor inside $X_2$.

Thus, by 4.42 we have the following short exact sequence

$$0 \to \mathcal{O}_{X_2}(-(X_1 \cap X_2)) \to \mathcal{O}_{X_1 \cup X_2} \to \mathcal{O}_{X_1} \to 0$$

hence we have the isomorphism

$$\mathcal{O}_{X_1 \cup X_2} \simeq Cone(\mathcal{O}_{X_1}[1] \to \mathcal{O}_{X_2}(-(X_1 \cap X_1)))$$

and therefore in $D^b(A \times C \times E \times C \times A) \mathcal{O}_{A \times A(B \cap D) \times E \times E(B \cap D) \times A}(0, D, 0, B, 0)$ is isomorphic to

$$Cone(\mathcal{O}_{Y_1}[1] \to \mathcal{O}_{Y_2}(-(Y_1 \cap Y_1))) \otimes \mathcal{O}(0, D, 0, B, 0).$$

When we project via $\pi_*$ to $A \times A$ the first component $X_1$, it surjects onto the diagonal, while the second component $X_2$ surjects onto $\mathbb{P}^2 \times \mathbb{P}^2$. Since both maps are blow-downs, the projection is an isomorphism except over the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$ where it is a $\mathbb{P}^1$-bundle.

Thus, taking the derived pushforward $\pi_{15*}$ of $Cone(\mathcal{O}_{Y_1}[1] \to \mathcal{O}_{Y_2}(-(Y_1 \cap Y_1))) \otimes \mathcal{O}(0, D, 0, B, 0)$ and applying Corollary 4.5 of [3] to the map

$$R_{12}^{12} F_3^{12} \to R_{111}^{111} F_{21}^{111} R_{111}^{21} F_{12}^{111}$$

we have that

$$F_3^{12} R_{12}^{12} \to R_{111}^{111} F_{21}^{111} R_{111}^{21} F_{12}^{111} \to \text{Id}_A[-2]$$

is a distinguished triangle.

The following Lemma holds in a more general context.

**Lemma 4.5.4.** Let $X$ be smooth projective variety over $k$, $\pi : X \to \text{Spec } k$ be the structure morphism, and $\iota : X \hookrightarrow T^*(X)$ be the embedding of the zero section:

$$\begin{array}{ccc}
X & \xhookrightarrow{\iota} & T^*(X) \\
\pi & \downarrow & \\
pt & & \\
\end{array}$$

(4.52)
Let $P^*$ and $P_*$ be the standard Fourier-Mukai kernels of the exact functors
\[ \pi^*: D(pt) \to D(X), \]
\[ \pi_*: D(X) \to D(pt), \]
and let $I_s$ and $I^1$ be the standard Fourier-Mukai kernels of the exact functors
\[ \iota_*: D(X) \to D(T^*X), \]
\[ \iota^1: D(T^*X) \to D(X). \]

See [7], Section 2.6.2 for the details on the standard kernels.

Then we have an isomorphism in $D(pt)$:
\[ P_* I^1 I_* P^* \simeq \Delta_* H^*(X,k). \]

**Proof.** By Lemma 2.19 of [7], we have
\[ P_* I^1 I_* P^* \simeq (\pi \times \pi)_* I^1 I_* . \quad (4.53) \]

By Proposition 7.8 of [7], the object $I^1 I_* \in D(X \times X)$ has the cohomology sheaves:
\[ H^i(I^1 I_*) \simeq \Delta_* \wedge^i \mathcal{N}_{X/T^*X}, \]
in degrees $0 \leq i \leq n$ and 0 elsewhere.

Moreover, by Theorem 1.8(6) of [9], the object $I^1 I_*$ is formal, and hence
\[ I^1 I_* \simeq \bigoplus_{i=0}^n \Delta_* \wedge^i \mathcal{N}_{X/T^*X}. \]

Since $\mathcal{N}_{X/T^*X} \simeq \Omega^1_{X/k}$, we conclude that
\[ I^1 I_* \simeq \Delta_* \left( \bigoplus_{i=0}^n \Omega^i_{X/k} \right). \]

Thus we have
\[ P_* I^1 I_* P^* \simeq (\pi \times \pi)_* I^1 I_* \simeq (\pi \times \pi)_* \Delta_* \left( \bigoplus_{i=0}^n \Omega^i_{X/k} \right) \simeq \Delta_* \pi_* \left( \bigoplus_{i=0}^n \Omega^i_{X/k} \right). \]

Since $\pi_*$ is isomorphic to the derived global section functor $R \Gamma(-)$, the assertion of the lemma follows by the degeneration of the Hodge-de-Rham spectral sequence. \qed

The main theorem of the thesis is the following:

**Theorem 4.5.5.** Assume that Conjecture 4.2.1 holds; then the assignment of:

1. the partition $(111)$ to the category $D^b(T^*Fl_3)$;
4.5. A skein triangulated action of $\mathcal{GB}_3$: main theorem

2. the partition $(12)$ to the category $D^b(T^* \mathbb{P}^2)$;
3. the partition $(21)$ to the category $D^b(T^* \mathbb{P}^{2\vee})$;
4. the partition $(3)$ to the category $D^b(pt)$;

and the assignment of:

1. the fork $f^{111}_{21}$ to the functor $F^{111}_{21}$;
2. the fork $f^{111}_{12}$ to the functor $F^{111}_{12}$;
3. the fork $f^{21}_3$ to the functor $F^{21}_3$;
4. the fork $f^{12}_3$ to the functor $F^{12}_3$;

define a skein triangulated representation of $\mathcal{GB}_3$ on $T^* \text{Fl}_3(i)$.

Proof. By Examples 3.5.6 and 3.7.5, $F^{111}_{21}$ and $F^{111}_{12}$ are split spherical functors with cotwist $[-2]$, while $F^{21}_3$ and $F^{12}_3$ are split $\mathbb{P}^2$ functors with $H = [-2]$.

As a consequence of Lemma 4.5.1, there exists a multifork isomorphism. By Lemma 4.5.4, there exists an isomorphism

$$R^3_{12}R^{12}_{111}F^{111}_{12}F^{12}_3 \simeq \text{Id}_3 \oplus [-2] \oplus [-2] \oplus [-4] \oplus [-4] \oplus [-6] \simeq R^3_{21}R^{21}_{111}F^{111}_2F^{21}_3$$

and moreover, from Theorem 7.2 of [7], it identifies together with the $\mathbb{P}^2$ functor structure of $F^{21}_3$ and $F^{12}_3$ the maps 4.8, 4.9 with

$$\text{Id}_3 \oplus [-2] \oplus [-4] \rightarrow \text{Id}_3 \oplus [-2] \oplus [-2] \oplus [-4] \oplus [-4] \oplus [-6].$$

Finally, by Lemma 4.5.3, the following two diagrams can be completed to two distinguished triangles

$$F^{12}_3R^3_{12} \rightarrow R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{12} \rightarrow \text{Id}_{12}[-2],$$

$$F^{21}_3R^3_{21} \rightarrow R^{21}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21} \rightarrow \text{Id}_{21}[-2].$$

Thus, by Theorem 4.2.3 such assignment define a categorical action of $\mathcal{GB}_3$ on $T^* \text{Fl}_3(i)$. \qed

Remark 4.5.6. From Remark 4.2.2, a categorical action of $\mathcal{GB}_2$ needs only to satisfy relation 4.2.

Therefore, it is a case already covered by Theorem 3.10.4 of Section 3.10.
4.6 Further developments

As a continuation of the results of the previous section, the long term research plan is to generalise our theory to the $\mathcal{GB}r_n$ case.

In this section we present some computations that we expect to be helpful to understand how to generalise Theorem 4.5.5 in arbitrary dimensions.

In particular, for computing higher dimensional multiple crossings we use the invariance of the generalised braids under isotopies and an induction on the construction of figure 4.14.

![Figure 4.14: Higher dimensional induction](image)

The idea behind this computation is a conjectural program which allows us to inductively compute the multiple crossing functors in higher dimensions and obtain equivalences of type $(pq) - (qp)$.

To motivate the general argument, in this thesis we provide the case $n = 3$.

Therefore, we construct the functor $T_{21}^{12}$ as the difference

$$T_{21}^{12} \circ R_{111}^{12} F_{12}^{111} \simeq R_{111}^{21} T_{111}^{111} T_{111}^{111} F_{12}^{111}.$$  \hfill (4.54)

We first compute the loop functor $R_{111}^{21} F_{21}^{111}$

- $R_{111}^{21}$ is represented by the Fourier-Mukai kernel $\mathcal{O}_{D \times E}(D, 0)[-1]$.
- $F_{21}^{111}$ is represented by the Fourier-Mukai kernel $\mathcal{O}_{E \times E \times E}$. 

**Lemma 4.6.1.** The Fourier-Mukai kernel associated to $R_{111}^{21} F_{21}^{111}$ is isomorphic to $\pi_{13*}(\mathcal{E})$ where the cohomologies of $\mathcal{E}$ are

$$H^r(\mathcal{E}) = \begin{cases} 
\mathcal{O}_{E \times E \times E}(0, D, 0), & \text{if } r = 0; \\
\mathcal{O}_{E \times E \times E}, & \text{if } r = -1; \\
0, & \text{otherwise.}
\end{cases}$$
4.6. Further developments

and where $\pi_{13}$ is the natural projection $\pi_{13} : A \times C \times A \to A \times A$.

