Integr. Equ. Oper. Theory (2019) 91:41
https://doi.org/10.1007/s00020-019-2540-4

Integral Equations and Operator Theory

Positive Self-adjoint Operator Extensions with Applications to Differential Operators

B. M. Brown, W. D. Evans and I. G. Wood

Abstract. In this paper we consider extensions of positive operators. We study the connections between the von Neumann theory of extensions and characterisations of positive extensions via decompositions of the domain of the associated form. We apply the results to elliptic second order differential operators and look in particular at examples of the Laplacian on a disc and the Aharonov–Bohm operator.

Mathematics Subject Classification. Primary 47A20; Secondary 35J15, 47A07, 47B25, 47F05.

Keywords. Operator extensions, Von Neumann theory, Sesquilinear form, Elliptic operators, Aharonov–Bohm operator.

1. Introduction

Let A be a closed strictly positive symmetric operator with dense domain $D(A)$ and range $R(A)$ in a Hilbert space H. In [11,12], Krein proved that there is a one to one correspondence between the set of positive self-adjoint extensions A_B of A and a set of pairs $\{N_B, B\}$, where N_B is a subspace of the kernel N of A^* and B is a positive self-adjoint operator with domain and range in N_B. Krein's result was subsequently developed further by Visik [15] and Birman [3]; this work of the three authors will be referred to as the KVB theory. An important extension of the KVB theory was made in [8] to a pair of closed densely defined operators A, A', which form a dual pair in the sense that $A \subset (A')^*$ and are such that $A \subset A_\beta \subset (A')^*$ for an operator A_β with a bounded inverse. The results in [8] include those of KVB when $A = A'$. Of particular interest to us in [8] is the application of the abstract theory to the case when A is generated by an elliptic differential expression acting in a bounded smooth domain Ω in $\mathbb{R}^n$. In this case the self-adjoint extensions of A are determined by boundary conditions on the boundary $\partial\Omega$ of Ω.

The Authors are grateful to the anonymous referees for providing useful comments that helped improve the paper.

Birkhäuser

Published online: 10 September 2019

In [5], results in Rellich [13], Kalf [9] and Rosenberger [14] were applied to the KVB theory to determine all the positive self-adjoint extensions of a positive Sturm–Liouville operator with minimal conditions on the coefficients. Our objective in this paper is to investigate what can be achieved by applying similar methods to two problems on bounded domains in $\mathbb{R}^n, n \geq 2$; in the first A is generated by a second order elliptic expression, and in the second it is the Aharonov–Bohm operator on a punctured disc. Our analysis depends on an abstract result which incorporates the von Neumann theory concerning all the self-adjoint extensions of any symmetric operator.

Denote by $A_F,\ a_F[\cdot,\cdot]$ the Friedrichs extension and associated sesquilinear form of A. Then for all $u \in D(A_F)$ and $v \in Q(A_F)$ we have

$$a_F[u,v] = (A_F u, v),$$

where $(\cdot,\cdot)$ is the inner product of H, and $D(A_F)$ is dense as a subspace of $Q(A_F)$ with inner product $a_F[\cdot,\cdot]$ (see [7, Chapter IV] for more on the relation between sesquilinear forms, operators and their Friedrichs extension). By the KVB theory, $\hat{A}$ is a positive self-adjoint extension of A if and only if, $\hat{A} = A_B$, where B is a positive self-adjoint operator acting in a subspace N_B of N and $A_B,\ B$ have associated forms $a_B,\ b$, respectively which satisfy

$$a_B = a_F + b, \quad Q(A_B) = Q(A_F) \dotplus Q(B). \tag{1.1}$$

Thus any $u \in Q(A_B)$ can be uniquely written as $u = u_F + u_N$, where $u_F \in Q(A_F),\ u_N \in Q(B)$. There are two distinguished positive self-adjoint extensions of A, namely the Friedrichs (or strong) extension A_F and the Krein–von Neumann (or weak) extension A_K. These are extremal in the sense that any positive self-adjoint extension $\hat{A}$ of A satisfies $A_K \leq \hat{A} \leq A_F$ in the form sense. In (1.1), the Krein–von Neumann extension A_K corresponds to $B = 0,\ N_B = N$, and the Friedrichs extension A_F to $B = \infty,\ Q(B) = 0$, that is, B acts trivially on a zero dimensional space.

2. Positive Extensions and the Von Neumann Theory

The von Neumann theory characterises the self-adjoint extensions of any closed densely defined symmetric operator T. Denoting the *deficiency spaces* $\ker(T^* \mp iI)$ by $N_\pm$, we have

$$D(T^*) = D(T) \dotplus N_+ \dotplus N_-, \tag{2.1}$$

and T_S is a self-adjoint extension of T if and only if there is a unitary operator $U(T_S)\colon N_+ \to N_-$ such that

$$D(T_S) = D(T) \dotplus (I + U(T_S))N_+. \tag{2.2}$$

Let $u, v \in D(T^*)$. Then by the von Neumann theory, there exist unique $u_0, v_0 \in D(T)$ and $u_\pm, v_\pm \in N_\pm$ such that $u = u_0 + u_+ + u_-$ and $v = v_0 + v_+ + v_-$.

It follows that

$$\begin{aligned}(T^*u,v)-(u,T^*v) &= (Tu_0+iu_+-iu_-,v_0+v_++v_-)\\ &\quad -(u_0+u_++u_-,Tv_0+iv_+-iv_-)\\ &= (Tu_0,v_++v_-)+i(u_+-u_-,v_0+v_++v_-)\\ &\quad +i(u_0+u_++u_-,v_+-v_-)-(u_++u_-,Tv_0)\\ &= -i(u_0,v_+-v_-)+i(u_+-u_-,v_0+v_++v_-)\\ &\quad +i(u_0+u_++u_-,v_+-v_-)-i(u_+-u_-,v_0)\\ &= 2i\big[(u_+,v_+)_{N_+}-(u_-,v_-)_{N_-}\big].\end{aligned}$$

Let P_+ and P_- denote the projections from $D(T^*)$ to N_+ and N_- with respect to the decomposition (2.1) and let $U\colon N_+ \to N_-$ be unitary. Set $\tilde\Lambda_0 = UP_+ + P_-$ and $\tilde\Lambda_1 = -iUP_+ + iP_-$. Then, for any $u, v \in D(T^*)$

$$(T^*u,v)-(u,T^*v) = \left(\tilde\Lambda_0 u, \tilde\Lambda_1 v\right) - \left(\tilde\Lambda_1 u, \tilde\Lambda_0 v\right) \tag{2.3}$$

(see [10, Theorem 3]). The triple $(N_+, \tilde\Lambda_0, \tilde\Lambda_1)$ is a boundary triple (also known as a space of boundary values) for T.

