The impact of temporal aggregation on supply chains with ARMA(1,1) demand processes

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Abstract.

Various approaches have been considered in the literature to improve demand forecasting in supply chains. Among these approaches, non-overlapping temporal aggregation has been shown to be an effective approach that can improve forecast accuracy. However, the benefit of this approach has been shown only under single exponential smoothing (when it is a non-optimal method) and no theoretical analysis has been conducted to look at the impact of this approach under optimal forecasting. This paper aims to bridge this gap by analysing the impact of temporal aggregation on supply chain demand and orders when optimal forecasting is used. To do so, we consider a two-stage supply chain (e.g. a retailer and a manufacturer) where the retailer faces an autoregressive moving average demand process of order (1,1) -ARMA(1,1)- that is forecasted by using the optimal Minimum Mean Squared Error (MMSE) forecasting method. We derive the analytical expressions of the mean squared forecast error (MSE) at the retailer and the manufacturer levels as well as the bullwhip ratio when the aggregation approach is used. We numerically show that, although the aggregation approach leads to an accuracy loss at the retailer's level, it may result in a reduction of the MSE at the manufacturer level up to 90% and a reduction of the bullwhip effect in the supply chain that can reach up to 84% for high lead-times.

Keywords: Forecasting, Temporal aggregation, Forecast accuracy, Mean Square Error, Bullwhip effect, MMSE forecasting method.

1. Introduction

Demand uncertainty is among the most important challenges facing modern companies. High variability in demand poses considerable difficulties in terms of forecasting and stock control. There are many approaches that may be used to reduce demand uncertainty and thus to improve the forecasting (and supply chain) performance of a company. An intuitively appealing approach that is known to be effective is demand aggregation.

Aggregation across time or temporal aggregation refers to the process by which a low frequency time series (e.g. quarterly) is derived from a high frequency time series (e.g. monthly) (Nikolopoulos et al., 2011). This is achieved through the summation (bucketing) of every *m* periods of the high frequency data, where *m* is called the aggregation level.

There are two types of temporal aggregation: *i*) overlapping and *ii*) non-overlapping. Overlapping temporal aggregation resembles the mechanism of a moving window technique where the window's size equals the aggregation level. At each period, the window is moved one-step ahead, so the oldest observation is dropped and the newest is included. The focus of this

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paper is on non-overlapping temporal aggregation. Therefore, in this paper temporal aggregation (TA) refers to the non-overlapping case. Please refer to Boylan and Babai (2016) for more information about the overlapping temporal aggregation case.

In non-overlapping temporal aggregation, the time series are divided into consecutive non-overlapping buckets of time where the length of the time bucket equals the aggregation level. As shown in Figure 1, the non-overlapping aggregated series is created by summing up the values inside each bucket. The number of aggregate periods is $[N/m]$, where N is the number of the original periods, m the aggregation level and the $[x]$ operator returns the integer part of x. We recommend creating time buckets that include the most recent observation, as this is needed for auto-regressive forecasts.

	Jan	Feb	March	April	May	June	July	Aug	Sep.	Oct.	Nov.	Dec.
Non-aggregate series	2	1	9	3	1	20	10	1	5	10	2	5
Non-overlapping aggregated series	12			24			16			17		
	Quarter 1			Quarter 2			Quarter 3			Quarter 4		

Figure 1 : Non-overlapping temporal aggregation

Because of using TA, the number of periods of aggregated demand is less than that of the original demands. Additionally, temporal aggregation may be used to align decision levels to forecast output. An important assumption that is often made in demand forecasting is that the level of the required forecasting matches the level of available collected data. However, this is not often true. In fact, in many organisations, managers from several departments are involved in forecast generation that supports decisions for production, inventory management (Argilaguet-Montarelo, 2017), logistics, procurement, finance, and marketing, with each function having different decision horizons. For example, budget forecasts are not required at the weekly horizon decision that is typical of inventory management, but they are needed at much longer horizons (Lapide, 2004).

Recent advances have shown the benefits associated with TA in terms of forecast accuracy and stock control improvements when non-optimal forecasting methods are used (Babai et al., 2012; Kourentzes et al., 2017; Rostami-Tabar et al., 2014). However, it should be noted that the benefit of this approach has been shown in the literature only under single exponential smoothing, which is optimal (minimum Mean Square Error) for an ARIMA(0,1,1) process. No analysis has been conducted to look at the impact of this approach under optimal forecasting for other processes. This paper aims to bridge this gap by analysing the impact of temporal aggregation on demand forecasts and orders in a two-stage supply chain involving a retailer and a manufacturer when optimal forecasting is used.

In this paper, we focus on ARMA(1,1) demand processes, which include AR(1), MA(1) and i.i.d. processes as special cases. Of course, this means that our analysis is not fully comprehensive and many demand processes are not addressed, such as seasonal processes. Nevertheless, the literature does support the applicability of an ARMA(1,1) process in supply chain forecasting and inventory management (Alwan et al., 2003; Chen et al., 2000; Lee et al., 2000; Rostami-Tabar et al., 2014; Zhang, 2004). Hosoda et al. (2008) show real supply chain contexts where retailers and suppliers follow autoregressive order one, AR(1) and ARMA(1,1) demand processes. Additionally, Disney et al. (2006) show that the demand processes for Procter & Gamble products can be modelled as an ARMA(1,1). Thus, although an ARMA(1,1) model is by no means comprehensive, it does represent the demand for a wide variety of industrial products. It should be noted that this research does not focus on the ARIMA(0,1,1) demand process, for which the SES method is optimal. However, numerical experimentation with an ARIMA(0,1,1) process bring us to similar conclusions as the case of ARMA(1,1) process, which is analysed in this paper.

In this paper, our objectives are threefold:

1) To analyse the impact of the use of the TA approach on the forecast accuracy (measured through the Mean Square Error) at the retailer level to forecast its lead-time demand. We analyse whether it is beneficial for the retailer to use the non-aggregate demand or the aggregated demand to produce the forecasts.
2) To analyse the impact of the use of the TA approach on the transmission of orders to the upstream link and on the bullwhip effect in supply chains. We evaluate whether the bullwhip effect in the supply chains can be reduced by the use of aggregated demand.
3) To examine the impact of the use of the TA approach on the forecast accuracy at the manufacturer level. We determine whether a manufacturer should use the orders received from the retailer (based on the non-aggregation approach) to generate its lead-time demand forecasts or should use the orders generated from the aggregated demand to generate the forecasts.

The rest of the paper is organised as follows. Section 2 briefly reviews the literature that deals with temporal aggregation. Section 3 starts with the presentation of our supply chain model and presents assumptions and notations. Section 4 focuses on the first objective of the paper. We present the analytical derivations of MSEs before and after aggregation at the retailer level. In Section 5, we move towards our second objective of focusing on the supply chain orders and we derive expressions for the bullwhip effect measure under the aggregation and non-aggregation approaches. In Section 6, we derive expressions of the MSEs at the manufacturer level under both approaches. The paper concludes in Section 7 with a summary of the findings and directions for future research.

2. Research Background

The analysis of temporal aggregation started with the work of Amemiya and Wu (1972) and has been the subject of continued research work (e.g. Athanasopoulos et al., 2011; Brewer, 1973; Stram & Wei, 1986; Tiao, 1972). Most researchers modelled demand as an AutoRegressive Integrated Moving Average (ARIMA) process of some form. They have analysed the impact of TA on the ARIMA process. They have characterised the aggregated process and determined the relationship between process parameters of the original and the aggregated process. Moreover, they showed that the aggregation approach results generally in an improvement of the forecast accuracy. The main limitations of this literature is that the forecasting methods and the performances measures have not been investigated in the supply chain context.

More recently, there has been substantial research to overcome these limitations (Syntetos et al., 2016). Nikolopoulos et al. (2011) have empirically analysed the effects of TA on forecasting intermittent demand requirements. Their main motivation was to reduce the number of zeros present in the original intermittent series and then forecast the series with conventional forecasting methods once the intermittency has been reduced substantially. The paper showed that the proposed methodology may indeed offer considerable improvements in terms of forecast accuracy. These findings have then been confirmed by Babai et al. (2012) and Petropoulos & Kourentzes (2015). Spithourakis et al. (2012) extended the application of Nikolopoulos et al. (2011) to fast-moving demand data. Results support forecast accuracy improvement by temporal aggregation.

Rostami-Tabar et al. (2013; 2014) further explored factors that impact the effect of TA on forecast accuracy. Assuming an ARMA (1,1) demand model and Single Exponential Smoothing (SES) forecasting method, they analytically show that the benefits of using temporal aggregation on the forecast accuracy depend on three factors: *i*) autoregressive and moving average parameters, *ii*) the aggregation level and *iii*) the smoothing constant for SES. The results show that for high levels of positive autocorrelation in the original series, the aggregation approach is outperformed by the non-aggregation approach. Secondly, the performance of aggregation was generally found to improve as the aggregation level increases.

Kourentzes et al. (2017) contrasted the effectiveness of using a multiple aggregation level or a single optimal aggregation level in forecast accuracy improvement. They conclude that using TA for demand forecasting is beneficial and argue that further research in identifying the optimal aggregation level is required. The current study contributes to the literature by analytically evaluating the impact of temporal aggregation on the supply chain's bullwhip effect and forecast accuracy when an optimal forecasting approach is used for the underlying demand process.