Proof. Let $\pi_{12}, \pi_{23}, \pi_{13}$ be the natural projections

$$
\begin{array}{c}
A \times C \\
\pi_{12} \\
\downarrow \\
A \times C \\
\pi_{13} \\
\downarrow \\
A \times A \\
\pi_{23} \\
\downarrow \\
C \times A \\
\end{array}
$$

(4.55)

By the standard technique of Proposition 3.2.3, the Fourier-Mukai kernel of the composition $R_{111}^{21} F_{21}^{11}$ is isomorphic to

$$
\pi_{13}^* (\pi_{12}^* \mathcal{O}_{E \times E D} \otimes \pi_{23}^* \mathcal{O}_{D \times E D}(D, 0)[-1]) \simeq \\
\simeq \pi_{13}^* (\mathcal{O}_{E \times E D \times E} \otimes \mathcal{O}_{E \times D \times E}(0, D, 0)[-1]).
$$

We observe that the supports $E \times E D \times E$ and $E \times D \times E$ of the sheaves involved in the derived tensor product are both of codimension 5 inside $E \times C \times E$.

The intersection $E \times E D \times E = (E \times E D \times A) \cap (E \times D \times E D)$ is smooth and codimension 9 inside $E \times C \times E$, therefore with a rank 1 excess bundle $\mathcal{E}$.

The excess line bundle $\mathcal{E}$ is equal to $\mathcal{O}(0, -D, 0)$; this follows from the adjunction formula and by fact that $\mathcal{E}$ is of rank 1. By Proposition 3.9.2 the cohomologies of $\mathcal{O}_{E \times E D \times E} \otimes \mathcal{O}_{E \times D \times E}(0, D, 0)$ are therefore

$$
H^r (\mathcal{O}_{E \times E D \times E} \otimes \mathcal{O}_{E \times D \times E}(0, D, 0)) = \\
\begin{cases} 
\mathcal{O}_{E \times E D \times E}(0, D, 0), & \text{if } r = 0, \\
\mathcal{O}_{E \times E D \times E}, & \text{if } r = -1; \\
0, & \text{otherwise.}
\end{cases}
$$

The following Lemma holds for general varieties and vector bundles.

**Lemma 4.6.2.** Let $Y$ be a variety and let $E \to Y$ be a vector bundle. If $X = \mathbb{P}(E)$ is the projectivisation of $E$ and is the projection $\pi : X = \mathbb{P}(E) \to Y$ then $\pi_*(\Omega_{X/Y}^k) = \mathcal{O}_Y[k]$.

Proof. Consider the short exact sequence

$$
0 \to \Omega_{X/Y} \to \pi^* (E^\vee) (-1) \to \mathcal{O}_X \to 0
$$

(4.56)

which is the dual of the Euler sequence.

If we apply the functor $\pi_*$ to (4.56) we obtain the short exact sequence

$$
0 \to \pi_* (\Omega_{X/Y}) \to 0 \to \mathcal{O}_Y \to 0
$$

(4.57)

indeed, since $X$ is a projective bundle, $\pi_* \mathcal{O}_X = \mathcal{O}_Y$ and $\pi_* F(-1) = 0$. 

Thus, from the long exact sequence associated to (4.57) we get
\[ \pi_*(\Omega^k_{X/Y}) \simeq \mathcal{O}_Y[k]. \]

\[ \Box \]

**Lemma 4.6.3.** \( \pi_{13*}(\mathcal{O}_{E \times D \times E} \otimes \mathcal{O}_{E \times D \times E}(0, D, 0))[-1] \) is formal in \( E \times E \).

**Proof.** From the adjunction (4.43) we have the distinguished triangle
\[ \text{id} \to R^{21}_{111}F^{111}_{21} \to \text{id}[-2] \]
which at level of Fourier-Mukai kernels correspond to the distinguished triangle
\[ \Delta_*\mathcal{O}_E \to X \to \Delta_*\mathcal{O}_E[-2] \quad (4.58) \]
where \( \Delta : E \to E \times E \) is diagonal embedding and \( X \) is the Fourier-Mukai kernel of \( R^{21}_{111}F^{111}_{21} \). The abelian group of equivalence classes of distinguished triangles of the form
\[ \mathcal{O}_{E \times E} \to Y \to \mathcal{O}_{E \times E}[-2] \]
with \( Y \in D^b(E \times E) \) is isomorphic to the group
\[ \text{Ext}^1(\mathcal{O}_{E \times E}[-2], \mathcal{O}_{E \times E}) \simeq \text{Ext}^3(\mathcal{O}_{E \times E}, \mathcal{O}_{E \times E}). \]

Therefore, in order to prove that \( \pi_{13*}(\mathcal{O}_{E \times D \times E} \otimes \mathcal{O}_{E \times D \times E}(0, D, 0))[-1] \) is formal in \( E \times E \), it is sufficient to prove that the distinguished triangle (4.58) correspond to the zero class in \( \text{Ext}^3(\mathcal{O}_{E \times E}, \mathcal{O}_{E \times E}). \)
The functor \( F^{111}_{21}R^{21}_{111}F^{111}_{21} \) is a retract, so it correspond to the zero class of
\[ \text{Ext}^1(F^{111}_{21}[-2], F^{111}_{21}) \]

Let \( \pi_{12}, \pi_{23}, \pi_{13} \) be the natural projections
\[ E \times E \quad \xrightarrow{\pi_{12}} \quad E \times E \times C \quad \xrightarrow{\pi_{13}} \quad E \times C \quad \xrightarrow{\pi_{23}} \quad E \times C. \quad (4.59) \]

The Fourier-Mukai kernel of \( F^{111}_{21}R^{21}_{111}F^{111}_{21} \) is then by Proposition 3.2.3 by base change around the commutative square
\[ \text{id} \quad \xrightarrow{(\pi_{12}, \pi_{13})} \quad E \times E \times C \quad \xrightarrow{(\pi_{23})} \quad E \times C \quad \xrightarrow{\pi_{23}} \quad E \times C. \quad (4.60) \]
4.6. Further developments

isomorphic to

\[ \pi_{13*}(\pi_{12}^*(X) \otimes \pi_{23}^*(\pi, i)_*\mathcal{O}_D) \simeq \pi_{13*}(\pi_{12}^*(X) \otimes \pi_{23}^*(id, \pi, i)_*\mathcal{O}_{E \times D}) \]

by projection formula we have the isomorphism

\[ \pi_{13*}(\pi_{12}^*(X) \otimes \pi_{23}^*(id, \pi, i)_*\mathcal{O}_{E \times D}) \simeq \pi_{13}(id, \pi, i)^* (id, \pi, i)_* \pi_{12}^* X. \]

Moreover, since

\[ \pi_{13}(id, \pi, i)_* = (\pi_{13} \circ (id, \pi, i))_* = (\pi, i)_* \]

and

\[ (id, \pi, i)^* \pi_{12}^* = (\pi_{12} \circ (id, \pi, i))^* = (id, \pi)^* \]

we have the isomorphism

\[ \pi_{13}(id, \pi, i)^*(id, \pi, i)_* \pi_{12}^* X \simeq (\pi, i)_* (id, \pi)^* X. \]

Consider now the following commutative diagram

\[
\begin{array}{ccc}
\text{Ext}^3_{A \times C}(\mathcal{O}_D, \mathcal{O}_D) & \xrightarrow{g} & \text{Ext}^3_{A \times C}(\mathcal{O}_D, \mathcal{O}_D) \\
\downarrow \quad \pi_2^* \circ i_* & & \quad \pi_2^* \circ i_* \\
\text{Ext}^3_{A \times C}(\mathcal{O}_D, \mathcal{O}_D) & \xrightarrow{h} & \text{Ext}^3_{A \times C}(\mathcal{O}_D, \mathcal{O}_D)
\end{array}
\] (4.61)

The zero element \( 0_{A \times C} \in \text{Ext}^3_{A \times C}(\mathcal{O}_D, \mathcal{O}_D) \) is sent by \( h \) to the zero element of \( \text{Ext}^3_{C}(i_* \mathcal{O}_D, i_* \mathcal{O}_D) \); showing that the homomorphism \((\pi, i)_*\) is injective proves therefore that \( g \) is injective and that \( g^{-1}(0_{A \times C}) = 0_{A \times B} \).

Indeed, since \( \pi_2^* i_* \pi_{2*}(\mathcal{O}, i)_* = \pi_2^* i_* i_* \), we have the isomorphism,

\[ \pi_2^* i_* \pi_{2*}(\mathcal{O}, i)_* \mathcal{O}_D \simeq \pi_2^* i_* \mathcal{O}_D \]

but \( i : D \to C \) is a divisorial inclusion, hence \( i^* \mathcal{O}_D = \mathcal{O}_D \oplus \mathcal{O}_D(D)[1] \) and

\[ \pi_2^* i_* \mathcal{O}_D \simeq \pi_2^* (\mathcal{O}_D \oplus \mathcal{O}_D(D)[1]) \]

which correspond to the zero element as it splits and therefore \( i_* \pi_{2*} \) is injective.