Given a self-adjoint extension T_S of T, we now choose

$$\Lambda_0(T_S) = U(T_S)P_+ + P_-, \tag{2.4}$$

$$\Lambda_1(T_S) = -iU(T_S)P_+ + iP_-. \tag{2.5}$$

Then, from (2.2), $\ker \Lambda_1(T_S) = \mathcal{D}(T_S)$ and we obtain, for all $u, v \in D(T^*)$

$$(T^*u,v)-(u,T^*v) = (\Lambda_0(T_S)u, \Lambda_1(T_S)v) - (\Lambda_1(T_S)u, \Lambda_0(T_S)v). \tag{2.6}$$

Let $T = A$ be positive and B a positive self-adjoint operator on a subspace N_B of the kernel of A^* with domain $D(B)$. By [2, Theorem 3.1], the domain of the self-adjoint extension A_B of A corresponding to B is

$$\mathcal{D}(A_B) = \left\{u_0 + A_F^{-1}(Bv+f) + v \colon u_0 \in \mathcal{D}(A), v \in \mathcal{D}(B), f \in N \cap \mathcal{D}(B)^{\perp}\right\}. \tag{2.7}$$

Remark 2.1. The special case $B = 0, N_B = N$ gives the domain of the Krein–von Neumann extension A_K, namely

$$\mathcal{D}(A_K) = \mathcal{D}(A) \dot{+} N, \tag{2.8}$$

the sum being a direct sum since A is strictly positive. It follows that

$$\ker(A_K) = N. \tag{2.9}$$

The Friedrichs extension is characterised by the choice of B as acting trivially on $N_B = \{0\}$. Following the approach of [2], we can set $b[u] = \infty$ for $u \in N\backslash Q(B)$. It follows from (1.1) that $Q(A_B) = Q(A_F)$ if and only if $Q(B) = \{0\}$. Since A_F is the only self-adjoint extension of A with domain in $Q(A_F)$ it follows that its domain is determined by $b[u] = \infty$ for all $u \in N\backslash\{0\}$.

Theorem 2.2. *Let A_B be a positive self-adjoint extension of the positive operator A associated with the pair $\{B, N_B\}$. Let $u \in D(A_B)$, where $u = u_F + w$, $u_F = u_0 + A_F^{-1}(Bw+v), u_0 \in \mathcal{D}(A), w \in \mathcal{D}(B), v \in N \cap \mathcal{D}(B)^{\perp}$. Then*

$$b[w,\zeta] = (\Lambda_0(A_B)u, \Lambda_1(A_B)\zeta), \quad \forall\ \zeta \in Q(B), \tag{2.10}$$

where $\Lambda_0(A_B) = U(A_B)P_+ + iP_-$ *and* $\Lambda_1(A_B) = -iU(A_B)P_+ + iP_-$*.*

Proof. Let $\varphi = \theta + \zeta \in Q(A_B)$ with $\theta \in Q(A_F)$ and $\zeta \in Q(B)$. Then on the one hand, we have

$$(A_B u, \varphi) = (A^* u, \varphi) = (A^* u_F, \varphi) \tag{2.11}$$

since $w \in N$, and on the other hand,

$$\begin{aligned}(A_B u, \varphi) &= a_B[u, \varphi] = a_F[u_F, \theta] + b[w, \zeta] \\ &= (A_F u_F, \theta) + b[w, \zeta] \\ &= (A^* u_F, \theta) + b[w, \zeta].\end{aligned} \tag{2.12}$$

On combining (2.11) and (2.12) we get

$$b[w, \zeta] = (A^* u_F, \varphi - \theta) = (A^* u_F, \zeta) = (A^* u, \zeta),$$

and as $A^*\zeta = 0$, Eq. (2.6) yields

$$b[w, \zeta] = (\Lambda_0(A_B)u, \Lambda_1(A_B)\zeta) - (\Lambda_1(A_B)u, \Lambda_0(A_B)\zeta)\,. \tag{2.13}$$

Since $\ker \Lambda_1(A_B) = D(A_B)$, (2.10) follows. □

Let $\{\psi_k\}$ be an orthonormal-basis of $Q(B)$, where B is a positive self-adjoint operator in $N_B \subset N$, and let $w = \sum_j w_j \psi_j,\ \zeta = \sum \zeta_k \psi_k$ and $b_{jk} = b[\psi_j, \psi_k]$. Then $b[w, \zeta] = \sum_{j,k} b_{jk} w_j \overline{\zeta_k}$ and from (2.10) and the fact that $\ker \Lambda_1(A_B) = D(A_B)$, $u = u_F + w \in D(A_B)$ if and only if

$$\forall k.\ (\Lambda_0(A_B)u, \Lambda_1(A_B)\psi_k) = \sum_j b_{jk} w_j \overline{\zeta_k}. \tag{2.14}$$

3. Elliptic Differential Operators of Second Order

In this section we shall apply the above abstract theory to the case when A is the closure of a symmetric second-order differential operator in $L^2(\Omega)$ defined by

$$A' u := (-\nabla \cdot p \nabla + q) u = \left(- \sum_{i,j=1}^{n} D_i p_{ij} D_j + q \right) u, \quad u \in C_0^\infty(\Omega), \tag{3.1}$$

subject to conditions on the coefficients $p_{ij},\ q$ and the domain Ω. The assumptions are the ones made in [1] which weaken the smoothness requirements on the coefficients and the boundary of Ω made by Grubb [8]. In the following definition of a boundary regularity class, $B^s_{p,q}$ is the Besov space of order s (see [1, Section 2]), and we set $x = (x', x_n), x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}$.

Definition 3.1. The boundary $\partial\Omega$ is said to be of class $B^{M-\frac{1}{2}}_{p,q}$ if for each $x \in \partial\Omega$ there exist an open neighbourhood U satisfying the following: for a suitable choice of coordinates on $\mathbb{R}^n$, there is a function $\gamma \in B^{M-\frac{1}{2}}_{p,q}(\mathbb{R}^{n-1})$ such that $U \cap \Omega = U \cap \mathbb{R}^n_\gamma$ and $U \cap \partial\Omega = U \cap \partial\mathbb{R}^n_\gamma$, where $\mathbb{R}^n_\gamma = \{x \in \mathbb{R}^n : x_n > \gamma(x')\}$.

In the list of assumptions to be made, we shall denote the boundary of Ω by Σ, and H_t^s is a Bessel potential space (a Sobolev space for $s \in \mathbb{N}$), which we write as H^s when $t = 2$; see [1, Section 2] for definitions of $H_t^s(\Omega)$ and $H_t^s(\Sigma)$.

