Gaalman, G., Disney, S.M., and Wang, X., (2018), "Bullwhip behaviour as a function of the lead-time for the order-up-to policy under ARMA demand", Pre-prints of the 20th International Working Seminar of Production Economics, Innsbruck, Austria, 19th–23rd February, Vol. 2, pp249–260.

Bullwhip Behaviour as a Function of the Lead-Time for the Order-Up-To Policy Under ARMA Demand

Gerard Gaalman[1], Stephen M. Disney[2], and Xun Wang[2]

[1] Department of Operations, Faculty of Economics and Business, University of Groningen, P.O. Box 800, 9700 AV Groningen, The Netherlands. g.j.c.gaalman@rug.nl
[2] Logistics Systems Dynamics Group, Cardiff Business School, Aberconway Building, Colum Drive, Cardiff, CF10 3EU, United Kingdom. DisneySM@cardiff.ac.uk, WangX46@cardiff.ac.uk

Abstract

We consider how lead-times create the bullwhip effect in an inventory replenishment system. The inventory system is a combination of a specific demand process, a forecasting method, and a replenishment policy. Previous studies have evaluated the bullwhip effect when some of the system parameters such the forecasting model, demand process, or the lead-time are changed. This type of analyses has proved practically valuable, indicating which improvement measures can be taken. However, the specific causes of a bullwhip behavior are often difficult to grasp. It is often assumed that longer lead times lead to more bullwhip; herein we show this is not always so. We study the order-up-to (OUT) replenishment policy, with general auto-regressive moving average (ARMA(p,q)) demand processes, conditionally expected forecasting, and general lead-times. Using the eigenvalues of the demand process we study the effect of eigenvalue ordering on the bullwhip metric. The positivity of the demand impulse response determines whether the bullwhip produced is increasing in the lead-time. We illustrate our results by studying the ARMA(2,2) demand process.

Keywords: Bullwhip effect, order-up-to policy, ARMA(p,q) demand, lead times, eigenvalues.

1. Introduction

The bullwhip effect produced by supply chain replenishment policies has been studied extensively. Usually, a combination of a specific lead-time, demand process, forecasting method, and a replenishment policy, is selected and the bullwhip effect is measured by means of a metric. Parameters of the demand process, the forecasting method, the replenishment policy, and the lead-time are varied, and the bullwhip effect is found to exist or not. Sometimes, discrete modeling choices such as the forecasting method are made. At other times, modeling choices are made that restrict more "fluid" variables such as the lead-time and demand process structure.

Often the AR(1) demand process is considered in a bullwhip study; see Zhang (2004), Urban (2005), or Luong (2007). The effect of ARMA(1,1) processes was studied by Chen and Disney (2003), Gaalman and Disney (2006), Duc et al., (2008). Gaalman and Disney (2006) studied the ARMA(2,2) process, while Luong and Phien (2007) consider the AR(2). Gilbert (2005) considered how the general ARIMA(p,d,q) demand evolves as it is passed echelon-to-echelon up a supply chain. Together with the ARIMA demand model, forecasts based on the conditional expectation (a.k.a. Minimum Mean Squared Error, MMSE) of demand are often used. Alwan et al., (2003) argue that MMSE should be used when ARIMA demand processes are present. Less frequently, empirically popular methods such as exponential smoothing and moving average are studied (see Chen et al., 2000; Dejonckheere et al., 2003; Zhang, 2004).

The most often used replenishment policy is the linear order-up-to (OUT) policy, Lee et al., (2000). However, several variations and generalization have also been considered. Common variations include the proportional OUT policy, (Deziel and Eilon, 1967; Dejonckheere et al., 2003) and the full-state OUT policy (Gaalman, 2006; Gaalman and Disney, 2009).

Lead-times are frequently a restricted set of numerical values. By far the most common bullwhip metric is the ratio of the order and demand variance. Gilbert (2002) used the ratio of the order and demand forecast error. Gaalman and Disney (2007, 2012) used the difference between the order and demand variances.