The necessity of conducting further research on the effect of TA on the bullwhip effect is also highlighted in the literature. For example, using real industry-level data, Cachon et al. (2007) indicate that the bullwhip effect may be more prevalent with aggregated series. Moreover, Bray and Mendelson (2012) mention the need to use temporal aggregation to gain further understanding of the bullwhip. Jin et al. (2015) empirically examined the effect of aggregation on the bullwhip effect at weekly and monthly levels by using retailer sales and order data, at the distribution center level. They demonstrate the masking effect of temporal aggregation and the damping effect of seasonality on the measurement of the bullwhip effect.

3. Supply chain model, notation and assumptions

For the remainder of the paper, we use the following notations:

m: Aggregation level, i.e. number of periods used for aggregated demand.

l : Lead time for the non-aggregate demand at both retailer and manufacturer level.

l' : Lead time for the aggregated demand at both retailer and manufacturer level.

n: Total number of periods available in the demand history.

t: Time unit for the non-aggregate demand series, $t=1, 2, \dots, n$.

T: Time unit for the aggregated demand series, $T=1,2,\dots, [n/m]$.

d_t: Non-aggregate demand in period t.

o_t: Order placed by retailer to manufacturer using the non-aggregate demand in period t.

D_T: Aggregated demand in period T, m periods aggregated demand.

O_T: Order placed by retailer to manufacturer using the aggregated demand in period T.

ε_t : Independent random variables for non-aggregate demand in period t, normally distributed with zero mean and standard deviation σ.

ε'_T : Independent random variables for aggregated demand in period T, normally distributed with zero mean and standard deviation σ'.

$\sigma_M{}^2$: Variance of the independent random variables for non-aggregate demand at the manufacturer in period t.

$\sigma_M'{}^2$: Variance of the independent random variables for aggregated demand at the manufacturer in period T.

$f_{t,l}^R$: Forecast of non-aggregate demand in period t, the forecast produced in period t for the demand over lead time l at the retailer.

$F_{T,l'}^R$: Forecast of aggregated demand in period T, the forecast produced in period T over lead-time l' at the retailer.

$f_{t,l}^M$: Forecast of non-aggregate demand in period t, the forecast produced in period t for the demand over lead time l at the manufacturer.

$F_{T,l'}^M$: Forecast of aggregated demand in period T, the forecast produced in period T over lead-time l' at the manufacturer.

MSE_{BA}^{R}: Theoretical Mean Squared Error (*MSE*) resulted from non-aggregate demand by the MMSE forecasting method at the retailer.
MSE_{AA}^{R}: Theoretical Mean Squared Error (*MSE*) resulted from aggregated demand by the MMSE forecasting method at the retailer.
MSE_{BA}^{M}: Theoretical Mean Squared Error (*MSE*) resulted from non-aggregate demand by the MMSE forecasting method at the manufacturer.
MSE_{AA}^{M}: Theoretical Mean Squared Error (*MSE*) resulted from aggregated demand by the MMSE forecasting method at the manufacturer.
γ_k : Covariance of lag *k* of non-aggregate demand, $\gamma_k = Cov(d_t, d_{t-k})$.
γ'_k : Covariance of lag *k* of aggregated demand, $\gamma'_k = Cov(D_T, D_{T-k})$.
ϕ : Autoregressive parameter of non-aggregate demand process at the retailer, $|\phi| < 1$.
ϕ_M : Autoregressive parameter of non-aggregate demand process at the manufacturer, $|\phi_M| < 1$.
ϕ' : Autoregressive parameter of aggregated demand process at the retailer, $|\phi'| < 1$.
ϕ'_M : Autoregressive parameter of aggregated demand process at the manufacturer, $|\phi'_M| < 1$.
θ : Moving average parameter of non-aggregate demand process at the retailer, $|\theta| < 1$.
θ_M : Moving average parameter of non-aggregate demand process at the manufacturer, $|\theta_M| < 1$.
θ' : Moving average parameter of aggregated demand process at the retailer, $|\theta'| < 1$.
θ'_M : Moving average parameter of aggregated demand process at the manufacturer, $|\theta'_M| < 1$.
C : Constant parameter of non-aggregate demand in any time period.
C' : Constant parameter of aggregated demand in any time period.
$B(\phi, \theta, l)$: Bullwhip effect for the non-aggregate demand.
$B(\phi', \theta', l')$: Bullwhip effect for the aggregated demand.

We consider a two-stage supply chain, with one player at each level. For simplicity, we call the first level player the retailer and the second level player the manufacturer. Both players use the conditional expectation to provide optimal (MMSE) forecasts and exploit a periodic review system, and the replenishment lead-time is constant and known. We assume the retailer knows its own demand process, which is reflected in its orders on the manufacturer. The Order Up To (OUT) policy is used to place replenishment orders. The OUT level is adjusted in each time period according to the latest demand forecast. Both players have access to the original demand and the aggregated demand process. This is shown in Figure 2. Figure 2 (top) indicates the use of a non-aggregate demand process in the model. The retailer receives demands following an ARMA(1, 1) process and then calculates forecasts using a conditional expectation. The OUT policy is used to generate an order to the manufacturer stage. The order received by the manufacturer also follows an ARMA(1,1) process (Lee et al., 2000). The manufacturer uses the same forecasting method and inventory policy as the retailer to calculate forecast and orders.

The retailer and manufacturer may use an aggregated demand process instead of using the original demand to calculate forecasts and orders. Amemiya and Wu (1972) show that using TA transforms an ARMA(1,1) process into another ARMA(1,1) process with different parameters (please refer to Amemiya and Wu (1972); Rostami-Tabar et al. (2013) for details). Figure 2 (bottom), illustrates the supply chain model when the aggregated demand process is used. In this situation, the retailer (manufacturer) has the option either to: *(i)* calculate the forecast and the OUT level and order to the upper stage based on non-aggregate demand (order) data (i.e. using daily or weekly data), or (*ii*) calculate the aggregated forecast and the OUT level and order based on aggregated demand (order) data (monthly or lead-time level demand data).

Additionally, it has been shown that the OUT policy transforms an ARMA(1, 1) demand process at the retailer level into another ARMA(1, 1) process at manufacturer level, which is valid for both non-aggregate demand and aggregated demand process; those processes are represented by o_t and O_T respectively (Hosoda & Disney, 2006). Therefore, o_t and O_T follow an ARMA(1,1) process as well. The manufacturer also has the same options either to calculate the forecast and the OUT level and order to the upper stage based on non-aggregate orders or aggregated orders.

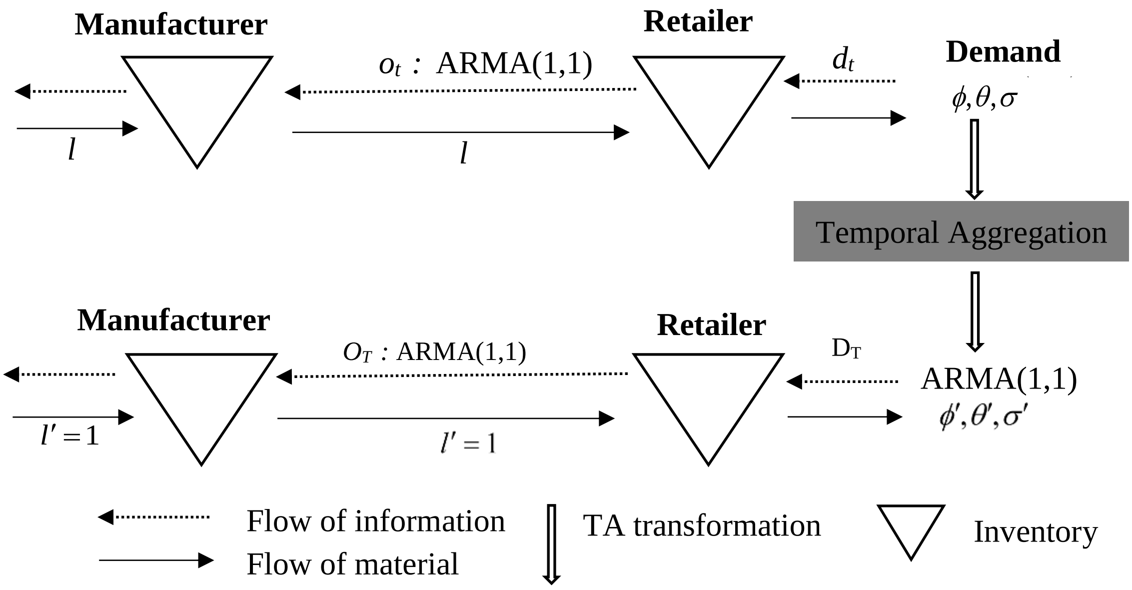

Figure 2 : Demand propagation in a two-stage supply chain with temporal aggregation

We assume that the non-aggregate demand series, d_t follows a mixed autoregressive moving average demand process of order (1, 1) - ARMA(1,1) - that can be mathematically written in period t by (1).

$$d_t = C + \varepsilon_t + \phi d_{t-1} - \theta \varepsilon_{t-1}. \tag{1}$$

Constraining θ and ϕ to lie between -1 and 1 in (1), means that the process is stationary and invertible.

The forecasting method considered in this study is the conditional expectation that provides the Minimum Mean Squared Error (MMSE) unbiased forecast. Using the Auto-Regressive Integrated Moving Average (ARIMA) methodology, we can mathematically specify the optimal MMSE forecasting method for any demand process. This optimality holds only on the basis of minimising the MSE.

The modelling assumes that the forecaster has knowledge of the demand process and its parameters. This is an ideal case, and future research is needed to relax these assumptions.