Assumptions

1. There exists $c_0 > 0$ such that for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$
$$\sum_{i,j=1}^{n} p_{ij}(x)\xi_i\xi_j dx \geq c_0\|\xi\|^2.$$
2. There exists $c > 0$ such that
$$\|u\|_1^2 = \int_\Omega \left(p|\nabla u|^2 + q|u|^2\right) dx \geq c\|u\|^2, \quad u \in C_0^\infty(\Omega).$$
The completion of $C_0^\infty(\Omega)$ with respect to the norm $\|\cdot\|_1$ is the form domain $Q(A_F)$ of A.
3. The boundary Σ is of class $B^{\frac{3}{2}}_{r,2}$ and the coefficients p and q of A lie in $H_t^1(\Omega)$ and $L_t(\Omega)$, respectively, under the constraints $n \geq 2$, $2 < r < \infty$, $2 < t \leq \infty$, and
$$1 - \tfrac{n}{t} \geq \tfrac{1}{2} - \tfrac{n-1}{r} > 0. \tag{3.2}$$

Remark 3.2. Our third assumption is Assumption 2.18 in [1]. Therefore, we have that for $v \in Q(A_F)$, $\gamma_0 v = 0$, where γ_0 is the trace operator which maps v into its value on Σ (see [1, Theorem 2.11]). Moreover, in the notation of [1,6], denote the solution of
$$Aw = 0 \text{ in } \Omega, \quad w = u \text{ on } \Sigma. \tag{3.3}$$
by
$$w = K_\gamma^0 \gamma_0 u. \tag{3.4}$$
Then by [1, Theorem 5.4], for all $s \in [0, 2]$,
$$K_\gamma^0 \colon H^{s-1/2}(\Sigma) \to H^s(\Omega) \tag{3.5}$$
is continuous,
$$K_\gamma^0 \colon H^{s-1/2}(\Sigma) \to Z_0^s(A) := \{u \in H^s(\Omega) \colon Au = 0\} \tag{3.6}$$
is a homeomorphism, and
$$\gamma_0 : Z_0^s(A) \to H^{s-1/2}(\Sigma) = (K_\gamma^0)^{-1}. \tag{3.7}$$
We remark that under the more restrictive assumptions that Ω is a bounded domain whose boundary is an $(n-1)$-dimensional C^∞ manifold, and the coefficients p_{jk}, q of A' in (3.1) lie in $C^\infty(\overline{\Omega})$ these properties were already shown by Grubb in [8].

Theorem 3.3. *Let the above assumptions hold and let A_B be a positive self-adjoint extension of A. For $u \in D(A_B)$, we have $u = u_F + w$ for some $u_F \in D(A_F)$, $w \in Q(B)$, and for all $\zeta \in Q(B)$*
$$b[w, \zeta] = \int_\Omega (-\nabla p \nabla + q) u_F \,\overline{\zeta}\, dx. \tag{3.8}$$

If $\{\psi_k\}$ *is an orthonormal basis of* $Q(B)$ *then, with* b_{jk} *as in* (2.14),

$$\forall k. \sum_j b_{jk} w_j + \int_\Omega (\nabla p \nabla - q) u_F \ \overline{\psi_k} dx = 0. \tag{3.9}$$

Proof. Let $a_B[\cdot,\cdot], a_F[\cdot,\cdot], b[\cdot,\cdot]$ denote the forms associated with A_B, A_F, B, respectively. For $u, \varphi \in Q(A_B)$ we have the decompositions

$$\begin{aligned} u &= u_F + w, \quad (u_F \in Q(A_F),\ w = K_\gamma^0 \gamma_0 u \in Q(B)), \\ \varphi &= \varphi_F + \zeta, \quad (\varphi_F \in Q(A_F),\ \zeta = K_\gamma^0 \gamma_0 \varphi \in Q(B)). \end{aligned} \tag{3.10}$$

If $u \in D(A_B)$, it has the decomposition $u = A_F^{-1} A^* u + (u - A_F^{-1} A^* u)$, i.e., $u_F = A_F^{-1} A^* u$ and $w = u - A_F^{-1} A^* u$, since $u_F \in D(A^*) \cap Q(A_F) = D(A_F)$ and $w \in Q(A_B) \cap N = Q(B)$. Now, let $\varphi = \varphi_F + \zeta \in Q(A_B)$. Then

$$(A_B u, \varphi) = \int_\Omega (-\nabla p \nabla + q) u \ \overline{\varphi} dx \tag{3.11}$$

and furthermore,

$$\begin{aligned} (A_B u, \varphi) = (u, \varphi)_1 = a_B[u, \varphi] &= a_F[u_F, \varphi_F] + b[w, \zeta] \\ &= \int_\Omega (-\nabla p \nabla + q) u_F \ \overline{\varphi_F} dx + b[w, \zeta] \\ &= \int_\Omega (-\nabla p \nabla + q) u \ \overline{\varphi_F} dx + b[w, \zeta]. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we get

$$b[w, \zeta] = \int_\Omega (-\nabla p \nabla + q) u \ (\overline{\varphi} - \overline{\varphi_F}) dx = \int_\Omega (-\nabla p \nabla + q) u_F \ \overline{\zeta} dx. \tag{3.13}$$

Now let $\{\psi_k\}$ be an orthonormal-basis of $Q(B)$, $w = \sum w_j \psi_j$ and $\zeta = \sum \zeta_k \psi_k$. Then

$$b[w, \zeta] = \sum_{j,k} b_{jk} w_j \overline{\zeta_k},$$

while the right-hand side of (3.13) is

$$\sum_k \int_\Omega (-\nabla p \nabla + q) u \ \overline{\zeta_k \psi_k} dx.$$

Consequently

$$\sum_k \overline{\zeta_k} \left[\sum_j b_{jk} w_j + \int_\Omega (\nabla p \nabla - q) u \ \overline{\psi_k} dx \right] = 0,$$

and equivalently,

$$\sum_k \overline{\zeta_k} \left[\sum_j b_{jk} w_j + \int_\Omega (\nabla p \nabla - q) u_F \ \overline{\psi_k} dx \right] = 0.$$

As $\{\zeta_k\}$ is an arbitrary sequence in ℓ^2, the 'boundary condition' associated with A_B is given by

$$\forall k. \sum_j b_{jk} w_j + \int_\Omega (\nabla p \nabla - q) u_F \, \overline{\psi_k} = 0. \tag{3.14}$$

□

Corollary 3.4. *The boundary condition associated with A_B is given by*

$$\forall k \cdot \sum_j b_{jk} w_j = (\nu_1 u_F, \gamma_0 \psi_k), \tag{3.15}$$

where $\nu_1 u = \sum_{j,k=1}^n n_j \gamma_0 [p_{jk} D_k u]$ and $\mathbf{n} = (n_1, \dots, n_n)$ is the interior unit normal.