These bullwhip analyses are valuable as they indicate which improvement measures can be taken (such as using a different replenishment policy, or information sharing between stages in the supply chain). However, deep insights into the precise cause of the bullwhip effect are frequently lacking. For example: Under what conditions is the bullwhip effect an increasing function of the lead-time? What causes the bullwhip to be present with a short lead-time, but to disappear when a longer lead-time is present? Why does the bullwhip sometimes alternate over the lead-time? General statements about the interaction between the bullwhip effect and the lead-time are missing. Probably the only exception is the work of Dejonckheere et al., (2003) who state that when using exponential smoothing or moving average forecasts within the OUT policy, no matter what the lead-time, or the demand process, bullwhip is always generated.

This research aims to better understand the link between bullwhip and the lead-time by considering the eigenvalues of the AR and MA components of the ARMA(*p*,*q*) demand process. An OUT policy with MMSE forecasts is used to represent the replenishment policy. We first introduce some general statements that hold for all demand processes and all lead-times, then we focus specifically on the hitherto unstudied ARMA(2,2) process.

In §2 we present our basic modeling set-up: the demand process, the OUT replenishment policy, its impulse response, and eigenvalue representation. In §3, we present conditions under which bullwhip is an increasing function of the lead-time. §4 considers the special case of ARMA(2,2) demand. §5 concludes. Appendix A contains the transformation used to obtain our eigenvalue state space representation of the system. Appendix B contains the proof of Theorem 2.

2. Modelling Assumptions

2.1. The Demand Process

We assume that the demand process follows an ARMA(p,q) process, Box et al., (1994),

$$d_{t+1} = \bar{d} + z_{t+1};\; z_{t+1} - \sum_{i=1}^{p} \phi_i z_{t+1-i} = \eta_{t+1} - \sum_{i=1}^{q} \theta_i \eta_{t+1-i}\,. \tag{1}$$

Let $m = \max\left(p,q\right)$, then

$$z_{t+1} = \sum_{i=1}^{m} \left(\phi_i z_{t+1-i} - \theta_i \eta_{t+1-i}\right) + \eta_{t+1} \tag{2}$$

where, if $p > q$ then $\forall i > q,\ \theta_i = 0$ else $\forall i > p,\ \phi_i = 0$. Note, this formulation allows for all subsets of the ARMA(*p*,*q*) family of models. Gaalman (2006) shows that the ARMA demand process in (1) can be converted into a state space representation, characterized an $m \times m$ companion matrix D_ϕ, a unit row vector M of length *m*, and a column vector G, also of length *m*. The pair (D_ϕ, M) has the following observable canonical form (Kailath, 1980).

$$d_{t+1} = \bar{d} + z_{t+1};\; z_{t+1} = My_{t+1} + \eta_{t+1},\; y_{t+1} = D_\phi y_t + G\eta_t, \tag{3}$$

$$D_\phi = \begin{pmatrix} \phi_1 & 1 & 0 & 0 \\ \phi_2 & 0 & \ddots & 0 \\ \vdots & 0 & \ddots & 1 \\ \phi_m & 0 & 0 & 0 \end{pmatrix},\; G = \begin{pmatrix} \phi_1 - \theta_1 \\ \phi_2 - \theta_2 \\ \vdots \\ \phi_m - \theta_m \end{pmatrix}, \text{ and } M = \begin{pmatrix} 1 & 0 & \dots & 0 \end{pmatrix}. \tag{4}$$

The impulse response of the demand process allows us to determine if bullwhip is present. Let p_t, $t \in \mathbb{N}_0$ denote the impulse response of the demand process which can be calculated from the state space form in the following manner:

- At $t=0$ the (input) impulse $\eta_0 = 1$ enters an empty process, i.e. $y_0 = 0$. From (4) the output will be $z_0 = My_0 + \eta_0 = 1$, so the impulse response is $p_0 = 1$. Note, for all ARMA processes $p_0 = \eta_0 = 1$.
- At $t=1$, we have $y_1 = D_\phi y_0 + G\eta_0 = G$ and the impulse $z_1 = p_1 = My_1 + \eta_1 = MG$.
- At $t=2$: $y_2 = D_\phi y_1 + G\eta_1 = D_\phi G$ and $z_2 = p_2 = My_2 + \eta_2 = MD_\phi G$.
- For all future $t = 3, 4, \ldots$, $y_t = D_\phi^{t-1} G$, $z_t = p_t = MD_\phi^{t-1} G$. Here D_ϕ^{t-1} is a multiplication of D_ϕ to the power of $t-1$.

2.2. The Inventory Balance Equations and Sequence of Events

We consider a discrete time system. At the beginning of time period t + 1, the state of the system is observed: demand during period t is tallied, forecasts for future demands are generated, inventory and WIP levels are determined, and replenishment orders are calculated. Immediately after the order is generated, the order placed at in period $t-k$ arrives. k is the lead-time, when k = 0 the order placed is received within the same period it was placed. Eq. (5) describes the inventory balance equation;

$$\begin{aligned} i_{t+1+k} &= i_{t+k} + o_t - d_{t+1+k} \\ i_{t+1} &= i_t + o_{t-k} - d_{t+1}. \end{aligned} \tag{5}$$

2.3. The OUT Replenishment Policy

The linear approximation to the OUT policy is represented by the following set of difference equations (Gaalman and Disney 2009).

$$\hat{i}_{t+k+1,t} = 0;\ o_t = \hat{d}_{t+1+k,t} - \hat{i}_{t+k,t} = \hat{d}_{t+1+k,t} + E(k)\eta_t \tag{6}$$

The conditional expected future demand forecasts, $\hat{d}_{t+1+k,t}$, can be obtained using a Kalman filter approach. In the stationary situation (assuming that an infinite number of past demand values are present) the one period ahead forecast error of an ARMA process is equal to $(z_t - \hat{z}_{t,t-1}) = \eta_t$. In the stationary situation, the following relation holds,

$$\hat{d}_{t+1,t} = \overline{d} + \hat{z}_{t+1,t}, \tag{7}$$

where

$$\hat{z}_{t+1,t} = M\hat{y}_{t+1,t};\ \hat{y}_{t+1,t} = D_\phi \hat{y}_{t,t-1} + G(z_t - \hat{z}_{t,t-1}) \text{ and } \hat{z}_{t+k+1,t} = MD_\phi^k \hat{y}_{t+1,t}. \tag{8}$$

At time t, the demand z_t and the one period forecast error $(z_t - \hat{z}_{t,t-1}) = \eta_t$ are known, resulting in $y_t = \hat{y}_{t,t-1}$. Gaalman and Disney (2009) show that the k periods ahead forecasted inventory, $\hat{i}_{t+k} = -E(k)\eta_t$, where the function $E(k) = 1 + M\sum_{j=0}^{k-1}(D^j)G,\ E(0) = 1$. From the impulse response discussed earlier, we can also express $E(k)$ as a sum of the demand impulse response, $E(k) = p_0 + \sum_{j=0}^{k-1} p_{j+1} = \sum_{j=0}^{k} p_j$.

Later we will require the impulse response of the orders. This can be obtained from (8):

- At $t=0$ the (input) impulse $p_0=\eta_0=1$ enters the empty process. From (6) the order will be $o_0=\hat{z}_{1+k,0}+E(k)\eta_0$. For all ARMA processes, $\eta_0=1$. From (8), $\hat{z}_{k+1,0}=MD_\phi^k\hat{y}_{1,0}=MD_\phi^k y_1=MD_\phi^k G=p_{k+1}$. Here the last two relations come from the demand impulse response. This leads to $o_0=p_{k+1}+E(k)=\sum_{j=0}^{k+1}p_j$.
- Then $\forall t\geq 1$, the order impulse response follows $o_t=MD_\phi^{k+t}G=p_{k+t+1}$.