Since \((\pi, i)_* \mathcal{O}_{E \times D} \simeq \mathcal{O}_{E \times E} \), because \( D \twoheadrightarrow E \) is \( \mathbb{P}^1 \)-bundle, and \((id, \pi)^*\) is fully faithful, we have that under the isomorphism

\[ f : \text{Ext}^3((\pi, i)_* \mathcal{O}_E, (\pi, i)_* \mathcal{O}_E) \xrightarrow{\simeq} \text{Ext}^3((\pi, i)_* \mathcal{O}_D, (\pi, i)_* \mathcal{O}_D) \]

the Fourier-Mukai the element \( 0_{A \times C} \in \text{Ext}^3_{A \times C}(\mathcal{O}_D, \mathcal{O}_D) \) that correspond to the Fourier-Mukai kernel representing \( F_2^{111} R_2^{211} F_2^{111} \) has preimage under \( f \circ g \) the zero element of \( \text{Ext}^3(\Delta_* \mathcal{O}_E, \Delta_* \mathcal{O}_E) \) and therefore \( X \) is formal. \( \square \)
Proposition 4.6.4. The Fourier-Mukai kernel associated to $R_{111}^{21} F_{21}^{111}$ is $\mathcal{O}_{E \times E} \oplus \mathcal{O}_{E \times E}[−2]$

Proof. Consider the projections $\pi_{12}, \pi_{23}, \pi_{13}$ as in (4.66).

By Lemma 4.6.1 the non-zero cohomologies of $\mathcal{O}_{E \times E} \oplus \mathcal{O}_{E \times E}(0, D, 0)$ are $\mathcal{O}_{E \times E}(0, D, 0)$ in degree -1 and $\mathcal{O}_{E \times E}$ in degree zero.

We have $\pi_{13} \mathcal{O}_{E \times E} \mathcal{O}_{E \times E} \simeq \mathcal{O}_{E \times E}$ since $E \times E$ is a $\mathbb{P}^1$ bundle, while by Lemma 4.6 we have that $\pi_{13} \mathcal{O}_{E \times E} \mathcal{O}_{E \times E}(0, D, 0)[-1] \simeq \mathcal{O}_{E \times E}[−2]$.

Finally by Lemma 4.6.3 we conclude that $F_1 R_1 \simeq \mathcal{O}_{E \times E} \oplus \mathcal{O}_{E \times E}[−2]$. 

By Proposition 4.6.4 we know that the Fourier Mukai kernel associated to the functor $R_{11}^{12} F_{12}^{111}$ is $\mathcal{O}_{E \times E} \oplus \mathcal{O}_{E \times E}[−2]$. While $R_{111}^{121} T_{111}^{111} T_{111}^{111} F_{12}^{111}$ is by definition of $T_{111}^{111}$ and $T_{111}^{111}$ isomorphic to

$$R_{111}^{21} T_{111}^{111} T_{111}^{111} F_{12}^{111} \simeq R_{111}^{121} Cone(F_{12}^{111} R_{111}^{12} \xrightarrow{\text{tr}^2} \text{id}) Cone(F_{111}^{121} R_{111}^{21} \xrightarrow{\text{tr}^1} \text{id}) F_{12}^{111}$$  \hspace{1cm} (4.62)

where $T_{111}^{111} = Cone(F_{12}^{111} R_{111}^{12} \xrightarrow{\text{tr}} \text{id})$ and tr is the counit of the adjunction $F_{12}^{111} \dashv R_{111}^{121}$.

Therefore the functor $R_{111}^{21} T_{111}^{111} T_{111}^{111} F_{12}^{111}$ is the convolution of the diagram

$$R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111} \xrightarrow{\text{tr}_1} \bigoplus \xrightarrow{\text{tr}_2} R_{111}^{21} F_{21}^{111}$$

$$R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111} \xrightarrow{\text{tr}_1} \bigoplus \xrightarrow{\text{tr}_2} R_{111}^{21} F_{21}^{111}$$

Notice that by Lemma 4.6.3 we have the following the isomorphisms

$$R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} \simeq R_{111}^{12} F_{21}^{111} \oplus R_{111}^{12} F_{21}^{111}[−2]$$

and similarly for $R_{111}^{12} F_{12}^{111} R_{111}^{12} F_{21}^{111}$

$$R_{111}^{12} F_{12}^{111} R_{111}^{12} F_{21}^{111} \simeq R_{111}^{12} F_{21}^{111} \oplus R_{111}^{12} F_{21}^{111}[−2].$$

Thus, the convolution of diagram (4.63) is isomorphic to the convolution of diagram

$$R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111} \xrightarrow{\text{tr}_1} \bigoplus \xrightarrow{\text{tr}_2} R_{111}^{21} F_{21}^{111}$$

$$R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111} \xrightarrow{\text{tr}_1} \bigoplus \xrightarrow{\text{tr}_2} R_{111}^{21} F_{21}^{111}$$

(4.64)
4.6. Further developments

Thus, to compute the double crossing functor $T^{12}_{21}$ we have the following recipe:

1. Compute the Fourier-Mukai transform associated to

$$R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21}.$$

2. Determine the Fourier-Mukai kernel of $R^{21}_{111}T^{111}_{21}F^{111}_{12}$ as a convolution of diagram (4.64).

3. Define the double crossing functor $T^{12}_{21}$ using the isomorphism (4.54).

**STEP 1: the Fourier Mukai kernel of $R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21}$**

The first step of the recipe for the computation of the double crossing $T^{12}_{21}$ is to compute the Fourier-Mukai transform associated to $R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21}$.

- $F^{21}_{111}$ is represented by the Fourier-Mukai kernel $O_{E \times E D}$.
- $R^{111}_{12}$ is represented by the Fourier-Mukai kernel $O_{B \times A A(B, 0)[−1]}$.
- $F^{111}_{12}$ is represented by the Fourier-Mukai kernel $O_{A \times A B}$.
- $R^{21}_{21}$ is represented by the Fourier-Mukai kernel $O_{D \times D E(D, 0)[−1]}$.
- $F^{111}_{21}$ is represented by the Fourier-Mukai kernel $O_{E \times E D}$.
- $R^{111}_{12}$ is represented by the Fourier-Mukai kernel $O_{B \times A A(B, 0)[−1]}$.

Figure 4.15: The functor $R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21}$

We begin by splitting the big composition in smaller ones in order make it more accessible.

**Lemma 4.6.5. The Fourier-Mukai kernel associated to $R^{21}_{111}F^{111}_{21}$ and $R^{12}_{111}F^{111}_{21}$ are respectively $O_{D \times E D(D, 0)[−1]}$ and $O_{B \times A B(B, 0)[−1]}$.**

- $R^{111}_{12}$ is represented by the Fourier-Mukai kernel $O_{D \times D E(D, 0)[−1]}$.
- $F^{21}_{111}$ is represented by the Fourier-Mukai kernel $O_{E \times E D}$. 
Proof. Let \( \pi_{12}, \pi_{23}, \pi_{13} \) be the natural projections

\[
\begin{array}{ccc}
C \times E \times C & \overset{\pi_{12}}{\leftarrow} & C \times E \\
& \overset{\pi_{13}}{\rightarrow} & \overset{\pi_{23}}{\rightarrow} & E \times E & \rightarrow & E \times C.
\end{array}
\]

(4.65)

The Fourier-Mukai kernel associated to \( R_{111}^{211} F_{21}^{111} \) is by Proposition 3.2.3

\[ \pi_{13*}(\pi_{12}^* \mathcal{O}_{D \times E}(D, 0)[-1] \otimes \pi_{23}^* \mathcal{O}_{E \times E}(D, 0) \otimes \mathcal{O}_{C \times E \times E D})[-1] \]

Let us compare the codimensions of the supports of the sheaves involved in the derived tensor product: it is easy to see that \( D \times E \times E \times C \times E \times E D \) are both of codimension 5 inside \( C \times E \times C \).

Their intersection \( (C \times E \times E D) \cap (D \times E \times E C) = D \times E \times E D \) is smooth and of codimension 10, therefore their intersection is transverse and we can use Lemma 3.9.4 and obtain the isomorphism

\[ \mathcal{O}_{D \times E \times E C}(D, 0) \otimes \mathcal{O}_{C \times E \times E D} \simeq \mathcal{O}_{D \times E \times E D}(D, 0, 0). \]

Notice that the morphism \( \pi_{13} : D \times E \times E D \to C \times C \) is an embedding. Indeed, it is easy to see that the space \( D \times E \times E D \)

\[
\left\{ \begin{array}{c} 0 \subset V_1 \subset V_2 \subset \mathbb{C}^3 \end{array} \right\} \times \left\{ \begin{array}{c} 0 \subset V_2 \subset \mathbb{C}^3 \end{array} \right\} \times \left\{ \begin{array}{c} 0 \subset V_1' \subset V_2 \subset \mathbb{C}^3 \end{array} \right\}
\]

is naturally embedded into \( C \times C \)

\[
C \times C = \left\{ \begin{array}{c} \alpha \end{array} \right\} \times \left\{ \begin{array}{c} \alpha \end{array} \right\} \times \left\{ \begin{array}{c} \alpha \end{array} \right\}
\]

and the image \( \pi_{13}(D \times E \times E D) = D \times E D \).