Proof. Since $u, \psi_k \in D(A^*)$, we can use [1, Theorem 6.1] to write

$$\begin{aligned} &\int_\Omega (\nabla p \nabla - q) u_F \, \overline{\psi_k} dx - \int_\Omega u_F \, (\nabla p \nabla - q) \overline{\psi_k} dx \\ &\quad = (\gamma_0 u_F, \Gamma_1 \psi_k)_\Sigma - (\Gamma_1 u_F, \gamma_0 \psi_k)_\Sigma, \end{aligned} \tag{3.16}$$

where Γ_1 is the "regularised" Neumann operator given by $\Gamma_1 u = \nu_1 u - P_{\gamma_0,\nu_1} \gamma_0 u$ and $P_{\gamma_0,\nu_1} \gamma_0$ is the Dirichlet-to-Neumann map $P_{\gamma_0,\nu_1} \gamma_0 = \nu_1 K_\gamma^0$. Under the smoothness conditions assumed, the trace maps γ_0, Γ_1, map $D(A^*)$ continuously into $H^{-1/2}(\Sigma)$, $H^{1/2}(\Sigma)$, respectively. The terms on the right-hand side of (3.16) therefore represent in fact, $H^{-1/2}, H^{1/2}$-duality products over the boundary Σ, which are extensions of the $L^2(\Sigma)$ inner products (see [1, Theorem 6.1]).

Since $(\nabla p \nabla - q)\psi_k = 0$ and $u_F \in Q(A_F)$, two of the four terms in (3.16) vanish and, as $\Gamma_1 u_F = \nu_1 u_F$, we get that the boundary condition in (3.14) becomes

$$\forall k \cdot \sum_j b_{jk} w_j = (\nu_1 u_F, \gamma_0 \psi_k)_\Sigma. \tag{3.17}$$

□

Remark 3.5. The Friedrichs extension is determined by the boundary condition $\gamma_0 u = 0$. Under the additional smoothness assumptions on Ω and the coefficients of A' in (3.1) in [8], the Friedrichs extension has domain $H_0^1(\Omega) \cap H^2(\Omega)$.

Remark 3.6. The Krein–von Neumann extension corresponds to $B = 0, Q_B = N_B = N = \ker(A^*)$ and so

$$Q(A_B) = Q(A_F) \dotplus N, \quad a_B[u] = a_F[u_F],$$

when $u = u_F + w, u_F \in Q(A_F), w \in N$. Thus in (3.15), $b_{jk} = 0$ for all j, k and $\nu_1 u_F = \Gamma_1 u_F$. Since ν_1 maps $D(A^*)$ continuously into $H^{-1/2}(\Sigma)$ and γ_0 is a homeomorphism of N onto $H^{1/2}(\Sigma)$, it follows from (3.15) that the boundary condition satisfied by the Krein–von Neumann extension is

$$\nu_1 u_F = 0.$$

Since $w = K_\gamma^0 \gamma_0 u$ we have

$$\nu_1 u_F = (\nu_1 - \nu_1 K_\gamma^0 \gamma_0) u = \Gamma_1 u.$$

Remark 3.7. On combining (2.10) and (3.17) we have

$$(\Lambda_0(A_B)u, \Lambda_1(A_B)\psi_k) = (\nu_1 u_F, \gamma_0 \psi_k)_\Sigma. \tag{3.18}$$

For the Krein–von Neumann extension $\Lambda_K \psi_k = 0$, so we again get

$$\Gamma_1 u = \nu_1 u_F = 0$$

as the Krein–von Neumann boundary condition.

Example 3.8. We consider extensions of the positive operator $A = -\Delta + 1$ when Ω is the unit disc in $\mathbb{R}^2$. According to (3.15), $v = v_F + w$ lies in the domain of an extension A_B if and only if

$$(\nu_1 v_F, \gamma_0 \psi_k)_\Sigma = \sum_j b_{jk} \,(w, \psi_j) \tag{3.19}$$

for all k, where $\{\psi_k\}$ is an orthonormal-basis of the subspace $Q(B)$ in $N = \ker A^*$.

Let $-\Delta\psi + \psi = 0$ and put $\psi(r,\theta) = R(r)\Theta(\theta)$, where $x = (r,\theta)$ are polar co-ordinates. Then since

$$\Delta\psi = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = R\Theta = \psi,$$

we get

$$r^2\frac{R''}{R} + r\frac{R'}{R} - r^2 = n^2$$

and $\dfrac{\Theta''}{\Theta} = -n^2$ with constant n and $\Theta(0) = \Theta(2\pi)$; thus $\Theta_n(\theta) = e^{in\theta}, n \in \mathbb{Z}$ and we seek the $L^2(0,1;rdr)$ solutions of

$$r^2R'' + rR' - r^2R = n^2R, \quad r \in [0,1].$$

These solutions are given by the modified Bessel functions $I_n(r)$ and $K_n(r)$.

For $n \geq 1$, $K_n(r)$ does not lie in $L^2(0,1;rdr)$. The function $K_0(r)$ has a logarithmic singularity at 0, which means that ΔK_0 is not zero in the sense of distributions, excluding K_0 from N. Therefore

$$\psi_k(r,\theta) = I_k(r)e^{ik\theta}, \quad k \in \mathbb{Z}$$

is a basis for N; note that $I_{-k} = I_k$.

For $k \in \mathbb{Z}$

$$\gamma_0\psi_k(\theta) = \psi_k(1,\theta) = I_k(1)e^{ik\theta}.$$

and since $v_F \in D(A_F)$, we have $\nu_1 v_F = \frac{\partial v_F}{\partial \nu}$. On expanding v_F in θ in terms of its Fourier series,

$$v_F(r,\theta) = \sum_{n\in\mathbb{Z}} v_{F,n}(r)e^{in\theta},$$

we derive

$$\frac{\partial v_F}{\partial \nu}\Big|_{\partial\Omega} = \sum_{n\in\mathbb{Z}} \frac{\partial v_{F,n}}{\partial r}(1)e^{in\theta}.$$

Consequently $v = v_F + w \in D(A_B)$ if and only if for all $k \in \mathbb{Z}$

$$\sum_j b_{jk}\,(w, \psi_j) = (\nu_1 v_F, \gamma_0 \psi_k)_\Sigma$$
$$= \int_0^{2\pi} \sum_{n\in\mathbb{Z}} \frac{\partial v_{F,n}}{\partial r}(1) e^{in\theta} I_k(1) e^{-ik\theta} d\theta$$
$$= 2\pi \frac{\partial v_{F,k}}{\partial r}(1) I_k(1).$$

Remark 3.9. 1. For the Krein–von Neumann extension, $v = v_F + w \in D(A_K)$ if and only if for all $k \in \mathbb{Z}$ we have

$$0 = 2\pi \frac{\partial v_{F,k}}{\partial r}(1) I_k(1).$$

As $I_k(1) \neq 0$ for all $k \in \mathbb{Z}$, this implies that

$$v_F(1,\theta) = \frac{\partial v_F}{\partial r}(1,\theta) = 0$$

and hence $v_F \in D(A)$. As there are no restrictions on w, we get $D(A_K) = D(A) + N$, as expected. Also, the boundary condition satisfied by any $u \in D(A_K)$ is $\Gamma_1 u = 0$, where $\Gamma_1 = \nu_1 - P_{\gamma_0,\nu_1}\gamma_0$ is the regularised Neumann operator.