2.4. *Bullwhip Criterion and the Impulse Response*

A popular criterion to study the bullwhip effect of an inventory replenishment rule is the ratio of the ordering and demand variance. A bullwhip effect exists if the ratio

$$BI_1=\Sigma_{oo}/\Sigma_{dd}>1. \tag{9}$$

In (9), Σ_{dd} is the variance of the demand process, Σ_{oo} is the variance of the orders. These variances will only exist if the demand is stationary. In short-term demand forecasting, both the demand and order variance can become large (for instance, when the demand is highly correlated, (see Box et al., (1994) and Brown (1963)). As a consequence, $BI_1\to 1$, suggesting the demand and order variance are equal, and bullwhip is absent. However, a difference between demand and order variance may still exist (Gaalman and Disney, 2012)[1]. In this case, the bullwhip criterion, *CB*(*k*), often expresses the difference between these two variances better,

$$CB(k)=\left(\Sigma_{oo}-\Sigma_{dd}\right)/\Sigma_{\eta\eta}. \tag{10}$$

If $CB(k)>0$ we have a bullwhip effect. Note $BI_1=1+CB(k)/(\Sigma_{dd}/\Sigma_{\eta\eta})$. Expressions for the demand and order variances are needed to investigate $CB(k)$. A state space approach provides the variance and covariance expressions of the state variables, Gaalman and Disney (2009); however, as we are only interested in the demand and order variances, a more direct route is to apply Tsypkin's squared impulse response theorem. Using Tsypkin's relation, the demand variance is $\Sigma_{dd}/\Sigma_{\eta\eta}=\sum_{t=0}^{\infty}\left(p_t\right)^2$ (Li et al., 2014). For the order variance, $\Sigma_{oo}/\Sigma_{\eta\eta}=\left(\sum_{j=0}^{k+1}(p_j)\right)^2+\sum_{t=1}^{\infty}\left(p_{t+k+1}\right)^2$ and $CB(k)$ becomes

$$CB(k)=\left(\left(\sum_{j=0}^{k+1}\left(p_j\right)\right)^2+\sum_{t=1}^{\infty}\left(p_{t+k+1}\right)^2\right)-\sum_{t=0}^{\infty}\left(p_t\right)^2=2\sum_{j=0}^{k}\left(p_{j+1}\sum_{i=0}^{j}p_i\right). \tag{11}$$

Remark 1. $CB(0)=2p_1=2(\phi_1-\theta_1)$ for all ARMA processes.
Remark 2. $CB(k)-CB(k-1)=\ 2p_{k+1}\sum_{j=0}^{k}p_j$.

Theorem 1. Iff $\{p_1,p_2,...,p_{k+1}\}>0$ *then* $CB(k)$ *is positive and increasing in the lead-time.*

[1] Furthermore, under non-stationary demand with infinite variance, the order variance is also infinite, but it is possible the infinite order variance is smaller (or larger) than the infinite demand variance.

Proof. It follows immediately from the above remarks that $CB(k)-CB(k-1)>0$ if $\left\{p_{k+1}, \sum_{j=0}^{k} p_j\right\}>0$. ■

Remark 1. Only the first k + 1 ARMA coefficients influence the consequences of Theorem 1.

Corollary 1: *For MA(q) demand processes,* $p_k \geq 0, \forall k$ *if* $\forall i,\ \theta_i < 0$, *indicating that bullwhip always exists and increases in the lead-time for such MA processes.* ■

Corollary 2: For AR(p) demand processes, $p_k \geq 0, \forall k$ *if* $\forall i,\ \phi_i > 0$, *indicating that bullwhip always exists and increases in the lead-time for such AR processes.* ■

Corollary 3: For ARMA(1,1) demand processes, $p_k \geq 0, \forall k$ *iff* $\phi > 0$ *and* $\theta < \phi$ *indicating that bullwhip always exists and increases in the lead-time for such ARMA(1,1) processes.* ■

The proof of Corollary 1, 2 and 3 follows by simple inspection of the demand impulse response.