In conclusion, the Fourier-Mukai kernel associated to \( R_{111}^{211} F_{21}^{111} \) is isomorphic to

\[ \pi_{13*}(\mathcal{O}_{D \times E \times E D}(D, 0))[-1] \simeq \mathcal{O}_{D \times E D}(D, 0)[-1]. \]

The same argument shows that the kernel of \( R_{111}^{121} F_{12}^{111} \) is isomorphic to

\[ \mathcal{O}_{B \times A B}(B, 0)[-1]. \]

\( \square \)
Lemma 4.6.6. The Fourier-Mukai kernel associated to \( F_1^{11}R_1^{21}F_1^{11}R_1^{12} \) is isomorphic to \( \mathcal{O}_{Z_3}(D,0)[-2] \) where

\[
Z_3 = \left\{ \begin{array}{ccc}
0 & \subseteq & V_1 \\
V_1 & \subseteq & V_2 \\
V_2 & \subseteq & \mathbb{C}^3
\end{array} \right\} \times \left\{ \begin{array}{ccc}
0 & \subseteq & V_1' \\
V_1' & \subseteq & V_2' \\
V_2' & \subseteq & \mathbb{C}^3
\end{array} \right\}
\]

Proof. Let \( \pi_{12}, \pi_{23}, \pi_{13} \) be the natural projections

\[
\begin{array}{ccc}
C \times C \times C & \xrightarrow{\pi_{12}} & C \times C \\
& \searrow & \downarrow \pi_{13} \\
& & C \times C
\end{array}
\]

Denote by \( F_1R_1F_2R_2 \) the Fourier-Mukai kernel of \( F_1^{11}R_1^{21}F_1^{11}R_1^{12} \).

By Proposition 3.2.3 and by the previous Lemma 4.6.5 \( F_1R_1F_2R_2 \) is isomorphic in \( D^b(C \times C) \) to

\[
F_1R_1F_2R_2 \simeq \pi_{13*}(\pi_{12}^* \mathcal{O}_{B \times A B}(B,0)[-1] \otimes \pi_{23}^* \mathcal{O}_{D \times E D}(D,0)[-1]) = \pi_{13*}(\mathcal{O}_{B \times A B \times C}(B,0,0) \otimes \mathcal{O}_{C \times D \times E D}(0, D, 0))[-2]
\]

The supports of \( \mathcal{O}_{B \times A B \times C}(B,0,0) \) and \( \mathcal{O}_{C \times D \times E D}(0, D, 0) \) are both of codimension 5 inside \( C \times C \times C \), their intersection \( D \times E (B \cap D) \times_A B = (B \times_A B \times C) \cap (C \times D \times E D) \) is smooth and of codimension 10.

\[
\mathcal{O}_{B \times A B \times C}(B,0,0) \otimes \mathcal{O}_{C \times D \times E D}(0, D, 0) \simeq \mathcal{O}_{D \times E (B \cap D) \times_A B}(B, D, 0)
\]

The projection \( \pi_{13} : D \times E (B \cap D) \times_A B \to C \times C \) is an embedding.

Indeed, \( B \times_A (B \cap D) \times_E D \) is isomorphic to

\[
\left\{ \begin{array}{ccc}
0 & \subseteq & V_1 \\
V_1 & \subseteq & V_2 \\
V_2 & \subseteq & \mathbb{C}^3
\end{array} \right\} \times \left\{ \begin{array}{ccc}
0 & \subseteq & V_1' \\
V_1' & \subseteq & V_2' \\
V_2' & \subseteq & \mathbb{C}^3
\end{array} \right\} \times \left\{ \begin{array}{ccc}
0 & \subseteq & V_1'' \\
V_1'' & \subseteq & V_2'' \\
V_2'' & \subseteq & \mathbb{C}^3
\end{array} \right\}
\]

and therefore \( \text{id} \) a \( \mathbb{P}^1 \times \mathbb{P}^1 \)-bundle over the \( D \cap B \) copy in the central component.

Its image under \( \pi_{13} \) is the subvariety \( Z_3 \) of \( C \times C \)

\[
Z_3 = \left\{ \begin{array}{ccc}
0 & \subseteq & V_1 \\
V_1 & \subseteq & V_2 \\
V_2 & \subseteq & \mathbb{C}^3
\end{array} \right\} \times \left\{ \begin{array}{ccc}
0 & \subseteq & V_1' \\
V_1' & \subseteq & V_2' \\
V_2' & \subseteq & \mathbb{C}^3
\end{array} \right\}
\]
which is the same \( \mathbb{P}^1 \times \mathbb{P}^1 \) over the \( B \cap D \) viewed as a line bundle over the flag \( \{ 0 \subset V_1 \subset V_2 \subset \mathbb{C}^3 \} \)

\[
B \cap D = \left\{ \begin{array}{c}
\alpha \\
0 \subset V_1 \subset V_2 \subset \mathbb{C}^3
\end{array} \right\}.
\]

Thus, in conclusion we have that the Fourier-Mukai kernel of \( F_{11}^{11} R_{21}^{21} F_{11}^{11} R_{11}^{12} \) is

\[
\pi_{13}^* (O_{D \times E(D \cap B)} \times_A B(D, D, 0))[-2] = O_{Z_3}(D, 0)[-2].
\]

\[\square\]

Before merging all the Fourier-Mukai kernel together, we prove the following technical Lemma which holds in a more general context.

**Lemma 4.6.7.** Let \( W_1, W_2, W_3 \) be smooth subvarieties of the variety \( Z \). and consider the following commutative diagram of inclusion maps

\[
\begin{aligned}
W_1 \cap W_2 & \xleftarrow{r} W_1 \cap W_2 \cap W_3 \\
W_1 & \xrightarrow{p} W_1 \cap W_2 \cap W_3 \\
W_2 & \xrightarrow{q} W_1 \cap W_2 \cap W_3 \\
W_3 & \xrightarrow{s} W_1 \cap W_2 \cap W_3 \\
Z & \xrightarrow{w} W_1 \cap W_2 \cap W_3
\end{aligned}
\]

The relative canonical bundle \( \omega_{W_1 \cap W_2 \cap W_3 / Z} \) is isomorphic to the sheaf

\[
\omega_{W_1 \cap W_2 \cap W_3 / Z} \simeq r^* h^! O_Z \otimes s^* w^! O_Z.
\]

**Proof.** By base change, the relative canonical bundle \( \omega_{W_1 \cap W_2 / Z} \) is isomorphic to

\[
\begin{aligned}
\omega_{W_1 \cap W_2 / Z} & \simeq q^* i^! O_Z \otimes q^! O_{W_1} \\
& \simeq q^* i^! O_Z \otimes q^* i^! O_Z \\
& \simeq q^* i^! O_Z \otimes s^* j^! O_Z.
\end{aligned}
\]

Similarly we can iterate this argument for the relative canonical bundle on \( W_1 \cap W_2 \cap W_3 / Z \).
\[ \omega_{W_1 \cap W_2 \cap W_3 / Z} \simeq r^* h^i \mathcal{O}_Z \otimes h^j \mathcal{O}_{W_1 \cap W_2} \]
\[ \simeq r^* h^i \mathcal{O}_Z \otimes r^j h^j \mathcal{O}_Z \]
\[ \simeq r^* h^i \mathcal{O}_Z \otimes s^* w^l \mathcal{O}_Z. \] (4.70)

Proposition 4.6.8. The Fourier-Mukai kernel representing
\[ R_{11}^{12} F_1^{11} R_{21}^{11} R_{11}^{12} F_1^{11} \]
is isomorphic to \( \pi_{15*}(\mathcal{F}) \), where \( E \times C \times C \times C \times A \xrightarrow{\pi_{15}} E \times A \) and
\[ \mathcal{F} \in D^b(E \times C \times C \times C \times A) \]
is the derived tensor product of the Fourier-Mukai kernels involved in the composition with cohomologies
\[ H^r(\mathcal{F}) = \begin{cases} \mathcal{O}_{Y_1 \cup Y_2}(0, (1, -2), (-2, 1), (1, -2), 0), & \text{if } r = 0, \\ \mathcal{O}_{Y_1}(0, (3, -3), (-3, 0), (0, 0), 0), & \text{if } r = -1; \\ 0, & \text{otherwise.} \end{cases} \]

The subvarieties \( Y_1 \) and \( Y_2 \) are the total spaces respectively of the zero section and the complement of the zero section of \( E \times (B \cap D) \times_A (B \cap D) \times_E (B \cap D) \times_A A \).

Moreover, the cohomologies of \( \mathcal{F} \) split under \( \pi_{15*} \).

Proof. Let \( \pi_{12}, \pi_{234}, \pi_{45} \) and \( \pi_{15} \) the natural projections
\[ \begin{array}{ccc}
E \times C \times C \times C \times A & \xrightarrow{\pi_{15}} & E \times A \\
\pi_{12} & \downarrow & \pi_{45} \\
E \times C & \xrightarrow{\pi_{234}} & C \times C \times C \times C \times A
\end{array} \]

By Proposition 3.2.3 and the previous Lemmas 4.6.5 and 4.6.6 the Fourier-Mukai kernel \( R_2 F_1 R_1 F_2 R_2 F_1 \) associated to \( R_{11}^{12} F_1^{11} R_{21}^{11} R_{11}^{12} F_1^{11} \) is isomorphic to
\[ \pi_{15*}(\pi_{12}^* \mathcal{O}_{E \times E D} \otimes \pi_{234}^* \mathcal{O}_{B \times_A (D \cap E) \times E D}(B, D, 0))[-2] \otimes \pi_{45}^* \mathcal{O}_{B \times A A}(B, 0)[-1]) \]

Let’s deal first with the transverse part of the intersection, since the derived tensor product is commutative.