2. For the Friedrichs extension, we formally have $b_{jk} = \infty$ for all j, k in (3.19). This implies that w must be orthogonal to all the ψ_k. As $w \in N$, this gives $w = 0$.

4. Aharonov–Bohm Operator

Let $\Omega = \{x\colon |x| < 1\}\backslash\{0\} \subset \mathbb{R}^2$, and let A be the closure in $L^2(\Omega)$ of $A' \restriction_{C_0^\infty(\Omega)}$, where

$$A' := -(\nabla + iM)^2.$$

Here, the Aharonov–Bohm magnetic potential

$$M := \alpha \frac{1}{(x_1^2 + x_2^2)}(-x_2, x_1) = \alpha \frac{e_\theta}{r}, \quad \alpha \in (0,1), \tag{4.1}$$

where $x = (r\cos\,\theta, r\sin\,\theta)$ in polar co-ordinates and $e_\theta = (-\sin\,\theta, \cos\,\theta)$ is the unit vector orthogonal to $e_r = x/r$. Then

$$\text{curl } M = 0 \;\text{ in } \Omega, \text{ and } M \cdot e_r = 0. \tag{4.2}$$

For $u \in C_0^\infty(\Omega)$ we have

$$(A'u, u) = \int_\Omega |(\nabla + iM)u|^2 dx$$
$$= \int_0^1 \int_0^{2\pi} \left(\left|\frac{\partial u}{\partial\, r}\right|^2 + r^{-2} \left| i\frac{\partial u}{\partial\, \theta} + \alpha u\right|^2 \right) r dr d\theta. \tag{4.3}$$

The sequence $\{\varphi_k(\theta) : k \in \mathbb{Z}\}$, where $\varphi_k(\theta) = \frac{e^{-ik\theta}}{\sqrt{2\pi}}$, is an orthonormal basis for $L^2(0, 2\pi)$ and hence any $u \in L^2(\Omega)$ has the representation

$$u(r, \theta) = \Sigma_k u_k(r)\varphi_k(\theta), \tag{4.4}$$

where

$$u_k(r) = \int_0^{2\pi} u(r, \theta)\overline{\varphi_k(\theta)}d\theta.$$

On substituting in (4.3), we have, with $\lambda_k = k + \alpha$

$$(A'u, u) = \sum_k \int_0^1 \left(|u_k'(r)|^2 + \frac{\lambda_k^2}{r^2}|u_k(r)|^2\right) rdr.$$

Since $\min\{|\lambda_k|/r : k \in \mathbb{Z},\ 0 < r < 1\} \geq \min\{\alpha, 1 - \alpha\} > 0$, it follows that A is strictly positive and its form domain $Q(A_F)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm given by the square root of

$$a_F[u] := \sum_k \int_0^1 \left(|u_k'(r)|^2 + \frac{\lambda_k^2}{r^2}|u_k(r)|^2\right) rdr. \tag{4.5}$$

Let B be a positive self-adjoint operator acting in a closed subspace N_B of $N = \ker A^*$ which is associated with the self-adjoint extension A_B of A in the KVB theory, and let $a_B[\cdot, \cdot], a_F[\cdot, \cdot], b[\cdot, \cdot]$ be the forms of A_B, A_F, B, respectively.

For $u, \varphi \in Q(A_B)$, we have

$$\begin{aligned} u =& v + w,\ \ v \in Q(A_F),\ \ w \in Q(B) \\ \varphi =& \vartheta + \zeta, \vartheta \in Q(A_F),\ \ \zeta \in Q(B). \end{aligned} \tag{4.6}$$

and since $v(R, \theta) = \vartheta(R, \theta) = 0$ (which follows from the definition of $Q(A_F)$),

$$\begin{aligned} A^*w = 0 \text{ in } \Omega, \quad w(R) = u(R), \\ A^*\zeta = 0 \text{ in } \Omega, \quad \zeta(R) = \varphi(R). \end{aligned}$$

Remark 4.1. Since $v(1, \theta) = 0$ for any $v \in Q(A_F)$, $Q(A_F)$ coincides with Brasche and Melgaard's form domain of A_F in [4], and so A_F is determined in their Theorem 4.5.

We now proceed as in the proof of Theorem 2.2. For $u = u_F + w \in D(A_B)$ and $\varphi = \vartheta + \zeta \in Q(A_B)$

$$(A_B u, \varphi) = \int_\Omega (Au_F)\overline{\varphi}dx \tag{4.7}$$

and

$$(A_B u, \varphi) = \int_\Omega (Au_F)\overline{\vartheta}dx + b[w, \zeta]. \tag{4.8}$$

Consequently

$$b[w, \zeta] = \int_\Omega (Au_F)\overline{\zeta}dx. \tag{4.9}$$

If $\{\psi_k\}$ is an orthonormal basis of $Q(B)$, then we have with the same notation as in Sect. 3, that $u = u_F + w \in D(A_B)$ if and only if

$$\forall k : \quad \sum_j b_{jk} w_j = \int_\Omega (Au_F)\overline{\psi_k} dx. \tag{4.10}$$

The transformation

$$Wf(r) = r^{1/2} f(r), \quad f \in L^2(0,1; rdr)$$

is a unitary operator from $L^2(0,1; rdr)$ onto $L^2(0,1)$, and as $\{e^{im\theta}/\sqrt{2\pi}\}_{m\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{S}^1)$ we have

$$L^2(\Omega) = \bigoplus_{m\in\mathbb{Z}} W^{-1} L^2(0,1) \otimes \operatorname{Span}\left\{e^{im\theta}/\sqrt{2\pi}\right\}.$$

In terms of this decomposition it follows that

$$A = \bigoplus_{m\in\mathbb{Z}} W^{-1} T^{(m)} W \otimes 1, \tag{4.11}$$

where $T^{(m)}$ is the closure in $L^2(0,1)$ of the operator defined on $C_0^\infty(0,1)$ by the Sturm–Liouville expression

$$\tau^m y := -y'' + \left((m+\alpha)^2 - 1/4\right) r^{-2} y, \ m \in \mathbb{Z}, \ 0 < \alpha < 1, \tag{4.12}$$

i.e., $T^{(m)}$ is the minimal operator in $L^2(0,1)$ generated by τ^m. With $\nu = m + \alpha$, the set $\{r^{1/2+\nu}, r^{1/2-\nu}\}$ is a fundamental system for $\tau^m u = 0$. The expression τ^m is non-oscillatory. For $m = -1, \ 0$, it is in the limit-circle case at 0; for all other values of m, it is in the limit-point case at 0. It is regular at 1 for all values of m. Thus $T^{(m)}$ has deficiency indices $(2,2)$ for $m = -1, \ 0$ and $(1,1)$ otherwise. We shall now apply results from [5] to determine the positive self-adjoint extensions of $T^{(m)}$ in $L^2(0,1)$ for all $m \in \mathbb{Z}$. Note that the singular point here is at the left endpoint of the interval $[0,1]$, i.e., it is the point 0, unlike the analysis of [5], where it is at the right endpoint. If $S^{(m)}$ is one such extension, then

$$\bigoplus_{m\in\mathbb{Z}} W^{-1} S^{(m)} W \otimes 1 \tag{4.13}$$

is a positive self-adjoint extension of A.