2.5. A Novel Eigenvalue Representation

Given the importance of the demand impulse response on bullwhip behavior, one wonders how the demand parameters influence the impulse response. In general, this is rather complex, but by investigating the eigenvalues of the AR and MA components of the demand process, additional insights can be obtained. We start by transforming our demand impulse/state space model (as shown in Appendix A) into the following eigenvalue representation,

$$p_{k+1} = M(D_\phi^k)G = \sum_{i=1}^{m} r_i\ (\lambda_i^\phi)^k\ ,\ r_i = \frac{\Pi_{l=1}^{m}\left(\lambda_i^\phi - \lambda_l^\theta\right)}{\Pi_{\{l=1, l\neq i\}}^{m}\left(\lambda_i^\phi - \lambda_l^\phi\right)}. \quad (12)$$

Here $\{\lambda_i^\phi,\ \lambda_l^\theta\}$ are the eigenvalues of the AR and MA components of the demand process respectively. Note, we assume a stable demand process with only positive and real AR eigenvalues and real (possibly negative) MA eigenvalues exists. When we have negative AR eigenvalues the demand alternates between positive and negative values, creating complex bullwhip behavior that does not conform to the conditions in Theorem 1. Without any consequences, we assume $0 \leq \lambda_1^\phi < \ldots \lambda_{m-1}^\phi < \lambda_m^\phi < 1$ and $0 \leq \lambda_1^\theta < \ldots \lambda_{m-1}^\theta < \lambda_m^\theta < 1$.

Remark 1. $\phi_1 = \sum_{i=1}^{m} \lambda_i^\phi$ and $\phi_m = (-1)^{m-1} \prod_{j=1}^{m} \lambda_j^\phi$. The other coefficients have more a complex structure (see Kailath, 1980). Similar expressions hold for the MA eigenvalues.

Remark 2. If $p > q$ then $p_{k+1} = \sum_{i=1}^{p} r_i\ (\lambda_i^\phi)^k$ and $p-q$ eigenvalues $\lambda_j^\theta = 0$. Note: the AR(p) process has $q = p$ zero eigenvalues λ_j^θ. This means that properties of the AR(p) case can be found from the ARMA(p,p) case. If $q > p$, $q-p$ eigenvalues λ_j^ϕ are zero.

Remark 3. For all ARMA(p,q) processes, $p_1 = (\phi_1 - \theta_1) = \sum_{i=1}^{m} r_i = \sum_{i=1}^{m} (\lambda_i^\phi - \lambda_i^\theta)$.

Remark 4. In a stable system, the impulse response for large lead-times goes to zero. A positive impulse response must have $p_{k\to\infty} \to r_m (\lambda_m^\phi)^{k\to\infty} \to 0^+$, indicating that $r_m > 0$.

Remark 5. The impulse response, p_{k+1}, as a function of the lead time has at most $m-1$ changes of sign. The signs of r_i's can be positive or negative.

Remark 6. The summation of the impulse response $E(k)$ satisfies

$$E(0)=0,\ E(k)=1+\sum\nolimits_{l=1}^{m} r_l\left(\frac{1-(\lambda_l^{\phi})^k}{(1-\lambda_l^{\phi})}\right),\text{ and } E(\infty)=1+\sum\nolimits_{l=1}^{m}\frac{r_l}{(1-\lambda_l^{\phi})}=\frac{\prod_{j=1}^{m}(1-\lambda_j^{\theta})}{\prod_{j=1}^{m}(1-\lambda_j^{\phi})}>0. \quad (13)$$

Remark 7. Due to its complexity, the analytical expression of $CB(k)$ is not provided. Stability conditions guarantee the existence of $CB(\infty)$, which can be positive or negative.