Using MMSE, the forecast of demand over horizon l for the retailer and the manufacturer, calculated at time t, knowing the demands $d_{t-1}, d_{t-2},\ldots,$ for the retailer and orders $o_{t-1}, o_{t-2},\ldots,$ for the manufacturer is:

$$f_{t,l}^{R} = \sum_{i=0}^{l-1} E\left(d_{t+i} \middle| d_{t-1}, d_{t-2}, \ldots\right) \tag{2}$$

$$f_{t,l}^{M} = \sum_{i=0}^{l-1} E\left(o_{t+i} \middle| o_{t-1}, o_{t-2}, \ldots\right) \tag{3}$$

The demand over horizon l at the retailer level, $d_{t,l} = D_T$ can be expressed as a function of the non-aggregate demand series as follows

$$d_{t,l} = D_T = \sum_{i=0}^{l-1} d_{t+i} . \tag{4}$$

The demand over horizon l at the manufacturer level (same as orders received from the retailer), $o_{t,l} = O_T$ can be expressed as a function of the non-aggregate order series as follows

$$o_{t,l} = O_T = \sum_{i=0}^{l-1} o_{t+i} . \tag{5}$$

Using MMSE, the forecast of aggregated demand at period T over horizon l', knowing the demands D_{T-1}, D_{T-2} ,..., and orders O_{T-1}, O_{T-2} ,..., is:

$$F^R_{T,l'} = E\left(D_{T+l'} \middle| D_{T-1}, D_{T-2}, ...\right) \tag{6}$$

$$F^M_{T,l'} = E\left(O_{T+l'} \middle| O_{T-1}, O_{T-2}, ...\right) \tag{7}$$

The forecast of one period ahead at both retailer and manufacturer levels using aggregated data, $F^R_{T,1}$ and $F^M_{T,1}$ are the equivalent of the forecast over horizon l in using non-aggregate demand series $f^R_{t,l}$ and $f^M_{t,l}$. Therefore, in order to compare the demand forecasts over horizon/lead time l resulting from the non-aggregate and the aggregated demand at both retailer and manufacturer levels, we set $l' = 1$ and l and m are used interchangeably, $l=m$.

4. Impact of the temporal aggregation approach on the forecast accuracy at the retailer

Section 4 is focused on the first objective of our paper, namely the evaluation of the effect of TA on a retailer's forecast accuracy when an MMSE forecasting method is used and a forecast over lead-time l, is required.

In this section, we derive the expression of the MSE for the forecast made at the retailer stage for non-aggregate and aggregated demand. The original data series is used in the case of non-aggregate demand and comparisons are to be performed at the aggregate level. The following approach is used for the aggregated demand: firstly buckets of aggregated series are created based on the aggregation level; then the MMSE method is applied to these aggregated series to produce the aggregate forecasts. In sub-sections 4.1 and 4.2, we derive the expressions of the MSE before and after aggregation at the retailer level. We present the numerical results in sub-section 4.3.

4.1. MSE before aggregation

We begin the analysis by deriving the *MSE* of an aggregate forecast for the ARMA(1,1) process at the retailer level by using the non-aggregate demand series. It is known that when demand follows an ARMA(1, 1) process, the auto-covariance is (Box et al., 2015):

$$\gamma_k = Cov(d_t, d_{i,t-k}) = \begin{cases} \frac{(1-2\phi\theta+\theta^2)}{1-\phi^2}\sigma^2 & k=0 \\ \frac{(\phi-\theta)(1-\phi\theta)}{1-\phi^2}\sigma^2 & |k|=1 \\ \phi^{k-1}\gamma_1 & |k|>1 \end{cases}, \tag{8}$$

Now we derive the MSE of the aggregate forecast resulting from the non-aggregate demand, using for the optimal –MMSE- method:

$$MSE_{BA}^R = var(Forecast\,Error) = var\left(d_{t,l} - f_{t,l}^R\right) = var\left(\sum_{i=0}^{m-1} d_{t+i} - f_{t,m}^R\right), \tag{9}$$

The aggregate forecast at retailer stage when the non-aggregate demand follows an ARMA(1,1) process is:

$$f_{t,m}^R = \sum_{i=0}^{m-1} E\left(d_{t+i} \middle| d_{t-1}, d_{t-2}, \cdots\right), \tag{10}$$

Considering (1) and through the recursive substitution of d_{t+i} we get:

$$\begin{aligned} d_{t+i} &= C\left(1+\phi+\phi^2+\cdots+\phi^i\right)+\phi^{i+1}d_{t-1}+\varepsilon_{t+i}+(\phi-\theta)\varepsilon_{t+i-1}+\phi(\phi-\theta)\varepsilon_{t+i-2}+\cdots-\phi^i\theta\varepsilon_{t-1} = \\ & C\left(\frac{1-\phi^{i+1}}{1-\phi}\right)+\phi^{i+1}d_{t-1}+\varepsilon_{t+i}+(\phi-\theta)\varepsilon_{t+i-1}+\phi(\phi-\theta)\varepsilon_{t+i-2}+\cdots-\phi^i\varepsilon_{t-1}, \end{aligned} \tag{11}$$

By substituting (11) into (10) , we get:

$$\begin{aligned} f_{t,m}^R &= \sum_{i=0}^{m-1} E\left(C\left(\frac{1-\phi^{i+1}}{1-\phi}\right)+\phi^{i+1}d_{t-1}+\varepsilon_{t+i}+(\phi-\theta)\varepsilon_{t+i-1}+\phi(\phi-\theta)\varepsilon_{t+i-2}+\cdots-\phi^i\theta\varepsilon_{t-1} \middle| d_{t-1}, d_{t-2}, \cdots\right) = \\ & \sum_{i=0}^{m-1}\left(C\left(\frac{1-\phi^{i+1}}{1-\phi}\right)+\phi^{i+1}d_{t-1}-\phi^i\theta\varepsilon_{t-1}\right) = \frac{C}{1-\phi}\left(m-\phi\left(\frac{1-\phi^m}{1-\phi}\right)\right)+\phi d_{t-1}\left(\frac{1-\phi^m}{1-\phi}\right)-\theta\varepsilon_{t-1}\left(\frac{1-\phi^m}{1-\phi}\right), \end{aligned} \tag{12}$$

By substituting (11) and (12) into (9) and simplifying, we obtain the MSE_{BA}^R:

$$\begin{aligned} MSE_{BA}^R &= var\left(d_{t,m} - f_{t,m}^R\right) = var\begin{pmatrix} \sum_{i=0}^{m-1}\left(C\left(\frac{1-\phi^{i+1}}{1-\phi}\right)+\phi^{i+1}d_{t-1}+\varepsilon_{t+i}+(\phi-\theta)\varepsilon_{t+i-1}+\phi(\phi-\theta)\varepsilon_{t+i-2}+\cdots-\phi^i\varepsilon_{t-1}\right)- \\ \frac{C}{1-\phi}\left(m-\phi\left(\frac{1-\phi^m}{1-\phi}\right)\right)+\phi d_{t-1}\left(\frac{1-\phi^m}{1-\phi}\right)-\theta\varepsilon_{t-1}\left(\frac{1-\phi^m}{1-\phi}\right) \end{pmatrix} \\ &= \frac{\gamma_0(1-\phi^2)}{1+\theta^2-2\phi\theta}\left(1+\sum_{i=0}^{m-2}\left(1+\sum_{j=0}^{m-i-2}\phi^j(\phi-\theta)\right)^2\right) = \left(1+\sum_{i=0}^{m-2}\left(1+\sum_{j=0}^{m-i-2}\phi^j(\phi-\theta)\right)^2\right)\sigma^2. \end{aligned} \tag{13}$$

4.2. MSE after aggregation

In this section, we proceed with the derivation of the *MSE* of the aggregate forecasts at the retailer level resulting from the aggregated demand:

$$MSE_{AA}^R = var\left(D_T - F_{T,1}^R\right) = var(D_T) + var\left(F_{T,1}^R\right) - 2cov\left(D_T, F_{T,1}^R\right), \tag{14}$$

Demand is first aggregated and then we provide the aggregate forecasts based on *MMSE* method. The aggregate forecast for period *T* is defined as:

$$F_{T,1}^{R} = E\left(D_T \middle| D_{T-1}, D_{T-2}, \ldots\right), \tag{15}$$

If the non-aggregate demand follows an ARMA(1, 1) process then the aggregated series follows an ARMA(1, 1) process (Amemiya & Wu, 1972). It can be shown that the following properties hold when the aggregated process is ARMA(1, 1):

$$D_T = \mu'(1-\phi') + \varepsilon'_T + \phi' D_{T-1} - \theta'\varepsilon'_{T-1}, where\ |\theta'| < 1, |\phi'| < 1\,, \tag{16}$$

$$\gamma'_k = \begin{cases} \dfrac{1-2\phi'\theta'+\theta'^2}{1-\phi'^2}\sigma'^2 & k=0 \\ \dfrac{(\phi'-\theta')(1-\phi'\theta')}{1-\phi'^2}\sigma'^2 & |k|=1\,, \\ \phi'\gamma'_{k-1} = \phi'^{k-1}\gamma'_1 & |k|>1 \end{cases} \tag{17}$$