Remark 4.2. We note that it is unlikely that all positive self-extensions of A are obtained in this way. This assertion is based on the situation for $A_0 = -\Delta + 1$ from Example 3.8. As in (4.11),

$$A_0 = \bigoplus_{m\in\mathbb{Z}} W^{-1} T_{(m)} W \otimes 1 \tag{4.14}$$

where now, $T_{(m)}$ is the minimal operator generated by

$$\tau_m y = -y'' + \left([m^2 - 1/4] r^{-2} + 1\right) y, \ m \in \mathbb{Z}.$$

At 0, τ_m is non-oscillatory and in the limit-circle case for $m = 0$ and is otherwise limit-point. As above for A, if $S_{(m)}$ is a positive self-adjoint extension of $T_{(m)}$ then

$$\bigoplus_{m\in\mathbb{Z}} W^{-1}S_{(m)}W\otimes 1 \tag{4.15}$$

is a positive self-adjoint extension of A_0. All such extensions have boundary conditions which depend on behaviour at 0, in view of the presence of the extension $S_{(0)}$ of $T_{(0)}$ which has deficiency indices $(1,1)$. However in Remark 3.9 we saw that this is not so for the Krein–von Neumann extension of A_0!

We shall proceed to determine the extensions $T^{(m)}$ in (4.11).

4.1. The Case when τ^m is Limit Point at 0 ($m \neq -1, 0$)

Theorem 2.1 in [5] establishes a one-one correspondence between the positive self-adjoint extensions of $T^{(m)}$ in this case and the one-parameter family $\{T_l^{(m)}\}$, $0\le l\le\infty$ of restrictions of $(T^{(m)})^*$ to the domains

$$\mathcal{D}(T_l^{(m)}) = \Big\{v\colon v\in\mathcal{D}((T^{(m)})^*), v'(1) = [\psi'(1) - l\|\psi\|^2]v(1)\Big\}. \tag{4.16}$$

Here ψ is a real function in $L^2(0,1)$ which satisfies $\tau^m\psi = 0$ and $\psi(1) = 1$. We therefore have

$$\psi(r) = r^{1/2+|\nu|},\ \psi'(1) = 1/2+|\nu|,\ \|\psi\|^2 = [2(1+|\nu|)]^{-1}.$$

4.2. The Case when τ^m is Limit-Circle at 0 ($m = -1, 0$) and dim $N_B = 1$

From Theorem 2.2 in [5] and writing T^* instead of $\left(T^{(m)}\right)^*$ for simplicity, it follows that the positive self-adjoint extensions of the operator $T^{(m)}$ which correspond to the pair $\{B, N_B\}$ in the KVB theory with dim$N_B = 1$ form a one-parameter family T_β of restrictions of T^* with domains

$$\mathcal{D}(T_\beta) := \left\{ v\in\mathcal{D}(T^*)\colon \left[g^2\left\{\left(\frac{v}{g}\right)\left(\frac{\psi}{g}\right)' - \left(\frac{v}{g}\right)'\left(\frac{\psi}{g}\right)\right\}\right]_0^1 = \beta v(1)\|\psi\|^2\right\}, \tag{4.17}$$

where ψ is a real function in N_B with $\psi(1) = 1$, g is the non-principal solution of $\tau^m u = 0$ and $\beta\ge 0$. The non-principal solution is $r^{1/2-|\nu|}$, $\nu = m+\alpha$. The Wronskian W is given by

$$W(v,\psi) = g^2\left\{\left(\frac{v}{g}\right)\left(\frac{\psi}{g}\right)' - \left(\frac{v}{g}\right)'\left(\frac{\psi}{g}\right)\right\}. \tag{4.18}$$

The limits at 0 of the first and the second terms in (4.17) exist separately. To see this, let

$$\frac{g''(r)}{g(r)} = (\nu^2 - 1/4)r^{-2} =: q(r).$$

Hence by the Jacobi identity [5, Equation (1.10)], for $v\in\mathcal{D}(T^*)$ we get

$$-\frac{1}{g}\left[g^2\left(\frac{v}{g}\right)'\right]' = -v'' + qv = T^*v\in L^2(0,1). \tag{4.19}$$

Thus, since $g \in L^2(0,1)$,

$$-\left[g^2\left(\frac{v}{g}\right)'\right]' = g\,(T^*v) \in L^1(0,1),$$

which implies that

$$\lim_{r\to 0+}\left[g^2\left(\frac{v}{g}\right)'\right](r) \quad \text{and} \quad \lim_{r\to 0+} g^2(r)\left(\frac{\psi}{g}\right)'(r)$$

both exist. From [9, Remark 3] (see also [5, (2.9)]), $\lim_{r\to 0+}(v/g)(r)$ exists, which confirms our assertion that the separate limits exist.

We shall now determine the boundary conditions satisfied by the self-adjoint extensions of $T^{(m)}$ in the two cases corresponding to $\nu = m+\alpha,\ m = -1, 0,\ \alpha \in (0,1)$.