3. When is Bullwhip an Increasing Function of the Lead-Time?

The AR and MA eigenvalue values determine the bullwhip effect behavior. Given our assumptions, there are $(2m)!(m!)^{-2}$ possible orderings. The behavior of some orderings is obvious:

- If $m=1$ two eigenvalue orderings are possible. For $0<\lambda_1^{\theta}<\lambda_1^{\phi}<1$, the impulse response is positive (for the inverse, $0<\lambda_1^{\phi}<\lambda_1^{\theta}<1$, the impulse is negative).
- If $\forall i,\ r_i>0$, then $p_{k+1}>0$ holds for the (unique) ordering $0\le\lambda_1^{\theta}<\lambda_1^{\phi}<\ldots,\lambda_{m-1}^{\theta}<\lambda_{m-1}^{\phi}<\lambda_m^{\theta}<\lambda_m^{\phi}<1$.
- If $\forall i,\ r_i<0$, then $0\le\lambda_1^{\phi}<\lambda_1^{\theta}<,\ldots,\lambda_{m-1}^{\phi}<\lambda_{m-1}^{\theta}<\lambda_m^{\phi}<\lambda_m^{\theta}<1$, $p_{k+1}<0$, $E(k)>0$, and $CB(k)$ is decreasing in the lead time.
- The set of orderings with one r_i change of sign with $r_i>0$ for $i=m,m-1,..,l+1$, $r_i<0$ for $i=l,l-1,...,1$ and the necessary condition $p_1>0$ ensures $p_{k+1}>0$. The inverse ordering leads to $p_{k+1}<0$.
- Orderings where the sign of r_i changes more than once can also have $p_{k+1}>0$.

The following Theorem provides a sufficient condition for a more general set of orderings, including above orderings.

Theorem 2. The impulse response is positive if, for each AR eigenvalue λ_l^{ϕ}, the number of MA eigenvalues smaller than λ_l^{ϕ} is larger than the number of AR eigenvalues smaller then λ_l^{ϕ}.
Proof. The proof, housed in Appendix B, is based on the z-transform function of the demand process (23). The transfer function consists of the product of *m* elementary transform functions each having a positive impulse response. Convolution then guarantees the demand process also has a positive impulse response. ■

Theorem 2 shows the dominance of the λ_l^{ϕ} eigenvalues over the λ_l^{θ} eigenvalues. It is an attractive property because it depends purely on the eigenvalue ordering itself rather than the specific value of the eigenvalues. In control theoretical terms, the dominance of the λ_l^{ϕ} eigenvalues over the λ_l^{θ} eigenvalues means that the demand process behaves as a *low pass filter*; the higher frequencies contribute less to the demand process. The OUT policy is less able to filter low frequencies and by this the order variance increases and bullwhip increases over the lead-time exists. The statement is proved for $m=1,2,3$ and partly for larger m's.

The *inverse* of Theorem 2 is: If for each λ_l^θ the number of AR eigenvalues smaller than λ_l^θ is larger than the number of MA eigenvalues smaller than λ_l^θ then the impulse response is not always positive and increasing bullwhip over the lead time is not present. Here the λ_l^θ eigenvalues dominate and the demand process exhibits large high-frequency harmonics. The proof is trivial as always $p_1 < 0$.

From these insights, the whole set of $(2m)!(m!)^{-2}$ orderings can be split into 3 subsets: one set of orderings which satisfy Theorem 2, one set of orderings that satisfies the inverse of Theorem 2, and the remaining orderings. The remaining subset contains orderings where r_m and/or p_1 are positive or negative. Moreover, depending on the specific values of the eigenvalues some orderings may still show increasing bullwhip behavior. In the next section, we consider the $m = 2$ case.