As usual we want to use Corollary 3.9.4 in order to obtain the isomorphism
\[ \pi_{12}^* \mathcal{O}_{E \times E D} \otimes \pi_{45}^* \mathcal{O}_{B \times A A}(B, 0)[-1] \simeq \]
$\sim O_{E \times E D \times C \times C \times A} \otimes O_{E \times C \times C \times B \times A}(0, 0, B, 0)[-1]$

The subvariety $E \times E D \times C \times B \times A A = (E \times E D \times C \times C \times A) \cap (E \times C \times C \times B \times A A)$ is smooth and of codimension 10 in $E \times C \times C \times C \times A$, while the codimensions of $E \times E D \times C \times C \times A$ and of $E \times C \times C \times B \times A A$ of $E \times E D \times C \times C \times A$ are both of codimension 5.

The Fourier-Mukai kernel $R_2 F_1 R_1 F_2 R_2 F_1$ is isomorphic, since the intersection is transverse, to

$$\pi_{15*}(O_{E \times E D \times C \times B \times A}(0, 0, B, 0)[-1] \otimes O_{E \times B \times A (D \cap B) \times E D \times A}(0, B, D, 0))[-2]$$

The subvariety $E \times E D \times C \times B \times A A$ is of dimension 16 so is codimension 10 in $E \times C \times C \times C \times A$, the subvariety $E \times B \times A (D \cap B) \times E D \times A$ is of dimension 14 so is codimension 12 in $E \times C \times C \times C \times A$. Their intersection $(E \times E D \times C \times B \times A A) \cap (E \times B \times A (D \cap B) \times E D \times A) = E \times E (B \cap D) \times A (B \cap D) \times E (B \cap D) \times A A$ is a reducible variety; indeed

$$E \times E (B \cap D) \times A (B \cap D) \times E (B \cap D) \times A A = Y_1 \cup Y_2$$

where $Y_1$ is the zero section $\{\alpha = 0\}$

$$Y_1 = \{0 < V_2 \subset \mathbb{C}^3\} \times \{0 < V_1 \subset V_2 \subset \mathbb{C}^3\} \times \{0 < V_1 \subset V_2 \subset \mathbb{C}^3\} \times \{0 < V_1' \subset V_2 \subset \mathbb{C}^3\} \times \{0 < V_1' \subset \mathbb{C}^3\}$$

which is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over $Fl_3$ and of codimension 21 in $E \times C \times C \times C \times C \times A$, so with one dimensional excess bundle $E_1$;

$Y_2$ is the component outside of the zero section

$$Y_2 = \left\{ \begin{array}{l}
\{0 < V_3 \subset \mathbb{C}^3\} \times \{0 < V_3 \subset V_2 \subset \mathbb{C}^3\} \\
\{0 < V_3 \subset V_2 \subset \mathbb{C}^3\} \times \{0 < V_3 \subset \mathbb{C}^3\} \\
\{0 < V_3 \subset \mathbb{C}^3\} \times \{0 < V_3 \subset \mathbb{C}^3\}
\end{array} \right\}$$

which is isomorphic to $B \cap D$, so smooth and of codimension 22: therefore the intersection is transverse on this component.

The intersection between the two components of $E \times E (B \cap D) \times A (B \cap D) \times E (B \cap D) \times A A$ is

$$Y_1 \cap Y_2 = \{0 < V_2 \subset \mathbb{C}^3\} \times \{0 < V_1 \subset V_2 \subset \mathbb{C}^3\} \times \{0 < V_1 \subset V_2 \subset \mathbb{C}^3\} \times \{0 < V_1 \subset V_2 \subset \mathbb{C}^3\} \times \{0 < V_1 \subset \mathbb{C}^3\}$$

so isomorphic to $Fl_3$, hence a Cartier divisor in $D \cap B \simeq Y_2$.

Thus, by 4.42 we have the following short exact sequence

$$0 \to O_{Y_2}(-(Y_1 \cap Y_1)) \to O_{Y_1 \cup Y_2} \to O_{Y_1} \to 0$$

hence we have the isomorphism

$$O_{Y_1 \cup Y_2} \simeq Cone(O_{Y_1}[1] \to O_{Y_2}(-(Y_1 \cap Y_1)))$$
4.6. Further developments

therefore in $D^b(E \times C \times C \times C \times A) \mathcal{O}_{E \times E(B \cap D) \times A(B \cap D) \times E(B \cap D) \times A}(0, B, D, B, 0)$ is isomorphic to

$$\text{Cone}(\mathcal{O}_{Y_1}[1] \to \mathcal{O}_{Y_2}(-(Y_1 \cap Y_1))) \otimes \mathcal{O}(0, B, D, B, 0).$$

For computing the excess conormal bundle

$$\mathcal{E}_1 = (\mathcal{N}_{E \times E(D \times C \times B \times A)} \oplus \mathcal{N}_{E \times B \times A(D \cap B) \times E(D \times A)})/\mathcal{N}_{Y_1}$$

we will use the fact that it is a line bundle, hence $\text{det}(\mathcal{E}_1) = \mathcal{E}_1$ and the short exact sequence

$$0 \to \mathcal{N}_{Y_1} \to \mathcal{N}_{E \times E(D \times C \times B \times A)} \oplus \mathcal{N}_{E \times B \times A(D \cap B) \times E(D \times A)} \to \mathcal{E}_1 \to 0 \quad (4.71)$$

thus,

$$\mathcal{E}_1 = \text{det}(\mathcal{N}_{E \times E(D \times C \times B \times A)}) \otimes \text{det}(\mathcal{N}_{E \times B \times A(D \cap B) \times E(D \times A)}) \otimes \text{det}(\mathcal{N}_{Y_1})^{-1}$$

So the excess bundle will be

$$\mathcal{E}_1 = \mathcal{O}(0, -D, 0, -B, 0) \otimes \mathcal{O}(0, -B, -B-D, -D, 0) \otimes \mathcal{O}(0, \omega_{Fl_3} \otimes \omega_{F_2}^{-1}, \omega_{Fl_3}, \omega_{Fl_3} \otimes \omega_{F_2}^{-1}, 0).$$

Recall that the tautological bundles can be written as $\mathcal{O}(-1, 0) = \mathcal{V}_1$ and $\mathcal{O}(0, -1) = \Lambda^2 \mathcal{V}_2$.

The canonical bundles of $Fl_3, \mathbb{P}^2, \mathbb{P}^{21}$ are respectively

$$\omega_{Fl_3} = \mathcal{O}(-3, -3) \otimes \mathcal{O}(1, 1) = \mathcal{O}(-2, -2), \quad \omega_{F_2} = \mathcal{O}(-3, 0), \quad \omega_{F_2}^{-1} = \mathcal{O}(0, -3)$$

The normal bundles of $B$ and $D$ inside $C$ could be written respectively as

$$\mathcal{O}(B) = (\Lambda_2 \mathcal{V}_2)^{-2} \otimes \mathcal{V}_1^* = \mathcal{O}(1, -2), \quad \mathcal{O}(D) = \Lambda_2 \mathcal{V}_2^* \otimes (\mathcal{V}_1^*)^{-2} = \mathcal{O}(-2, 1)$$

and we have that $\omega_{Fl_3} \otimes \omega_{F_2}^{-1} = \mathcal{O}(1, -2) = \mathcal{O}(B), \omega_{Fl_3} \otimes \omega_{F_2}^{-1} = \mathcal{O}(-2, 1) = \mathcal{O}(D)$

Then the excess conormal bundle is $\mathcal{E}_1 = \mathcal{O}(0, (2, -1), (-1, -1), (-1, 2), 0)$

and if $L = \mathcal{O}(0, B, D, B, 0) = \mathcal{O}(0, (1, -2), (-2), 1), (1, -2), 0)$ then

$$\mathcal{E}_1 \otimes L = \mathcal{O}(0, (3, -3), (-3, 0), (0, 0), 0)$$

So the cohomologies of the derived tensor product are

$$H^r(\mathcal{O}_{E \times E(D \times C \times B \times A)}(0, 0, 0, B, 0) \otimes \mathcal{O}_{E \times B \times A(D \cap B) \times E(D \times A)}(0, B, D, B, 0))) =$$

$$= \begin{cases} 
\mathcal{O}_{Y_1 \cup Y_2}(0, (1, -2), (-2, 1), (1, -2), 0), & \text{if } r = 0, \\
\mathcal{O}_{Y_1}(0, (3, -3), (-3, 0), (0, 0), 0), & \text{if } r = -1; \\
0, & \text{otherwise}.
\end{cases}$$

Consider the map

$$\pi_{13} : Y_1 \longrightarrow E \times A$$

As $Y_1$ lies inside the zero section of $E \times C \times C \times C \times A$, its image under $\pi_{13}$ is $\mathbb{P}^2 \times \mathbb{P}^{21}$. Moreover:
1. If \((V'_1, V'_2) \not\in Fl_3\) then \(V_1\) is forced to be different from \(V'_1\), thus \(V'_2\) is determined: so \(Y_1\) is a \(\mathbb{P}^1\) bundle over the open \(\mathbb{P}^2 \times \mathbb{P}^{2\vee} \setminus Fl_3\).