The relations between the aggregated and the initial parameters can be represented as follows (Rostami-Tabar et al., 2014):

$$\gamma'_0 = m\gamma_0 + \gamma_1\left(\sum_{k=1}^{m-1} 2(m-k)\phi^{k-1}\right) \tag{18}$$

$$\gamma'_1 = \gamma_1\left(\sum_{k=1}^{m} k\phi^{k-1} + \sum_{k=2}^{m}(k-1)\phi^{2m-k}\right) \tag{19}$$

$$\phi' = \phi^m\,, \tag{20}$$

The aggregate forecast resulting for the aggregated demand at the retailer stage for period T by using the MMSE method is:

$$F_{T,1}^{R} = E\left(D_T \middle| D_{T-1}, D_{T-2}, \ldots\right) = \mu'(1-\phi') - \theta'\varepsilon'_{T-1} + \phi' D_{T-1}, \tag{21}$$

By substituting (16) and (21) into (14) we have:

$$\begin{aligned} MSE_{AA}^{R} &= var\left(\mu'(1-\phi') + \varepsilon'_T + \phi' D_{T-1} - \theta'\varepsilon'_{T-1} - \left(\mu'(1-\phi') - \theta'\varepsilon'_{T-1} + \phi' D_{T-1}\right)\right) \\ &= var\left(\varepsilon'_{T,1}\right) = \sigma'^2, \end{aligned} \tag{22}$$

By considering γ'_0 in (18) and substituting (17) into it, we get:

$$\sigma'^2 = \frac{\left(1-\phi^{2m}\right)\left(m\gamma_0 + \gamma_1\sum_{k=1}^{m-1} 2(m-k)\phi^{k-1}\right)}{\left(\left(1-2\phi^m\theta' + \theta'^2\right)\right)}\,, \tag{23}$$

Therefore, by substituting (23) into (22) we have:

$$MSE_{AA}^{R} = \frac{\left(1-\phi^{2m}\right)\left(m\gamma_0 + \gamma_1\sum_{k=1}^{m-1} 2(m-k)\phi^{k-1}\right)}{\left(\left(1-2\phi^m\theta' + \theta'^2\right)\right)}, \tag{24}$$

From (A9) and (A10) in Appendix A, the moving average parameters of the aggregated process are as follows:

$$\theta_1' = \frac{-\left(\mathrm{X} + \phi^{2m}\mathrm{X} - 2\phi^m\right) + \sqrt{\left(\mathrm{X} + \phi^{2m}\mathrm{X} - 2\phi^m\right)^2 - 4\left(1 - \phi^m \mathrm{X}\right)^2}}{2\left(1 - \phi^m \mathrm{X}\right)}, \quad \phi > \theta \tag{25}$$

$$\theta_2' = \frac{-\left(\mathrm{X} + \phi^{2m}\mathrm{X} - 2\phi^m\right) - \sqrt{\left(\mathrm{X} + \phi^{2m}\mathrm{X} - 2\phi^m\right)^2 - 4\left(1 - \phi^m \mathrm{X}\right)^2}}{2\left(1 - \phi^m \mathrm{X}\right)}, \quad \phi < \theta \tag{26}$$

where $\mathrm{X} = \dfrac{\left(m + \sum_{k=1}^{m-1} 2(m-k)\phi^k\right)}{\left(\sum_{k=1}^{m} k\phi^k + \sum_{k=1}^{m-1} k\phi^{2m-k}\right)}$

By substituting (25) and (26) into (24) we get:

$$MSE_{AA}^{R} = \sigma'^2 = \begin{cases} \dfrac{\left(1-\phi^{2m}\right)\varpi}{\left(\left(1 - 2\phi^m\left(\dfrac{-\eta + \sqrt{\eta^2 - 4\upsilon^2}}{2\upsilon}\right) + \left(\dfrac{-\eta + \sqrt{\eta^2 - 4\upsilon^2}}{2\upsilon}\right)^2\right)\right)} & \phi > \theta \\ \dfrac{\left(1-\phi^{2m}\right)\varpi}{\left(\left(1 - 2\phi^m\left(\dfrac{-\eta - \sqrt{\eta^2 - 4\upsilon^2}}{2\upsilon}\right) + \left(\dfrac{-\eta + \sqrt{\eta^2 - 4\upsilon^2}}{2\upsilon}\right)^2\right)\right)} & \phi < \theta \end{cases} . \tag{27}$$

where $\varpi = \left(m\gamma_0 + \gamma_1 \sum_{k=1}^{m-1} 2\left(m-k\right)\phi^{k-1}\right)$, $\eta = \left(\mathrm{X} + \phi^{2m}\mathrm{X} - 2\phi^m\right)$, $\upsilon = \left(1 - \phi^m \mathrm{X}\right)$.

Given expressions of MSE_{BA}^{R} and MSE_{AA}^{R} by Equation (13) and (27), we can analyse the conditions under which temporal aggregation provides more accurate forecasts. However, these equations are too mathematically complex to derive exact proofs. Therefore, they require a numerical investigation to provide insights that will be presented in sub-section 4.3.

4.3. Numerical analysis

In this section, the effect of TA on forecast accuracy at the retailer level is evaluated when an MMSE forecasting method is used. We numerically analyse the ratio of $MSE_{AA}^{R}/MSE_{BA}^{R}$ for different values of the aggregation level m and the process parameters ϕ and θ. We are interested to determine under which conditions, if any, the ratio of MSE after aggregation to MSE before aggregation is less than one, meaning that TA improves the forecast accuracy of the retailer. Figure 3 shows the ratio $MSE_{AA}^{R}/MSE_{BA}^{R}$ for m = 2 to 12 and various combinations of ϕ and θ between -1 and +1.

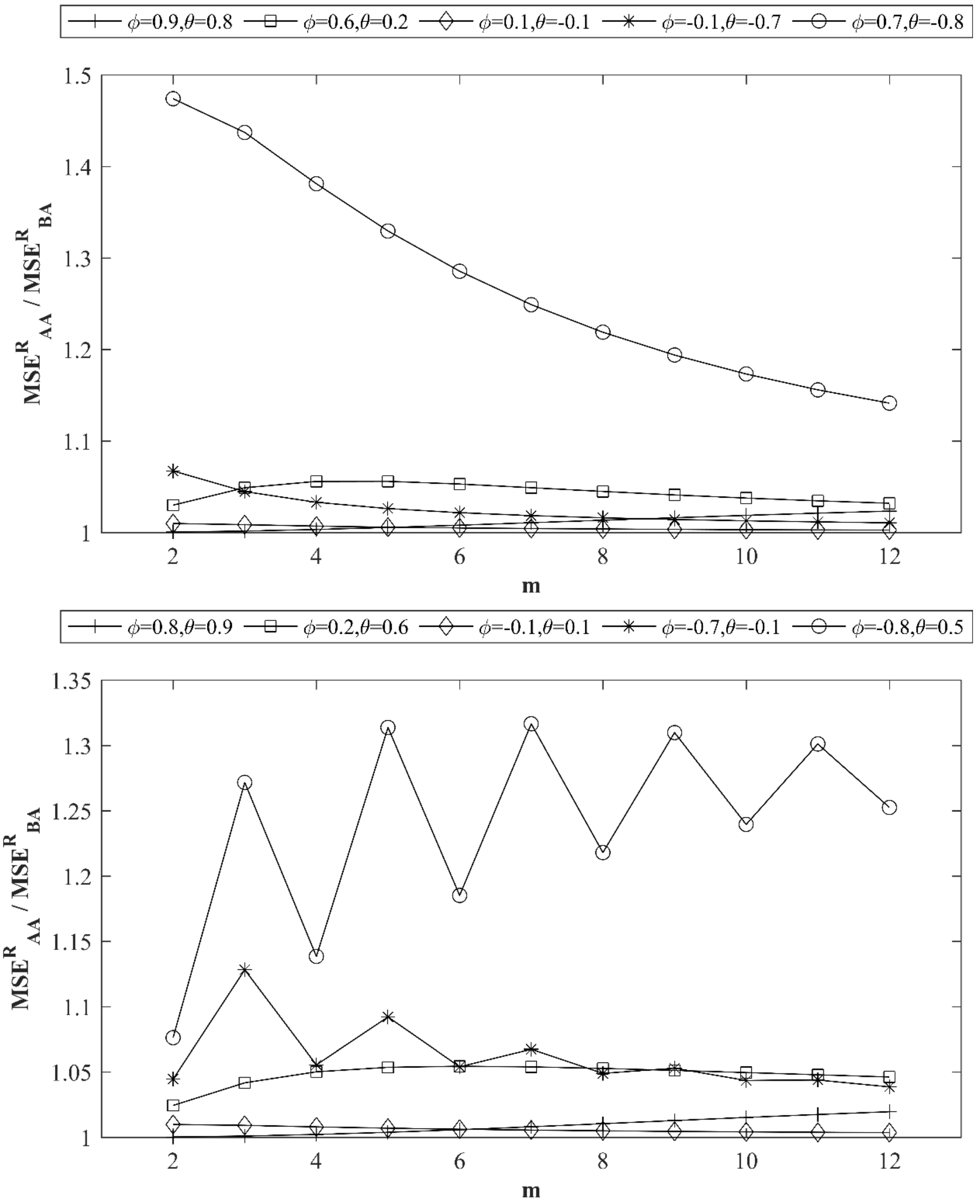

Figure 3: Ratio of MSEs after aggregation to before aggregation, $\phi > \theta$ (top) and $\phi < \theta$ (bottom)

Figure 3 presents the impact of the process parameters, ϕ and θ, and the aggregation level, m on the ratio of $MSE_{AA}^{R}/MSE_{BA}^{R}$. If the ratio is greater than one, then TA does not improve the forecast accuracy at the retailer level. Although we only present some values of the autoregressive and moving average parameters and $2 \leq m \leq 12$, in Figure 3, it has been checked that the ratio is always greater than one regardless of the aggregation level and the process parameters. This means that TA does not improve the forecast accuracy at the retailer and the aggregation approach is outperformed by the non-aggregation one when an MMSE forecasting method is used by the retailer. The underperformance of the MMSE forecasting method with the aggregated series can be attributed to the fact that by applying the temporal aggregation to the original data series we increase the uncertainty of the underlying model and process parameter estimations. Hence, using the MMSE forecasting method with higher uncertainty of aggregated demand series, in comparison to the original, will increase the variance of the noise and consequently the variance of forecast error. We observe in Figure 3 (bottom), when the autoregressive parameter is negative, the effect of the aggregation approach depends on whether the aggregation level is odd or even. Clearly, there is a greater amplification of MSEs when the aggregation level is odd.