4.2.1. The Case $m = -1,\ \nu = -1+\alpha \in (-1,0)$. In this case, the non-principal solution is $g(r) = r^{1/2+\nu}$ and $\psi(r) = \gamma\left(C_1 r^{1/2-\nu} + C_2 r^{1/2+\nu}\right)$, where $\gamma = (C_1+C_2)^{-1}$ for C_1, C_2 are constants and $C_1 \neq 0$. Thus,

$$\left(\frac{\psi}{g}\right)(r) = \gamma\left(C_1 r^{-2\nu} + C_2\right), \quad g^2\left(\frac{\psi}{g}\right)'(r) = -2\gamma\nu C_1, \tag{4.20}$$

and so using (4.18)

$$W(r) = \left(\frac{v(r)}{g(r)}\right)(-2\gamma\nu C_1) - \left[g^2(r)\left(\frac{v}{g}\right)'(r)\right]\left[\gamma(C_1 r^{-2\nu} + C_2)\right]. \tag{4.21}$$

The value at $r = 1$ is

$$W(1) = -v'(1) - v(1)\left[2\gamma\nu C_1 - 1/2 - \nu\right]. \tag{4.22}$$

By (4.20) and since $\nu < 0$, the limits at 0 of both terms in (4.21) exist and

$$\begin{aligned}\lim_{r\to 0+} W(r) &= -2\gamma\nu C_1 \lim_{r\to 0+}\frac{v(r)}{r^{1/2+\nu}} - \gamma C_2 \lim_{r\to 0+} g^2(r)\left(\frac{v}{g}\right)'(r)\\ &= -2\gamma\nu C_1 \lim_{r\to 0+}\frac{v(r)}{r^{1/2+\nu}}\\ &\quad -\gamma C_2 \lim_{r\to 0+}\left[r^{1/2+\nu}v'(r) - (1/2+\nu)r^{-1/2+\nu}v(r)\right].\end{aligned} \tag{4.23}$$

Thus the boundary condition for A_B in this case is

$$2\gamma\nu C_1\left\{\lim_{r\to 0+}\frac{v(r)}{r^{1/2+\nu}} - v(1)\right\} + \left\{\gamma C_2 \lim_{r\to 0+} f_1(r) - f_1(1)\right\} = \beta v(1)\|\psi\|^2, \tag{4.24}$$

where $f_1(r) := \left[r^{1/2+\nu}v'(r) - (1/2+\nu)r^{-1/2+\nu}v(r)\right]$.

4.2.2. The Case $m = 0$, $\nu = \alpha \in (0,1)$. This time, the non-principal solution is $g(r) = r^{1/2-\nu}$ and, with ψ as above, we have

$$\left(\frac{\psi}{g}\right)(r) = \gamma\left(C_1 + C_2 r^{2\nu}\right), \quad g^2\left(\frac{\psi}{g}\right)'(r) = 2\gamma\nu C_2, \tag{4.25}$$

giving

$$W(r) = \left(\frac{v(r)}{g(r)}\right)(2\gamma\nu C_2) - \left[g^2(r)\left(\frac{v}{g}\right)'(r)\right]\left[\gamma(C_1 + C_2 r^{2\nu})\right]. \tag{4.26}$$

Therefore

$$W(1) = 2\gamma\nu C_2 v(1) - [v'(1) - (1/2-\nu)v(1)]. \tag{4.27}$$

By (4.25) and since $\nu > 0$, both limits at 0 in (4.26) exist and

$$\begin{aligned}\lim_{r\to 0+} W(r) &= 2\gamma\nu C_2 \lim_{r\to 0+}\frac{v(r)}{r^{1/2-\nu}} - \gamma C_1 \lim_{r\to 0+} g^2(r)\left(\frac{v}{g}\right)'(r)\\ &= 2\gamma\nu C_2 \lim_{r\to 0+}\frac{v(r)}{r^{1/2-\nu}}\\ &\quad - \gamma C_1 \lim_{r\to 0+}\left[r^{1/2-\nu}v'(r) - (1/2-\nu)r^{-1/2-\nu}v(r)\right].\end{aligned} \tag{4.28}$$

Thus the boundary condition in this case is

$$2\gamma\nu C_2\left\{\lim_{r\to 0+}\frac{v(r)}{r^{1/2-\nu}} - v(1)\right\} - \left\{\gamma C_1 \lim_{r\to 0+} f_2(r) - f_2(1)\right\} = \beta v(1)\|\psi\|^2, \tag{4.29}$$

where $f_2(r) := \left[r^{1/2-\nu}v'(r) - (1/2-\nu)r^{-1/2-\nu}v(r)\right]$.

4.3. The Case when τ^m is Limit-Circle at 0 ($m = -1, 0$) and $\dim N_B = 2$

From [5, Theorem 2.2], we have

$$\mathcal{D}(A_B) := \left\{v \in \mathcal{D}(A^*)\colon \left[g^2\left\{\left(\frac{v}{g}\right)\left(\frac{\psi_k}{g}\right)' - \left(\frac{v}{g}\right)'\left(\frac{\psi_k}{g}\right)\right\}\right]_0^1 = \sum_{j=1}^{2} b_{jk}c_j,\ j = 1,2\right\}, \tag{4.30}$$

where $B := (b_{jk})_{j,k=1,2}$ is a matrix of parameters which is non-negative, $\{\psi_1, \psi_2\}$ is a real orthonormal basis of N_B and $c_1,\ c_2$ are determined by the values of v/g at 0 and 1:

$$\frac{v}{g}(0) = \sum_{j=1}^{2} c_j \frac{\psi_j}{g}(0), \quad \frac{v}{g}(1) = \sum_{j=1}^{2} c_j \frac{\psi_j}{g}(1). \tag{4.31}$$

The main difference from the analysis of the previous section is that we now replace ψ by an orthonormal basis (ψ_1, ψ_2) obtained from the linearly independent basis elements

$$r^{1/2-|\nu|} \quad \text{and} \quad r^{1/2+|\nu|},\ \nu = m + \alpha.$$

On using the Gram–Schmidt procedure, we obtain the orthogonal vectors

$$r^{\frac{1}{2}-|\nu|} \text{ and } r^{|\nu|+\frac{1}{2}} - (1-|\nu|)r^{\frac{1}{2}-|\nu|},$$

and the orthonormal system

$$\psi_1 = \sqrt{2(1-|\nu|)}r^{\frac{1}{2}-|\nu|}, \quad \psi_2 = \frac{\sqrt{2(1+|\nu|)}}{|\nu|}\left(r^{|\nu|+\frac{1}{2}} - (1-|\nu|)r^{\frac{1}{2}-|\nu|}\right).$$

The non-principal solution is $g(r) = r^{1/2-|\nu|}$ and we have

$$\psi_1/g = \sqrt{2(1-|\nu|)}, \quad (\psi_1/g)' = 0$$

and

$$\psi_2/g = \frac{\sqrt{2(1+|\nu|)}}{|\nu|}(r^{2|\nu|} + |\nu| - 1), \quad g^2(\psi_2/g)' = 2\sqrt{2(|\nu|+1)}.$$

Let

$$W_k = g^2\left\{\left(\frac{v}{g}\right)\left(\frac{\psi_k}{g}\right)' - \left(\frac{v}{g}\right)'\left(\frac{\psi_k}{g}\right)\right\}, \quad k = 1, 2.$$

Then

$$W_1(r) = -\sqrt{2(1-|\nu|)}\, g^2(r)\left(\frac{v}{g}\right)'(r),$$

and we set

$$\Theta_1 := W_1(1) - W_1(0) = -\sqrt{2(1-|\nu|)}\left(\left(\frac{v}{g}\right)'(1) - \lim_{r\to 0} g^2(r)\left(\frac{v}{g}\right)'(r)\right). \tag{4.32}$$