2. If \((V'_1, V'_2) \in Fl_3\), then
   \[
   \begin{cases}
   V_1 \neq V'_1 \text{ implies } V_2 = V'_2; \\
   V_2 \neq V'_2 \text{ implies } V_1 = V'_1;
   \end{cases}
   \]
   This implies that over \(Fl_3 \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}\), \(Y_1\) is a \(\mathbb{P}^1 \cup \mathbb{P}^1\)-bundle.

The morphism \(\pi_{13}\) is flat on \(Y_1\).

Since flatness is a local open condition, on the open \(\mathbb{P}^2 \times \mathbb{P}^{2\vee} \setminus Fl_3\) is flat because it is a \(\mathbb{P}^1\)-bundle.

For the closed subvariety \((V'_1, V'_2) \in Fl_3\), by definition, the morphism \(\pi_{13}\) is flat if and only if \(O_{Y_1,p}\) is a flat \(\mathcal{O}_{E \times A, \pi_{13}(p)}\)-module.

Choose \(x, y\) and \(\alpha, \beta\) to be local coordinates respectively for \(\mathbb{P}^2\) and for \(\mathbb{P}^{2\vee}\), so that
\[
\mathcal{O}_{E \times A, \pi_{13}(p)} = \mathbb{C}[\alpha, \beta, x, y]_{(\alpha, \beta, x, y)}
\]
Denote moreover \(\delta, \gamma\) and \(w, z\) local coordinates for \(V_1\) and \(V_2\) respectively. In \(\mathbb{P}^{2\vee} \times \mathbb{P}^2\) we have the following relations that we want to be satisfied
\[
V_1 \subset V_2, \quad V'_1 \subset V_2, \quad V_1 \subset V'_2.
\]

Let’s take the local chart for \(V'_2 = (1 : \alpha : \beta)\), the argument for the other two charts will be completely specular.

Moreover, since all the linear spaces \(V_1, V'_1, V_2, V'_2\) lie in the same ambient space \(\mathbb{C}^3\), the incidence relations are independent by the action of \(GL(\mathbb{C}, 3)\); we can then assume that the point \((\alpha, \beta)\) correspond to the origin and
\[
V'_2 = (1 : 0 : 0)\perp
\]
If \(V_1 \neq V'_1\) then \(V_2 = V'_2\), so the local coordinates of \(V_1\) and \(V'_1\) can be:

- \(V'_1 = (0 : 1 : \beta)\), \(V_1 = (0 : 1 : \delta)\), with \(\beta\) forced to be different from \(\delta\); thus, the local ring is \(O_{Y_1,p} \simeq \mathbb{C}[\alpha, \beta, \gamma, \delta, x, y, w, z]_{(\alpha, \beta, x, y)}\)
  By symmetry, this case cover also the chart \(V'_1 = (0 : \alpha : 1)\), \(V_1 = (0 : \gamma : 1)\)

- \(V'_1 = (0 : 1 : \beta)\), \(V_1 = (0 : \gamma : 1)\) with \(\beta\) and \(\gamma\) are both forced to be different from 1. Then, the local ring is \(O_{Y_1,p} \simeq \mathbb{C}[\alpha, \beta, \gamma, \delta, x, y, w, z]_{(\gamma, \beta, x, y)}\)
  By symmetry, this case cover also the chart \(V'_1 = (0 : \alpha : 1)\), \(V_1 = (0 : 1 : \delta)\)

If \(V_1 \neq V'_2\) then \(V_1 = V'_1\): if we choose as local coordinates of \(V_1 = V'_1 = (0 : 1 : \beta)\), then \(V_2\) is forced to have the coordinates \(V_2 = (0 : 1 : \frac{-1}{\beta})\perp\) with \(\beta \neq 0\) (\(V_2 = (0 : 1 : 0)\perp\) otherwise) or \(V_2 = (0 : -\alpha : 1)\perp\).

Therefore, the local ring \(O_{Y_1,p} \simeq \mathbb{C}[\alpha, \beta, \gamma, \delta, x, y, w, z]_{(\frac{1}{\beta}, \beta, x, y)}\).

The argument is symmetric in the case we take coordinates \(V_1 = V'_1 = (0 : \alpha : 1)\).
Finally, if $V'_2 = V_2$ and $V_1 = V'_1$ then the two charts for $V'_1 = V'_1$ are $(0 : 1 : \beta)$ and $(0 : \alpha : 1)$.

So, $\mathcal{O}_{Y_1,p} \simeq \mathbb{C}[\alpha, \beta, \gamma, \delta, x, y, w, z]_{(\beta, x, y)}$ and $\mathcal{O}_{Y_1,p} \simeq \mathbb{C}[\alpha, \beta, \gamma, \delta, x, y, w, z]_{(\alpha, x, y)}$

Since $\pi_{13}$ is flat, using base change around the commutative square

$$
\begin{array}{ccc}
Y_{1,p} & \xrightarrow{u} & Y_1 \\
\downarrow v & & \downarrow \pi_{13} \\
P & \xrightarrow{w} & \mathbb{P}^2 \times \mathbb{P}^2
\end{array}
$$

The map $\pi_{13}$ is flat and the map $w$ is proper, so the commutative square (4.72) is Tor-independent.

By flat base change theorem $w^* \circ \pi_{13*} \simeq v_* \circ u^*$.

So $w^*(\pi_{13*}\mathcal{O}_{Y_1}) \simeq v_*(u^*\mathcal{O}_{Y_1})$.

Moreover, we have that $u^*\mathcal{O}_{Y_1} = \mathcal{O}_{Y_{1,p}}$, 

$v_*\mathcal{O}_{Y_{1,p}} = \Gamma(Y_{1,p}, \mathcal{O}_{Y_{1,p}}) = \mathbb{C}$.

Thus, $w^*\pi_{13*}\mathcal{O}_{Y_1} = \mathbb{C}$, therefore the fibers of $\pi_{13*}\mathcal{O}_{Y_1}$ are one dimensional, so $\pi_{13*}\mathcal{O}_D \in Pic(\mathbb{P}^2 \times \mathbb{P}^2)$.

In particular the the higher dimensional cohomologies of the derived pushforward $\pi_{13*}\mathcal{O}_{Y_1}$ vanish and

$\pi_{13*}\mathcal{O}_{Y_1} \simeq \pi_{13*}\mathcal{O}_{Y_1}$.

Since for every point $q$ of $E \times A$, the fiber $\pi_{13}^{-1}(q)$ is compact, then the only regular functions of $\pi_{13*}\mathcal{O}_{Y_1}$ on an open set $U$ containing $q$ are the constant. Therefore we have the local isomorphisms $(\pi_{13*}\mathcal{O}_{Y_1})_q \simeq (\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2})_q$ induced by $\pi_{13}^\#$.

Thus, we deduce that $\pi_{13*}\mathcal{O}_{Y_1} = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}$.

Since $\pi_{12} : Y_2 \rightarrow E \times A$ is an embedding, we conclude that

$$
\begin{align*}
\pi_{13*}(\mathcal{O}_{Y_1 \cup Y_2}) & \simeq \pi_{13*}Cone(\mathcal{O}_{Y_1}[1] \rightarrow \mathcal{O}_{Y_2}(-(Y_1 \cap Y_2))) \\
& \simeq Cone(\pi_{13*}\mathcal{O}_{Y_1}[1] \rightarrow \pi_{13*}\mathcal{O}_{Y_2}(-(Y_1 \cap Y_2))) \\
& \simeq Cone(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}[1] \rightarrow \mathcal{O}_{Z_4}(-(Fl_3)))
\end{align*}
$$

(4.73)

where $Z_4$ is the subvariety of $E \times A$

$$Z_4 = \left\{ \begin{array}{c}
\alpha \\
0 \subset V_2 \subset \mathbb{C}^3
\end{array} \right\} \times \left\{ \begin{array}{c}
\alpha \\
0 \subset V_1 \subset \mathbb{C}^3
\end{array} \right\}.$$
Since $Fl_3$ is a Cartier divisor inside $Z_1$ we have the isomorphism
\[
\mathcal{O}_{Z_1 \cup \mathbb{P}^2 \times \mathbb{P}^2} \simeq \text{Cone}(\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}[-1] \rightarrow \mathcal{O}_{Z_1}(-(Fl_3))),
\]
we conclude that
\[
\pi_{13*}(\mathcal{O}_{Y_1 \cup Y_2}) = \mathcal{O}_{Z_1 \cup \mathbb{P}^2 \times \mathbb{P}^2}
\]
Denote $\mathcal{L}$ the line bundle $\mathcal{O}(0, (1, -2), (1, 0))$
\[
\pi_{13*}(\mathcal{O}_{Y_1 \cup Y_2} \otimes \mathcal{L}) \simeq \pi_{13*} \text{Cone}(\mathcal{O}_{Y_1} \otimes \mathcal{L}[1] \rightarrow \mathcal{O}_{Y_2}(-(Y_1 \cap Y_2)) \otimes \mathcal{L})
\]
\[
\simeq \text{Cone}(\pi_{13*}\mathcal{O}_{Y_1} \otimes \mathcal{L}[1] \rightarrow \pi_{13*}\mathcal{O}_{Y_2}(-(Y_1 \cap Y_2)) \otimes \mathcal{L}) \quad (4.74)
\]
Indeed $\pi_{13*}\mathcal{O}_{Y_1} \otimes \mathcal{L}[1] \simeq 0$: by flat base change theorem around the commutative square (4.72), the following functors are isomorphic $w^* \circ \pi_{13*} \simeq v_* \circ u^*$.