5. Impact of the temporal aggregation on bullwhip effect in supply chains

In this section, we move towards the second objective of the paper, namely of analysing the effect of the TA approach on the supply chain dynamics (i.e. transmission of orders) and the bullwhip effect.

In this section, the bullwhip ratio at the retailer is expressed based on both the aggregated and the non-aggregate demand. One approach to measure the bullwhip effect at the retailer is to calculate the ratio of the variance of the orders made by the retailer to the manufacturer to the variance of the demand faced by the retailer. This measure has been used in previous studies (e.g. Chen et al., 2000; Duc et al., 2008). According to Duc et al. (2008), the variance of orders and the measure of the bullwhip effect for an ARMA(1, 1) demand process are calculated as follows, considering $\gamma_0 = Var(d_t)$:

$$var(o_t) = \left(1 + \frac{2(\phi-\theta)(1-\phi^l)\left(1-\phi^{l+1}-\phi\theta(1-\phi^{l-1})\right)}{(1-\phi)(1+\theta^2-2\theta\phi)}\right)\gamma_0 \tag{28}$$

$$B(\phi,\theta,l) = 1 + \frac{2(\phi-\theta)(1-\phi^l)\left(1-\phi^{l+1}-\phi\theta(1-\phi^{l-1})\right)}{(1-\phi)(1+\theta^2-2\theta\phi)}, \tag{29}$$

We evaluate the bullwhip effect when the retailer demand comes from an aggregated demand compared to that of non-aggregate demand, i.e. we compare the bullwhip ratios of the two cases. Because we are interested in evaluating the impact of TA on the bullwhip effect, we first aggregate the original demand series at the retailer level to obtain an aggregated series. Then, the bullwhip effect is calculated for the aggregated series. As discussed earlier in sub-section 4.2, if the non-aggregate series follows an ARMA(1, 1) process, then the aggregated series also follows an ARMA(1, 1) process (Amemiya & Wu, 1972).
Similarly to (28), the variance of the order quantity for an aggregated demand is calculated as:

$$var(O_T) = \gamma_0'\left(1 + \frac{2(\phi'-\theta')(1-\phi'^2)}{(1+\theta'^2-2\theta'\phi')}\right) \tag{30}$$

By substituting (18), (20) into (30) and considering (25) and (26) we get:

$$var(O_T) = \begin{cases} \left(m\gamma_0 + \gamma_1\left(\sum_{k=1}^{m-1} 2(m-k)\phi^{k-1}\right)\right)\left(1 + \frac{2(\phi^m-\theta_1')(1-\phi^{2m})}{(1+\theta_1'^2-2\theta_1'\phi^m)}\right) & \phi > \theta \\ \left(m\gamma_0 + \gamma_1\left(\sum_{k=1}^{m-1} 2(m-k)\phi^{k-1}\right)\right)\left(1 + \frac{2(\phi^m-\theta_2')(1-\phi^{2m})}{(1+\theta_2'^2-2\theta_2'\phi^m)}\right) & \phi < \theta \end{cases} \tag{31}$$

The bullwhip ratio is calculated for the aggregated series similarly to (29), as the aggregated series also follows an ARMA(1, 1) process and the same forecasting method- MMSE- is used. The bullwhip of the aggregated series is:

$$B(l',\phi',\theta') = 1 + \frac{2(\phi'-\theta')(1-\phi'^{l'})\left(1-\phi'^{l'+1}-\phi'\theta'(1-\phi'^{l'-1})\right)}{(1-\phi')(1+\theta'^2-2\theta'\phi')}, \tag{32}$$

By dividing (32) by (29), we obtain the ratio of the bullwhip effect after aggregation divided by the bullwhip effect before aggregation:

$$RBW=\frac{B(\phi',\theta',l')}{B(\phi,\theta,l)}=\frac{1+\frac{2(\phi'-\theta')(1-\phi'^{l'})\left(1-\phi'^{l'+1}-\phi'\theta'\left(1-\phi'^{l'-1}\right)\right)}{(1-\phi')(1+\theta'^2-2\theta'\phi')}}{1+\frac{2(\phi-\theta)(1-\phi^{l})\left(1-\phi^{l+1}-\phi\theta\left(1-\phi^{l-1}\right)\right)}{(1-\phi)(1+\theta^2-2\theta\phi)}}, \qquad (33)$$

The ratio in (33) helps us to analyse the effect of temporal aggregation on the bullwhip effect. A ratio smaller than one means that the aggregation approach reduces the bullwhip effect. A ratio greater than one means that aggregation increases the bullwhi effect.

By substituting (20) into (33) and considering θ_1' and θ_2' defined in (25) and (26), we get:

$$RBW=\frac{B(\phi',\theta',l')}{B(\phi,\theta,l)}=\begin{cases}\dfrac{1+\frac{2(\phi^m-\theta_1')(1-\phi^{ml'})\left(1-\phi^{m(l'+1)}-\phi^m\theta_1'\left(1-\phi^{m(l'-1)}\right)\right)}{(1-\phi^m)(1+\theta_1'^2-2\theta_1'\phi^m)}}{1+\frac{2(\phi-\theta)(1-\phi^{l})\left(1-\phi^{l+1}-\phi\theta\left(1-\phi^{l-1}\right)\right)}{(1-\phi)(1+\theta^2-2\theta\phi)}} & \phi>\theta \\[2em] \dfrac{1+\frac{2(\phi^m-\theta_2')(1-\phi^{m})\left(1-\phi^{(l'+1)m}-\phi^m\theta_2'\right)}{(1-\phi^m)(1+\theta_2'^2-2\theta_2'\phi^m)}}{1+\frac{2(\phi-\theta)(1-\phi^{l})\left(1-\phi^{l+1}-\phi\theta\left(1-\phi^{l-1}\right)\right)}{(1-\phi)(1+\theta^2-2\theta\phi)}} & \phi<\theta\end{cases}, \qquad (34)$$

It should be noted that, to make a comparison between the bullwhip effect of the non-aggregate demand series and the aggregated series, we set $m=l$ and $l'=1$.

Due to the complexity of Equation (34) and in order to analyse the impact of the process parameters and lead time/aggregation level on the bullwhip effect, we conduct a numerical analysis in sub-section 5.1.

5.1. Numerical Results

Previous research has shown that the bullwhip effect does not always occur, but its existence depends on the values of the autoregressive and moving average parameters of the ARMA model. Duc et al (2008) show that for an ARMA (1, 1) process, the bullwhip effect occurs when $\phi>\theta$; however when $\phi<\theta$ the supply chain system faces an anti-bullwhip effect. In this sub-section, the effect of the aggregation level m, and the process parameters θ and ϕ, on the bullwhip effect is numerically examined when using both the aggregated and the non-aggregate series. We are interested to determine under which conditions TA reduces the bullwhip effect.

Figure 4 (based on Equation (32)) shows the impact of the aggregation level m, autoregressive ϕ and moving average θ parameters on the bullwhip effect using the aggregated demand series. The two regions are shown in Figure 4 with white colour representing the area with bullwhip effect ($\phi > \theta$) and grey colour representing the area with anti-bullwhip effect ($\phi < \theta$).