Also

$$W_2(r) = 2\sqrt{2(1+|\nu|)}\left(\frac{v}{g}\right)(r) - \frac{\sqrt{2(1+|\nu|)}}{|\nu|}(|\nu| - 1 + r^{2|\nu|})g^2\left(\frac{v}{g}\right)'(r)$$

giving

$$\begin{aligned}\Theta_2 &:= W_2(1) - W_2(0)\\ &= \sqrt{2(1+|\nu|)}\left[2v(1) - \left(\frac{v}{g}\right)'(1)\right]\\ &\quad - \sqrt{2(1+|\nu|)}\lim_{r\to 0}\left\{2\left(\frac{v}{g}\right)(r) - \frac{1}{|\nu|}(|\nu| - 1 + r^{2|\nu|})g^2\left(\frac{v}{g}\right)'(r)\right\}.\end{aligned} \tag{4.33}$$

We also have

$$V := \begin{pmatrix} v(1)/g(1) \\ v(0)/g(0)\end{pmatrix} = \begin{pmatrix}\sqrt{2(1-|\nu|)} & \sqrt{2(1+|\nu|)} \\ \sqrt{2(1-|\nu|)} & \sqrt{2(1+|\nu|)}(1-1/|\nu|)\end{pmatrix}\begin{pmatrix} c_1 \\ c_2\end{pmatrix}.$$

Setting $V = \Psi c$, where

$$\Psi := \begin{pmatrix}\sqrt{2(1-|\nu|)} & \sqrt{2(1+|\nu|)} \\ \sqrt{2(1-|\nu|)} & \sqrt{2(1+|\nu|)}(1-1/|\nu|)\end{pmatrix}$$

is invertible and has inverse

$$\Psi^{-1} = \frac{|\nu|}{2\sqrt{(1-|\nu|^2)}} \begin{pmatrix} \sqrt{2(1+|\nu|)}(1/|\nu|-1) & \sqrt{2(1+|\nu|)} \\ \sqrt{2(1-|\nu|)} & -\sqrt{2(1-|\nu|)} \end{pmatrix}. \tag{4.34}$$

The boundary condition in (4.30) is therefore

$$\begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix} = B\Psi^{-1}V. \tag{4.35}$$

For the Krein–von Neumann extension $\left(T^{(m)}\right)_K$ of the one-dimensional operator, the boundary condition is determined by $\Theta_1 = \Theta_2 = 0$: Hence

$$\lim_{r\to 0} g^2(r)\left(\frac{v}{g}\right)'(r) = \left(\frac{v}{g}\right)'(1) \tag{4.36}$$

and

$$\lim_{r\to 0} 2\left(\frac{v}{g}\right)(r) = 2v(1) - \frac{1}{|\nu|}\left(\frac{v}{g}\right)'(1). \tag{4.37}$$

Following Remark 2.1, the Friedrichs extension $\left(T^{(m)}\right)_F$ is obtained by $c_1 = c_2 = 0$, so that the right hand side of (4.30) is finite. Thus, from (4.31), the boundary conditions are given by $(v/g)(0) = (v/g)(1) = 0$, i.e.,

$$\lim_{r\to 0} \frac{v(r)}{r^{1/2-|\nu|}} = v(1) = 0. \tag{4.38}$$

References

[1] Abels, H., Grubb, G., Wood, I.G.: Extension theory and Kreĭn-type resolvent formulas for nonsmooth boundary value problems. J. Funct. Anal. **266**(7), 4037–4100 (2014)

[2] Alonso, A., Simon, B.: The Birman–Kreĭn–Vishik theory of selfadjoint extensions of semibounded operators. J. Oper. Theory **4**(2), 251–270 (1980)

[3] Birman, M.Š.: On the theory of self-adjoint extensions of positive definite operators. Mat. Sb. N.S. **38**(80), 431–450 (1956)

[4] Brasche, J.F., Melgaard, M.: The Friedrichs extension of the Aharonov–Bohm Hamiltonian on a disc. Integral Equ. Oper. Theory **2**(3), 419–436 (2005)

[5] Brown, B.M., Evans, W.D.: Selfadjoint and m sectorial extensions of Sturm–Liouville operators. Integral Equ. Oper. Theory **85**(2), 151–166 (2016)

[6] Brown, B.M., Grubb, G., Wood, I.G.: M-functions for closed extensions of adjoint pairs of operators, with applications to elliptic boundary problems. Math. Nachr. **282**, 314–347 (2009)

[7] Edmunds, D.E., Evans, W.D.: Spectral Theory and Differential Operators. Clarendon Press, Oxford (1989)

[8] Grubb, G.: A characterization of the non-local boundary value problems associated with an elliptic operator. Ann. Scuola Norm. Sup. Pisa **3**(22), 425–513 (1968)

[9] Kalf, H.: A characterization of the Friedrichs extension of Sturm–Liouville operators. J. Lond. Math. Soc. (2) **17**(3), 511–521 (1978)

[10] Kochubei, A.N.: Extensions of symmetric operators and symmetric binary relations. Math. Notes **17**(1), 25–28 (1975)

[11] Kreĭn, M.: The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. I. Rec. Math. [Mat. Sb.] N.S. **20**(62), 431–495 (1947)

[12] Kreĭn, M.: The theory of self-adjoint extensions of semi-bounded Hermitian transformations and its applications. II. Mat. Sb. N.S. **21**(63), 365–404 (1947)

[13] Rellich, F.: Halbbeschränkte gewöhnliche Differentialoperatoren zweiter Ordnung. Math. Ann. **122**, 343–368 (1951)

[14] Rosenberger, R.: A new characterization of the Friedrichs extension of semibounded Sturm–Liouville operators. J. Lond. Math. Soc. (2) **31**(3), 501–510 (1985)

[15] Vishik, M.: On general boundary conditions for elliptic differential operators. Trudy Moskov. Mat. Obsc. (Russian). (English translation In: Amer. Math. Soc. Transl. 24, 107–172), 187–246 (1952)

B. M. Brown
Cardiff School of Computer Science and Informatics
Cardiff University
Queen's Buildings, 5 The Parade, Roath
Cardiff CF24 3AA
UK
e-mail: malcolm@cs.cf.ac.uk

W. D. Evans
School of Mathematics
Cardiff University
Senghennydd Road
Cardiff CF24 4AG
UK
e-mail: evanswd@cf.ac.uk

I. G. Wood (✉)
School of Mathematics, Statistics and Actuarial Science
University of Kent
Sibson Building
Canterbury CT2 7FS
UK
e-mail: i.wood@kent.ac.uk

Received: November 28, 2018.
Revised: August 5, 2019.