So $w^*(\pi_{13*}(\mathcal{O}_{Y_1} \otimes \mathcal{L}[1])) \simeq v_*(u^*(\mathcal{O}_{Y_1} \otimes \mathcal{L}[1]))$.

Consider the following commutative square
\[
\begin{array}{ccc}
Y_1 & \xrightarrow{i} & \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \\
\downarrow{p} & & \downarrow{\pi_{14}} \\
\mathbb{P}^2 \times \mathbb{P}^2 & & \\
\end{array}
\]

Then by Lemma 4.6.7, the relative canonical bundle $\omega_{Y_1/\mathbb{P}^2 \times \mathbb{P}^2} \simeq \pi_{13}^!\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}$ is isomorphic to
\[
\pi_{13}^!\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} \simeq i^*p^!\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} \otimes i^!\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2}
\]
\[
\simeq i^*\mathcal{O}(0, -3, -3, 0) \otimes i^*\mathcal{O}(1, 2, 2, 1) \quad (4.75)
\]
Recall that

\[
\mathcal{L} = \mathcal{O}(-2, -1, -1, 1)
\]

So

\[
\mathcal{L} = \pi_{13}^!\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} \otimes \pi_{13}^*\mathcal{O}(-3, 0)
\]

and therefore

\[
\pi_{13*}\mathcal{L} \simeq \pi_{13*}(\pi_{13}^!\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2} \otimes \pi_{13}^*\mathcal{O}(-3, 0))
\]
\[
\simeq \pi_{13*}\text{Hom}(\mathcal{O}_{Y_1}, \pi_{13}^!\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}) \otimes \mathcal{O}(-3, 0)
\]
\[
\simeq \pi_{13*}\text{Hom}(\pi_{13*}\mathcal{O}_{Y_1}, \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2})[-1] \otimes \mathcal{O}(-3, 0) \quad (4.77)
\]

Therefore we have the isomorphisms
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\[ \pi_{13*}(\mathcal{O}_Y \otimes \mathcal{L}) \cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-3, 0)[-1]. \]

and

\[ \pi_{13*}(\mathcal{O}_Y(0, (3, -3), (-3, 0), (0, 0), 0)[1]) = \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(-3, 0)[1] \cong \mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(2B + D)[1] \]

since on the fibers the line bundle is trivial.

The last statement of the proposition follows therefore from relative Bott vanishing.

\[ \square \]

**STEP 2: The Fourier Mukai kernel of** \( R_{12111}^{111} T_{111}^{111} T_{211}^{111} F_{111}^{21} \)

The second step of the recipe for constructing the double crossing \( T_{211}^{12} \) is computing the Fourier-Mukai kernel of the functor

\[ R_{12111}^{21} T_{111}^{111} T_{211}^{111} F_{21}^{11} = R_{111}^{21}(F_{21}^{11} R_{111}^{12} \rightarrow id)(F_{21}^{11} R_{111}^{12} \rightarrow id) F_{21}^{111} \quad (4.78) \]

as the convolution of diagram

\[ \begin{array}{ccc}
R_{111}^{21} F_{21}^{11} & \oplus & R_{111}^{21} F_{21}^{111}[-2] \\
R_{111}^{21} F_{21}^{11} & \oplus & R_{111}^{21} F_{21}^{111}[-2] \\
\end{array} \]

at level of Fourier-Mukai transforms.

**Proposition 4.6.9.** The cohomologies of the derived tensor product of Fourier-Mukai kernels involved in the composition of \( R_{111}^{12} F_{21}^{111} R_{111}^{21} \text{Id} F_{21}^{111} \) before taking the derived pushforward \( \pi_{15*} \) are

\[ H^r(\mathcal{O}_{E \times D \times C \times B \times A}(0, 0, 0, B, 0)) \otimes \mathcal{O}_{E \times D \times D \times E \times A}(0, 0, D, 0, 0)) = \]

\[ = \begin{cases} 
\mathcal{O}_{Y_3}(0, 0, D, B, 0), & \text{if } r = 0, \\
\mathcal{O}_{Y_3}(0, 0, 0, B, 0), & \text{if } r = -1; \\
0, & \text{otherwise.}
\end{cases} \]

where

\[ Y_3 := (E \times D \times E \times D \times E \times B \cap D \times A) \times A \]

and

\[ \pi_{15} : E \times C \times C \times C \times A \to E \times A. \]
Chapter 4. Categorical action of generalised braids

Proof. Consider the following diagram

\[
\begin{array}{c}
E \times C & \xrightarrow{\pi_{12}} & E \times C \times C \times C \times A \\
& \pi_{23} & \downarrow \pi_{34} \\
C \times C & \xrightarrow{\pi_{34}} & C \times C
\end{array}
\]

\[
E \times A \xrightarrow{\pi_{15}} E \times A
\]

By change base theorem and lemma 4.6.5 the Fourier-Mukai kernel \(K_5\) associated to \(R_{12}^{11}F_{21}^{111}I_{111}^{111}\) is

\[
K_5 \simeq \pi_{15*}(\pi_{12}^*\mathcal{O}_{E \times E \times D} \otimes \pi_{23}^*\mathcal{O}_{C \times C} \otimes \pi_{34}^*\mathcal{O}_{D \times D}(D, 0)[-1] \otimes \pi_{45}^*\mathcal{O}_{B \times A}(B, 0)[-1])
\]

\[
\simeq \pi_{15*}(\mathcal{O}_{E \times E \times D \times C \times B \times A}(0, 0, 0, B, 0)[-1] \otimes \mathcal{O}_{E \times D \times D \times D \times E \times D \times A}(0, 0, D, 0, 0)[-1])
\]

(4.80)

The subvariety \(E \times D \times C \times B \times A\) is of dimension 16 so is codimension 10 in \(E \times C \times C \times C \times A\) while the subvariety \(E \times D \times D \times E \times D \times A\) is of dimension 14 so is codimension 12 in \(E \times C \times C \times C \times A\).

\(Y_4 = (E \times D \times C \times B \times A) \cap (E \times D \times D \times E \times D \times A) \simeq D \times E (D \cap B)\) which is a \(\mathbb{P}^1\)-bundle over \(B \cap D\) and at the same time the blow up \(\text{Bl}_{\{\alpha = 0\}}(D)\) over the zero section of \(D\), so it is 5 dimensional and its codimension in \(E \times C \times C \times C \times A\) is 21, so with one dimensional excess bundle \(E_3\).

For computing \(E_3\) we will use the short exact sequence 4.71 to get the isomorphism

\[
E_3 \simeq \text{det}(\mathcal{N}_{E \times D \times D \times D \times A}) \otimes \text{det}(\mathcal{N}_{E \times D \times D \times D \times A}) \otimes \text{det}(\mathcal{N}_{Y_4})^{-1}
\]

\[
\simeq E_1 = \mathcal{O}(0, -D, 0, -B, 0) \otimes \mathcal{O}(0, -D, +\omega_D - D, -D, 0) \otimes \mathcal{O}(0, D, -\omega_D + D, +B + D, 0)
\]

\[
\simeq \mathcal{O}(0, -D, 0, 0, 0)
\]

So by Proposition 3.9.2 the cohomologies of the derived tensor product are

\[
H^r(\mathcal{O}_{E \times E \times D \times C \times B \times A}(0, 0, 0, B, 0) \otimes \mathcal{O}_{E \times D \times D \times E \times D \times A}(0, 0, D, 0, 0)) =
\]

\[
\begin{cases}
\mathcal{O}_{Y_3}(0, 0, D, B, 0), & \text{if } r = 0, \\
\mathcal{O}_{Y_3}(0, 0, 0, B, 0), & \text{if } r = -1; \\
0, & \text{otherwise}.
\end{cases}
\]

\(\square\)

Analogously a symmetric proof proves the following
Proposition 4.6.10. The cohomologies of the derived tensor product of Fourier-Mukai kernels involved in the composition of $R_{111}^1 \text{Id} F_{111}^{12} R_{111}^1 F_{211}^{12}$ before taking the derived pushforward $\pi_{15*}$ are

$$H^r(\mathcal{O}_{E \times E B \times C \times B \times A}(0,0,0,B,0) \otimes \mathcal{O}_{E \times B \times A B \times B B \times A}(0,B,0,0,0)) =$$

$$= \begin{cases} 
\mathcal{O}_{Y'_3}(0,B,0,B,0), & \text{if } r = 0, \\
\mathcal{O}_{Y'_3}(0,0,0,B,0), & \text{if } r = -1; \\
0, & \text{otherwise.}
\end{cases}$$

where

$$Y'_3 := E \times E (B \cap D) \times A B \times A B \times A A.$$ 

and

$$\pi_{15} : E \times C \times C \times C \times A \to E \times A.$$ 

The following Lemma holds in the context of triangulated categories.