4. Bullwhip Behaviour Over the Lead-Time Under ARMA(2,2) Demand

When the ARMA(2,2) demand is present, the OUT policy impulse response is given by

$$p_{k+1} = r_1(\lambda_1^\phi)^k + r_2(\lambda_2^\phi)^k,\; r_1 = \frac{(\lambda_1^\phi - \lambda_1^\theta)(\lambda_1^\phi - \lambda_2^\theta)}{(\lambda_1^\phi - \lambda_2^\phi)},\; r_2 = \frac{(\lambda_2^\phi - \lambda_1^\theta)(\lambda_2^\phi - \lambda_2^\theta)}{(\lambda_2^\phi - \lambda_1^\phi)}. \tag{14}$$

This follows from (13). There are 4!/(2!2!) = 6 possible eigenvalue orderings, see Fig. 1.

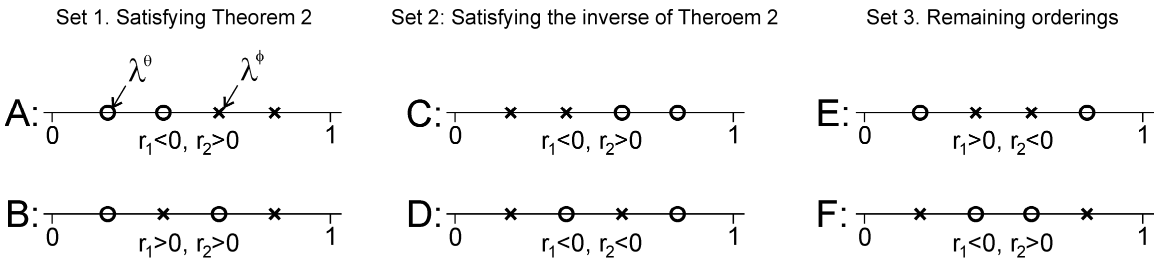


Figure 1. The six possible eigenvalue orderings when demand is ARMA(2,2)

We now consider each of the six possible eigenvalue ordering in turn:

Set 1. Satisfying Theorem 2

Case A: The r_i's change of sign once $(r_1 < 0,\, r_2 > 0)$ and $p_1 > 0$. Substituting $r_1 = p_1 - r_2$ into the impulse response gives

$$p_{k+1} = (p_1 - r_2)(\lambda_1^\phi)^k + r_2(\lambda_2^\phi)^k = p_1(\lambda_1^\phi)^k + r_2[(\lambda_2^\phi)^k - (\lambda_1^\phi)^k] > 0. \tag{15}$$

which shows that bullwhip increases in the lead-time.

Case B: Both $r_2 > 0,\, r_1 > 0$ so $p_{k+1} = \sum_{i=1}^{2} r_i(\lambda_i^\phi)^k > 0$, which shows that bullwhip is increasing in the lead-time. The demand behaves as a low pass filter in both case A and case B, but the filtering "strength" is much lower in case B; bullwhip is reduced as a consequence.

Set 2. Satisfying the inverse of Theorem 2

Case C: In this case $r_1 < 0,\ r_2 > 0$, and $-2 < p_1 < 0$. Since $r_2 > 0$ then $p_{k\to\infty} \to 0^+$. The impulse response is initially negative and becomes positive with a (single) maximum. If $-1 < p_1 < 0$ $E(k) > 0$ has a (single) minimum, $CB(k)$ initially decreases, increases, and then decreases. For $-2 < p_1 < -1$ $E(k)$ changes two times of sign +,-,+. The combination of p_1 and $E(k)$ changes of signs leads to negative $CB(k) - CB(k-1) = 2p_{k+1}E(k)$ and a decreasing $CB(k)$.

Case D: In this case $r_1 < 0,\ r_2 < 0$, so $p_{k+1} < 0$ and $E(k) > 0$ decreases monotonically to $E(\infty) > 0$. As a consequence $CB(k)$ is negative and decreases monotonically; bullwhip is not present, and the order variance decreases in the lead-time.