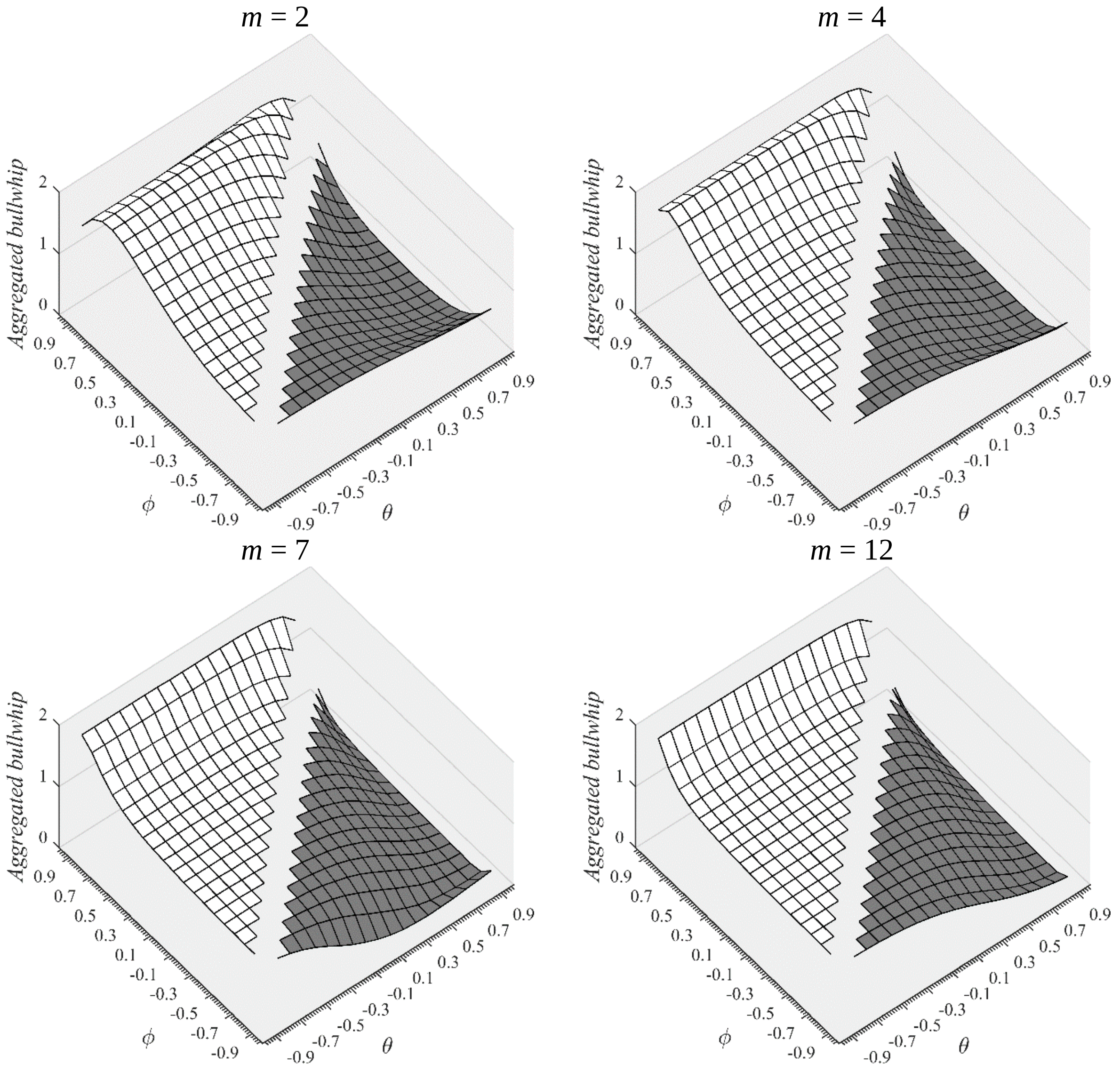

Figure 4 : The impact of *m*, ϕ and θ on the bullwhip effect of aggregated demand

We find that aggregation does not have an effect on the region where the bullwhip and anti-bullwhip exist for the original ARMA(1,1) demand series. After using temporal aggregation, we still find bullwhip when $\phi > \theta$ and anti-bullwhip effects when $\phi < \theta$. As one of the objectives of this paper is to consider the effect of TA and MMSE on the reduction of bullwhip effect, we only consider the parameter values of ARMA (1, 1) where the bullwhip effect occurs.

We analyse the effect of TA on the reduction of the bullwhip effect by evaluating the ratio of the bullwhip effect after aggregation and before aggregation (equation (34) from the previous sub-section). Detailed analysis of the numerical results shows that the bullwhip effect of the aggregated series generally decreases as the aggregation level increases with few exceptions given an odd aggregation level, *m*, and highly positive values of the autoregressive parameter ϕ and highly negative moving average parameter, θ, which represents highly positive autocorrelation.

It should be noted that, the range of the bullwhip effect corresponding to the original series is higher that the aggregated series and we observe a tremendous reduction in the range for the aggregated series. Considering $2 \le l \le 12$ and $-0.9 < \phi < 0.9, -0.9 < \theta < 0.9$ in the non-aggregate series, the range is [1.02, 10.9]. For the aggregated series considering the same range of process parameters and the aggregation level $2 \leq m \leq 12$, the range is [1.002, 1.9]. The retailer can consider using the non-aggregate or aggregated demand to calculate the OUT level and order to the upper level, but which option is more suitable and will lead to performance improvement? We can answer this question by analysing the bullwhip reduction if the retailer uses the aggregated demand series.

Table 1 : Values of the ratio of the aggregated bullwhip effect into the non-aggregate one in the case of $m = 2$

	ϕ								
θ	-0.80	-0.50	-0.20	-0.10	0.10	0.20	0.50	0.80	0.90
-0.90	0.97	0.87	0.73	0.69	0.65	0.64	0.67	0.79	0.86
-0.80	-	0.87	0.73	0.70	0.65	0.64	0.67	0.78	0.85
-0.70	-	0.89	0.75	0.71	0.65	0.64	0.67	0.78	0.85
-0.40	-	-	0.85	0.79	0.70	0.68	0.67	0.75	0.82
-0.10	-	-	-	-	0.83	0.78	0.70	0.73	0.79
0.10	-	-	-	-	-	0.91	0.75	0.73	0.78
0.40	-	-	-	-	-	-	0.92	0.78	0.78
0.70	-	-	-	-	-	-	-	0.94	0.88
0.80	-	-	-	-	-	-	-	-	0.95

Table 2 : Values of the ratio of the aggregated bullwhip effect into the non-aggregate one in the case of $m = 7$

	ϕ								
θ	-0.80	-0.50	-0.20	-0.10	0.10	0.20	0.50	0.80	0.90
-0.90	0.90	0.77	0.63	0.58	0.49	0.44	0.32	0.26	0.30
-0.80	-	0.79	0.64	0.59	0.49	0.45	0.32	0.26	0.29
-0.70	-	0.83	0.65	0.60	0.50	0.45	0.32	0.26	0.29
-0.40	-	-	0.78	0.70	0.56	0.49	0.33	0.26	0.29
-0.10	-	-	-	-	0.73	0.62	0.38	0.26	0.29
0.10	-	-	-	-	-	0.82	0.45	0.28	0.29
0.40	-	-	-	-	-	-	0.76	0.35	0.32
0.70	-	-	-	-	-	-	-	0.70	0.49
0.80	-	-	-	-	-	-	-	-	0.69

Table 3 : Values of the ratio of the aggregated bullwhip effect into the non-aggregate one in the case of m = 12

	ϕ								
θ	-0.80	-0.50	-0.20	-0.10	0.10	0.20	0.50	0.80	0.90
-0.90	0.93	0.76	0.61	0.57	0.47	0.42	0.28	0.16	0.16
-0.80	-	0.78	0.62	0.57	0.47	0.43	0.28	0.16	0.16
-0.70	-	0.82	0.64	0.59	0.48	0.43	0.28	0.16	0.16
-0.40	-	-	0.77	0.69	0.54	0.47	0.30	0.17	0.16
-0.10	-	-	-	-	0.71	0.60	0.34	0.17	0.16
0.10	-	-	-	-	-	0.81	0.41	0.19	0.17
0.40	-	-	-	-	-	-	0.74	0.25	0.19
0.70	-	-	-	-	-	-	-	0.59	0.34
0.80	-	-	-	-	-	-	-	-	0.54

Despite the existence of the bullwhip effect when using the aggregated series, it is important to investigate whether the bullwhip effect reduces using TA and what the reduction percentage is for different values of the aggregation level (lead-time) and process parameters.

To that end, we divide the bullwhip effect calculated for the aggregate series into the non-aggregate one as shown in Eq. (34). We are primarily interested in the range of parameters where the bullwhip effect exists, i.e. $\phi > \theta$. If the ratio is less than one, it means that using temporal aggregation reduces the bullwhip effect compared to the non-aggregate series and the retailer should consider using the aggregated demand series to calculate the OUT level and place orders based on that. The analysis has been conducted for the whole range of ϕ, θ, and aggregation level $2 \leq m \leq 12$. The results are presented in Table 1, Table 2 and Table 3.

The results indicate that the ratio is always smaller than one when the autoregressive process parameter is greater than the moving average one, $\phi > \theta$, regardless of the aggregation level. This shows that TA reduces the bullwhip effect. The amount of bullwhip effect reduction may reach 84% when the lead-time (aggregation level) is high and the autocorrelation is highly positive. This corresponds to the case where $m = 12$, $0.8 \leq \phi$ *and* $\theta \leq -0.4$. This is very interesting from the perspective of practitioners as using temporally aggregated series leads to less bullwhip effect in the system comparing to non-aggregate series. As the lead-time becomes longer, we observe more variabilities in upper stages when using the non-aggregate series to calculate OUT levels and orders, while variability is substantially lower in using the aggregated series.

From Section 4, we know that TA increases the variance of the forecast error at the retailer level and that may have negative consequences on the safety stock. However, this does not affect the bullwhip ratio since due to the stationarity assumption of the process, the safety stock is constant over time and the orders to the manufacturers are independent of the safety stock. Hence, the process parameters and the lead time/aggregation level only affect the bullwhip ratio. In addition, the bullwhip effect is reduced using the aggregated series because TA decreases the values of moving average and autoregressive process parameters towards zero and subsequently pushes the ARMA process towards an i.i.d process and the bullwhip effect for such a process is reduced. Our results in this section show that the application of a TA approach results in the reduction of the bullwhip effect in supply chains. Various previous studies have focussed on the issue of reduction of the bullwhip effect and have done so by the evaluation of strategies such as improving the forecasting method or sharing information (Chatfield et al., 2004, Mason-Jones et al., 2015, Tesfay, 2016, Wang & Disney, 2016). However, this is the first study of which we are aware, that considers the effect of TA on the reduction of the bullwhip effect.