Lemma 4.6.11. Let the following be a distinguished triangle in a triangulated category $\mathcal{C}$

$$V[-1]\overset{f}{\to} W \overset{g}{\to} U \overset{h}{\to} V$$

(4.81)

and let $\gamma \in \text{Hom}^0(A[-1], A')$ for $A' \in \mathcal{T}$. Let $X[1]$ be the cone of $\gamma \circ h : C \to A'[1]$ so that

$$V'[-1] \to W \to X \xrightarrow{\gamma \circ h} V'$$

(4.82)

is a distinguished triangle. Then $X \simeq \text{Cone}(V[-1] \xrightarrow{f \oplus \gamma} W \oplus V')$

Proof. Consider the following commutative square

$$V[-1]\overset{f \oplus \gamma}{\to} W \oplus V'$$

$$\downarrow \alpha \downarrow \text{(Id,0)}$$

$$W \xrightarrow{\text{Id}} W$$

and complete it to the following commutative diagram

$$V[-1]\overset{f \oplus \gamma}{\to} W \oplus V' \to Q$$

$$\downarrow \alpha \downarrow \text{(Id,0)} \downarrow \text{id}$$

$$W \xrightarrow{\text{Id}} W \xrightarrow{0} 0$$

$$U \xrightarrow{\gamma \circ h} V' \xrightarrow{\text{id}} Q[1]$$

$$\downarrow \gamma \downarrow \text{(0,Id)} \downarrow \text{id}$$

$$V \xrightarrow{f \oplus \gamma} W[1] \oplus V' \to Q[1]$$

(4.84)
Since all the rows and all the columns are distinguished triangle we can apply the octaedral axiom of triangulate categories in order to obtain the isomorphism $X \simeq Q$.

**Proposition 4.6.12.** The Fourier-Mukai kernel representing the functor

\[ R_{111}^{12} T_{111}^{111} T_{111}^{111} F_{21}^{111} \]

is the object $\mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)} \oplus \mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)}[-2]$ where

\[
Z_1 := \left\{ \begin{array}{c}
\alpha \\
0 \subset V_2 \subset \mathbb{C}^3
\end{array} \right\} \times \left\{ \begin{array}{c}
\alpha \\
0 \subset V_1 \subset \mathbb{C}^3
\end{array} \right\}.
\]

**Proof.** By Proposition 4.6.8, Lemmas 4.6.9, 4.6.10 and Proposition 4.4.7, the convolution of diagram (4.64) and therefore the Fourier-Mukai kernel of $R_{111}^{12} T_{111}^{111} T_{111}^{111} F_{21}^{111}$ is the derived pushforward of the projection on $E \times A$ of the diagram of the cohomologies.

We want to compute the cone of the subdiagram

\[
\begin{align*}
\mathcal{O}_{Y_1 \cup Y_2}(0, B, D, B) & \xrightarrow{g_2} \mathcal{O}_{Y_3}(0, 0, D, B) \\
& \oplus \\
\mathcal{O}_{Y_3}(0, 0, B, B) & \xrightarrow{g_1} \mathcal{O}_{Y_3}(0, 0, B, B)
\end{align*}
\]

using Lemma 4.6.11.

The intersection of $Y_3$ with $Y_1 \cup Y_2$ is a divisor in $Y_3$; indeed

\[ Y_3 \cap (Y_1 \cup Y_2) = (Y_3 \cap Y_1) \cup (Y_3 \cap Y_2) \]

where

\[ Y_3 \cap Y_2 = Y_2 \]
and $Y_2$ is a divisor in $Y_3$, and

$$Y_3 \cap Y_1 = Y_{3|\alpha=0}$$

so the restriction of $Y_3$ to the zero section, which is a divisor in $\tilde{Y}_3$ and will be denoted by $\tilde{Y}_3$.

So from 4.42 we have the short exact sequence

$$0 \to \mathcal{O}_{Y_3}(-\tilde{Y}_3 - Y_2) \to \mathcal{O}_{Y_1 \cup Y_2 \cup Y_3} \to \mathcal{O}_{Y_1 \cup Y_2} \to 0 \quad (4.87)$$

From Proposition 4.6.8 we have that the zero cohomology of $R_2F_1R_1F_2R_2F_1$ is

$$Y_1 \cup Y_2 = \mathcal{O}_{E \times E B \cap D \times A B \cap D \times E B \cap D \times A}(0, B, D, B, 0)$$

meanwhile from Lemma 4.6.10 we know that the zero cohomology of $R_2F_1R_1\text{Id} F_1$ is

$$Y_3 = \mathcal{O}_{E \times E D \times D \times E B \cap D \times A}(0, 0, D, B, 0)$$

Their intersection is the reducible variety

$$Y_3 \cap (Y_1 \cup Y_2) = \mathcal{O}_{E \times E D \times D \times E B \cap D \times A}(0, 0, D, B, 0)$$

which is a divisor in $Y$ with conormal bundle $N_1 = \mathcal{O}(0, -B, 0, 0, 0)$.

So from 4.42, they fit in the short exact sequence

$$0 \to \mathcal{O}_{E \times E D \times D \times E B \cap D \times A}(0, 0, D, B, 0) \to \mathcal{O}_{Y_1 \cup Y_2 \cup Y_3}(0, B, D, B, 0) \to \mathcal{O}_{Y_1 \cup Y_2}(0, B, D, B, 0) \to 0$$

Similarly the zero cohomology of $R_2\text{Id} F_2R_2F_1$

$$Y_3' = \mathcal{O}_{E \times E D \cap A B \times B \times A}(0, B, 0, 0, 0)$$

the intersection of $Y_3'$ with $Y_1 \cup Y_2$ is

$$Y_3' \cap (Y_1 \cup Y_2) = \mathcal{O}_{E \times E D \cap A B \times B \times D \cap A}(0, B, 0, B, 0)$$

which is a Cartier divisor in $Y_3'$ with conormal bundle $N_2 = \mathcal{O}(0, 0, -D, 0, 0, 0)$.

As before, the zero cohomologies of the functors fits in the short exact sequence

$$0 \to \mathcal{O}_{E \times E B \cap D \times E B \times B \times A}(0, B, 0, B, 0) \to \mathcal{O}_{Y_1 \cup Y_2 \cap Y_2}(0, B, D, B, 0) \to \mathcal{O}_{Y_1 \cup Y_2}(0, B, D, B, 0) \to 0 (4.88)$$

By Lemma 4.6.11 we have that

$$\text{Cone}(g_1 \oplus g_2) \simeq \mathcal{O}_{Y_1 \cup Y_2 \cup Y_3}(0, B, D, B, 0). \quad (4.89)$$

Similarly $f$ is the $\text{Ext}^1$ map coming from the short exact sequence (4.42) which glues $Y_1$ on top of $Y_3$, while $h$ became the identity after taking the derived pushforward since both the varieties $Y_3'$ and $E \times E (D \cap B) \times E$ are mapped to $Z_1$ under the projections to $E \times A$. Using Lemma , the derived pushforward of the cone of (4.85) and therefore the Fourier-Mukai kernel representing the functor $R_1^{12} T_{111}^{111} T_{111}^{111} F_{21}^{111}$ is isomorphic to

$$\mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)} \oplus \mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)}[-2].$$
STEP 3: The double crossing functor $T_{21}^{12}$

The last step for the construction of the double crossing functor $T_{21}^{12}$ is to defined it as the difference of Figure 4.16.

\[ \circ \quad \square \]

Figure 4.16: The double crossing functor $T_{21}^{12}$

**Theorem 4.6.13.** The Fourier-Mukai kernel representing the double crossing functor $T_{21}^{12}$ is $\mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)}$ where

\[
Z_1 := \left\{ \begin{array}{c} \alpha \subset V_2 \subset \mathbb{C}^3 \\ 0 \subset V_1 \subset \mathbb{C}^3 \end{array} \right\} \times \left\{ \begin{array}{c} \alpha \subset V_2 \subset \mathbb{C}^3 \\ 0 \subset V_1 \subset \mathbb{C}^3 \end{array} \right\}.
\]

**Proof.** By Proposition 4.6.12 the Fourier-Mukai kernel representing the functor

\[ R_{111}^{12} T_{111}^{111} R_{111}^{111} F_{21}^{111} \]

is the object $\mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)} \oplus \mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)}[-2]$, while by Proposition 4.6.4 the functor $R_{111}^{12} F_{21}^{111}$ is represented by $\mathcal{O}_{E \times E} \oplus \mathcal{O}_{E \times E}[-2]$.

Therefore

\[ X \star (\mathcal{O}_{E \times E} \oplus \mathcal{O}_{E \times E}[-2]) \simeq \mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)} \oplus \mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)}[-2]. \quad (4.90) \]

Indeed 4.90 leads to the isomorphism

\[ X \simeq \mathcal{O}_{Z_1 \cup (\mathbb{P}^2 \times \mathbb{P}^2)}. \quad (4.91) \]

\[ \square \]

**Remark 4.6.14.** The Fourier-Mukai kernel of Theorem 4.90 is the Kawamata-Namikawa Fourier-Mukai kernel of Theorem 3.8.2.
Bibliography


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