6. Impact of the temporal aggregation approach on the forecast accuracy at the manufacturer level

In this section, we focus on the third objective of the paper, namely the evaluation of the effect of the TA on the manufacturer's forecast accuracy. We are interested in determining whether it is beneficial to the manufacturer to generate its forecasts when the demand aggregation approach is considered. To do so, we derive the MSE of the forecasts at the manufacturer level by considering the non-aggregate and the aggregated demand. For the process under consideration, we calculate the MSE based on non-aggregate and aggregated demand at the manufacturer level by using the MMSE forecasting method and then we compare the forecast accuracy.

We start by deriving the MSE of the forecasts resulting from the non-aggregate demand. From Hosoda and Disney (2006), the demand process faced by the manufacturer (i.e. orders made by the retailer to the manufacturer) is also an ARMA(1, 1) process with the parameters given as follows:

$$\begin{aligned} \phi_M &= \phi \\ \theta_M &= \frac{\left(\theta + \phi\left(\frac{1-\phi^l}{1-\phi}\right) - \theta\left(\frac{1-\phi^l}{1-\phi}\right)\right)}{\left(1 + \phi\left(\frac{1-\phi^l}{1-\phi}\right) - \theta\left(\frac{1-\phi^l}{1-\phi}\right)\right)} \quad , \\ {\sigma_M}^2 &= \left(1 + \phi\left(\frac{1-\phi^l}{1-\phi}\right) - \theta\left(\frac{1-\phi^l}{1-\phi}\right)\right)^2 \sigma^2 \end{aligned} \tag{35}$$

By considering an aggregation level m - equals to the lead-time l - the MSE of forecast at the manufacturer level can be calculated as follows:

$$MSE_{BA}^{M} = var\left(Forecast\ Error\right) = var\left(o_{t,l} - f_{t,l}^{M}\right) = var\left(o_{t,m} - f_{t,m}^{M}\right), \tag{36}$$

Similar to sub-section 4.1, the MSE of forecast over lead-time l, using non-aggregate data can be calculated as follows:

$$MSE_{BA}^{M} = \left(1 + \sum_{i=0}^{m-2}\left(1 + \sum_{j=0}^{m-i-2} {\phi_M}^j \left(\phi_M - \theta_M\right)\right)^2\right){\sigma_M}^2. \tag{37}$$

By substituting (35) into (37), the MSE of forecast at manufacturer level using the non-aggregate demand results as:

$$MSE_{BA}^{M} = \left(1 + \sum_{i=0}^{m-2}\left(1 + \sum_{j=0}^{m-i-2}\phi^j\left(\phi - \frac{\left(\theta + \phi\left(\frac{1-\phi^l}{1-\phi}\right) - \theta\left(\frac{1-\phi^l}{1-\phi}\right)\right)}{\left(1 + \phi\left(\frac{1-\phi^l}{1-\phi}\right) - \theta\left(\frac{1-\phi^l}{1-\phi}\right)\right)}\right)\right)^2\right)\left(1 + \phi\left(\frac{1-\phi^l}{1-\phi}\right) - \theta\left(\frac{1-\phi^l}{1-\phi}\right)\right)^2 \sigma^2. \tag{38}$$

We now proceed with the derivation of the MSE of the aggregate forecasts made by the manufacturer when aggregated demand data are used.

$$MSE_{AA}^{M} = var\left(O_T - F_{T,1}^{M}\right) = var\left(O_{T,1}\right) + var\left(F_{T,1}^{M}\right) - 2cov\left(O_{T,1}, F_{T,1}^{M}\right), \tag{39}$$

Similar to the case of the non-aggregate case, the relationship between the aggregated process parameters at retailer and manufacturer levels can be obtained (Hosoda & Disney, 2006):

$$\begin{aligned} \phi'_M &= \phi' \\ \theta'_M &= \frac{\phi'}{1 + \phi' - \theta'} \quad , \\ {\sigma'_M}^2 &= \left(1 + \phi' - \theta'\right)^2 \sigma'^2 \end{aligned} \tag{40}$$

Because at the manufacturer level, O_T follows an ARMA(1, 1) process, we follow the same procedure as described in Section 4.2 to derive the MSE using the aggregated demand at the manufacturer level. This results in:

$$MSE_{AA}^{M} = var\left(\varepsilon'_M(T)\right) = {\sigma'_M}^2, \tag{41}$$

By substituting (40) into (41) we have:

$$MSE^{M}_{AA} = var\left(\varepsilon'_{M}(T)\right) = \left(1+\phi'-\theta'\right)^{2}\sigma'^{2}, \tag{42}$$

By substituting (23) into (42) we have:

$$MSE^{M}_{AA} = \left(1+\phi'-\theta'\right)^{2}\left(\frac{\left(1-\phi^{2m}\right)\left(m\gamma_0+\gamma_1\sum_{k=1}^{m-1}2\left(m-k\right)\phi^{k-1}\right)}{\left(\left(1-2\phi^{m}\theta'+\theta'^{2}\right)\right)}\right), \tag{43}$$

Finally, by substituting (20) into (43) and simplifying, we get:

$$MSE^{M}_{AA} = \begin{cases} \left(1+\phi^{m}-\theta'_1\right)^{2}\left(\dfrac{\left(1-\phi^{2m}\right)\left(m\gamma_0+\gamma_1\sum_{k=1}^{m-1}2\left(m-k\right)\phi^{k-1}\right)}{\left(\left(1-2\phi^{m}\theta'_1+\theta'^{2}_1\right)\right)}\right) & \phi>\theta \\ \left(1+\phi^{m}-\theta'_2\right)^{2}\left(\dfrac{\left(1-\phi^{2m}\right)\left(m\gamma_0+\gamma_1\sum_{k=1}^{m-1}2\left(m-k\right)\phi^{k-1}\right)}{\left(\left(1-2\phi^{m}\theta'_1+\theta'^{2}_1\right)\right)}\right) & \phi<\theta \end{cases}, \tag{44}$$

where θ'_1 and θ'_2 are defined in (25) and (26) respectively.

6.1. Numerical Results

In this section, the effect of temporal aggregation on the forecast accuracy at the manufacturer level is evaluated when an MMSE forecasting method is used. We numerically analyse the ratio $MSE^{M}_{AA}/MSE^{M}_{BA}$ for different values of the lead-time/aggregation level and the process parameters ϕ,θ.

Figure 5 presents the impact of the process parameters, ϕ *and* θ, and the aggregation level, *m* on the ratio of $MSE^{M}_{AA}/MSE^{M}_{BA}$. We are interested to determine under which conditions the ratio of MSE after aggregation to MSE before aggregation is less than one, meaning that TA improves the forecast accuracy at the manufacturer level. If the ratio is greater than one, then TA does not improve the forecast accuracy.

The results presented in Figure 5 show that TA generally improves the forecast accuracy at the manufacturer level and the TA approach outperforms the non-aggregation approach.

This is an interesting result, as TA does not improve the forecast accuracy at the retailer level. However, improvement can be achieved at the manufacturer level by using the aggregated demand and the rate of improvement can be as high as 90% for long lead times/forecast horizons.

Figure 5 (graph at the top) shows that when the autoregressive parameter ϕ is greater than the moving average parameter θ, TA improves forecast accuracy at the manufacturer level for all lead times. The rate of improvement varies from 50% to 90% depending on the value of the lead time/aggregation level. The manufacturer can benefit more from TA when dealing with longer lead-times.

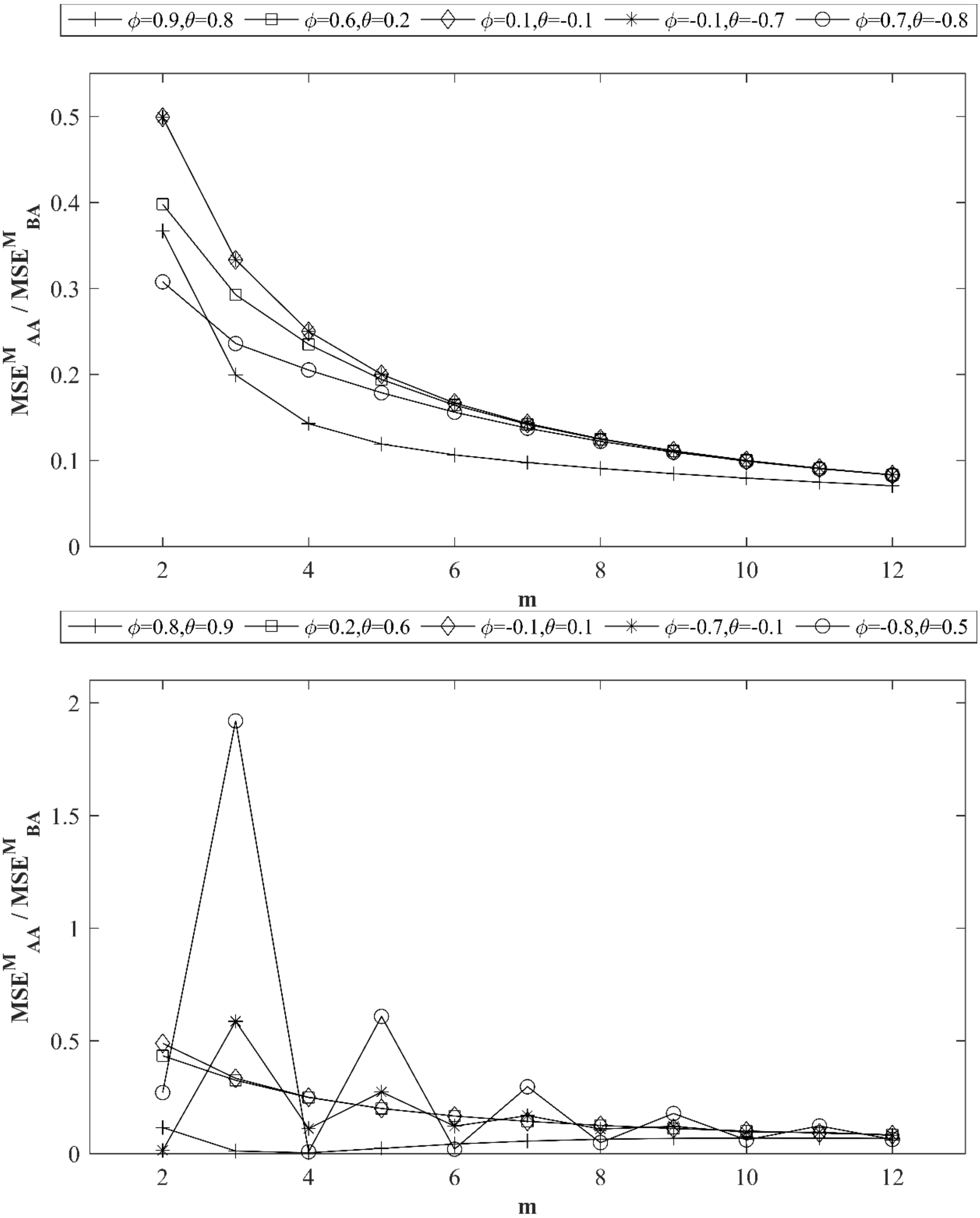

Figure 5: Ratio of MSEs after aggregation to before aggregation at the manufacturer level, $\phi>\theta$ (top) and $\phi<\theta$ (bottom)

On the other hand, when the autoregressive parameter is smaller than the moving average parameter, the ratio is less than one for most of the cases except for the few instances where the autoregressive parameter is highly negative and the moving average parameter is positive and only for lower odd values of *m as* shown in the Figure 5 (graph at the bottom).

Our results in this section show that the manufacturer can achieve up to 90% forecast accuracy improvement when the demand TA approach is used. Additionally, when the forecast horizon/lead time is longer, the forecast improvement gain for the manufacturer is substantial, regardless of the process parameter. In Section 5, we show that the bullwhip effect is reduced when the supply chain uses TA. As expected this reduction of the bullwhip effect in the supply chain results in an improved forecast for the manufacturer.

We have also conducted a simulation study based on the work of Graves(1999), when the retailer faces a non-stationary ARIMA(0,1,1) demand process (with a moving average parameter θ where $0 < \theta = 1-\alpha < 1$) for which the SES method is optimal. All the other settings are kept the same as in the case of the ARMA(1,1) process. The conclusion is similar to that of ARMA(1, 1), i.e. the ratio of MSE after aggregation into before aggregation at retailer is always higher than one which means aggregation does not improve the forecast accuracy at the retailer, whereas the bullwhip effect reduces when using temporally aggregated demand process especially when the aggregation level increases.

7. Conclusions

Previous studies have considered the effect of temporal aggregation on forecast accuracy using non-optimal forecasting methods. We extend this area of research by looking at the impact of TA using an optimal forecasting method. We also broaden the scope of the research by evaluating the effect of TA on the upstream supply chain link (via the forecast accuracy at the manufacturer) and on the overall dynamics of the supply chain (via the bullwhip effect).

In this paper, we have considered a two-stage supply chain and we have analytically evaluated the impact of temporal aggregation approach on the forecasting performance and the bullwhip when non-aggregate series follows an autoregressive moving average process of order (1, 1), [ARMA(1, 1)]. Forecasting is assumed to be relying upon an optimal MMSE procedure and the analytical results were complemented with a simulation experiment on artificial data. Analytical developments are based on the consideration of the Mean Squared Error and comparisons are undertaken when forecasting over lead-time is considered.

The main findings can be summarised as follows:

1. We found that at the retailer level combining two well-proven approaches does not necessarily lead to an improvement in the performance. We show that by combining an optimal forecasting method and temporal aggregation, the forecast accuracy at the retailer level may not be improved. However, there is a reduction of the bullwhip effect and an improvement of the forecast accuracy at the manufacturer level.
2. In terms of the reduction of forecast accuracy at the retailer level, the temporal aggregation approach is outperformed by the non-aggregation approach. By applying temporal aggregation, the uncertainty of the process is increased which results in the increase in the forecast error at the aggregated level. This result is valid in any time series forecasting context. If the series follows an ARMA(1,1) process and the MMSE forecasting method is used, it is not recommended to use temporal aggregation to produce cumulative forecast over the whole horizons.
3. We observe that when there is bullwhip effect in the original series, it still exists with the aggregated series. The range (the difference between the lowest and the highest value) of the bullwhip effect for the aggregated series is 0.9 while this is 9.9 for the non-aggregate series for the lead-time between 2 and 12 and the whole range of process parameters which were investigated.
4. We show that temporal aggregation reduces the bullwhip effect when it exists using the original series. The reduction rate varies depending on the characteristics of the demand series and the lead-time (set to be equal to the aggregation level). The reduction can reach up to 84% for a highly positive autocorrelation and long lead times. This is a very important insight for practitioners since managers know that by using temporally aggregated series to calculate the OUT level and the order to the upper stage, the bullwhip can be reduced. This is an important finding as previous literature has shown benefits to supply chain links to the reduction of the bullwhip effect (Hosoda et al., 2006).

5. We find that the temporal aggregation approach will result in improving the forecast accuracy at the manufacturer level. The accuracy improvements can be as high as 90% for longer lead-times. This is also an important result for the upstream supply chain members to devise strategies for improvements of their forecasting process.
6. Although our research shows no benefits of using temporal aggregation approach at the retailer level based on forecast accuracy, the reduction of bullwhip effect and an increase of the forecast accuracy have been observed at the upstream level.

It is important to note that, as the relationship between the forecast accuracy and the utility measures, especially in an inventory management setting, is not straightforward (Ali et al., 2011), future extensions of this work should evaluate the effect of the aggregation approach on utility measures such as inventory, production costs and service levels. Additionally, the analytical work discussed in this paper can be extended to consider higher order stationary demand processes and more importantly non-stationary demand processes, since this is a very important issue both from an academic and practitioner perspective. Another avenue for further research would be to evaluate the approaches considered in this paper at different levels of aggregation, and hierarchical approaches, when the demand process and its parameters are not known. A further useful investigation would be to conduct an empirical analysis for the supply chain model discussed in this paper.

Appendix A

By dividing (18) into (19) and then substituting (17) into it, we have:

$$\frac{\dfrac{\left(1-2\phi'\theta'+\theta'^2\right)}{1-\phi'^2}\sigma'^2}{\dfrac{(\phi'-\theta')(1-\phi'\theta')}{1-\phi'^2}\sigma'^2}=\frac{m\gamma_0+\gamma_1\left(\sum_{k=1}^{m-1}2(m-k)\phi^{k-1}\right)}{\gamma_1\left(\sum_{k=1}^{m}k\phi^{k-1}+\sum_{k=2}^{m}(k-1)\phi^{2m-k}\right)} \tag{A1}$$

Assuming

$$\mathrm{X}=\frac{m\gamma_0+\gamma_1\left(\sum_{k=1}^{m-1}2(m-k)\phi^{k-1}\right)}{\gamma_1\left(\sum_{k=1}^{m}k\phi^{k-1}+\sum_{k=2}^{m}(k-1)\phi^{2m-k}\right)} \tag{A2}$$

We get a quadratic equation

$$\mathrm{X}=\frac{\left(1-2\phi'\theta'+\theta'^2\right)}{(\phi'-\theta')(1-\phi'\theta')} \tag{A3}$$

$$(1-\phi'\mathrm{X})\theta'^2+\left(\mathrm{X}+\phi'^2\mathrm{X}-2\phi'\right)\theta'+(1-\phi'\mathrm{X})=0 \tag{A4}$$

By solving (A4), we obtain θ_1' *and* θ_2'.

$$\Delta=\left(\mathrm{X}+\phi'^2\mathrm{X}-2\phi'\right)^2-4(1-\phi'\mathrm{X})^2 \tag{A5}$$

$$\theta'_{1,2} = \frac{-\left(X + \phi'^2 X - 2\phi'\right) \pm \sqrt{\Delta}}{2 * \left(1 - \phi' X\right)} \quad \text{(A6)}$$

$$\theta'_1 = \frac{-\left(X + \phi'^2 X - 2\phi'\right) + \sqrt{\Delta}}{2\left(1 - \phi' X\right)}, \quad \phi > \theta \quad \text{(A7)}$$

$$\theta'_2 = \frac{-\left(X + \phi'^2 X - 2\phi'\right) - \sqrt{\Delta}}{2\left(1 - \phi' X\right)}, \quad \phi < \theta \quad \text{(A8)}$$

By substituting (A5) and (20) into (A7) and (A8), we get:

$$\theta'_1 = \frac{-\left(X + \phi^{2m} X - 2\phi^m\right) + \sqrt{\left(X + \phi^{2m} X - 2\phi^m\right)^2 - 4\left(1 - \phi^m X\right)^2}}{2\left(1 - \phi^m X\right)}, \quad \phi > \theta \quad \text{(A9)}$$

$$\theta'_2 = \frac{-\left(X + \phi^{2m} X - 2\phi^m\right) - \sqrt{\left(X + \phi^{2m} X - 2\phi^m\right)^2 - 4\left(1 - \phi^m X\right)^2}}{2\left(1 - \phi^m X\right)}, \quad \phi < \theta \quad \text{(A10)}$$

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