Set 3. Other orderings

Case E: Here $r_1 > 0,\ r_2 < 0$, $-1 < p_1 = \sum_{i=1}^{2}(\lambda_i^\phi - \lambda_i^\theta) < 1$ can be positive or negative. Since $r_2 < 0$ the impulse response $p_{k\to\infty} \to 0^-$ implying $p_{k+1} > 0, \forall k$ is not possible. If $p_1 < 0$ we can make the substitution $r_1 = p_1 - r_2$ to find

$$p_{k+1} = (p_1 - r_2)(\lambda_1^\phi)^k + r_2(\lambda_2^\phi)^k = p_1(\lambda_1^\phi)^k + r_2[(\lambda_2^\phi)^k - (\lambda_1^\phi)^k] < 0. \quad (16)$$

Since $p_{k+1} < 0,\ \forall k$ $E(k)$ decreases monotonically and converges to $E(\infty) > 0$. $CB(k)$ is negative and decreases monotonically. If $p_1 > 0$ the impulse response changes sign. $E(k)$ increases to a single maximum, decreases, and converges to $E(\infty) > 0$. $CB(k)$ initially increases and then decreases.

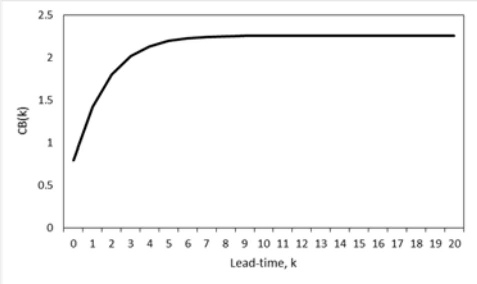
Case F: Here $r_1 < 0,\ r_2 > 0$, and $-1 < p_1 = \sum_{i=1}^{2}(\lambda_i^\phi - \lambda_i^\theta) < 1$. Since $r_2 > 0$ the impulse response $p_{k\to\infty} \to 0^+$. If $p_1 > 0$ we substitute $r_1 = p_1 - r_2$ to find

$$p_{k+1} = (p_1 - r_2)(\lambda_1^\phi)^k + r_2(\lambda_2^\phi)^k = p_1(\lambda_1^\phi)^k + r_2[(\lambda_2^\phi)^k - (\lambda_1^\phi)^k] > 0, \quad (17)$$

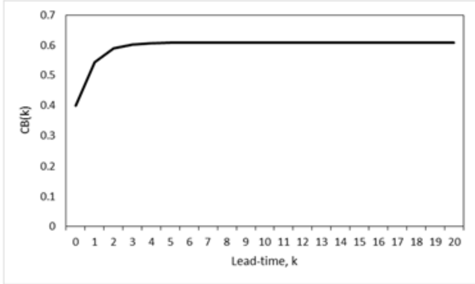
indicating that $CB(k)$ is increasing in the lead-time. This case is an example where Theorem 2 does not hold, but the impulse response can be positive, by satisfying an extra condition. If $p_1 < 0$ then the impulse response changes sign. $E(k) > 0$ decreases to a minimum and converges to $E(\infty) > 0$. Initially $CB(k)$ is negative and decreasing, before increasing and eventually becoming positive.

$CB(k)$ for the six cases has been plotted in Fig. 2 for visualisation and verification purposes. Note for brevity, we have only plotted the $-1 < p_1 < 0$ situation of case C. The AR(2), ARMA(2,1), MA(0,2), and ARMA(1,2) demand models can be seen as special variants of the ARMA(2,2) case. For example, AR(2) equals ARMA(2,2) with λ_i^ϕ's and two $\lambda_i^\theta = 0$. The analytical expressions of the impulse response is similar to case A. The ARMA(1,2) equals ARMA(2,2) with $\lambda_1^\phi = 0$ and is similar to case C, D or F. Complex conjugate λ_i^θ's in demand forecasting can exist. Depending on the real value of the λ_i^θ's, case A, C and F are similar. We note that complex conjugate λ_i^ϕ's lead to an impulse response made of a dampened cosine.

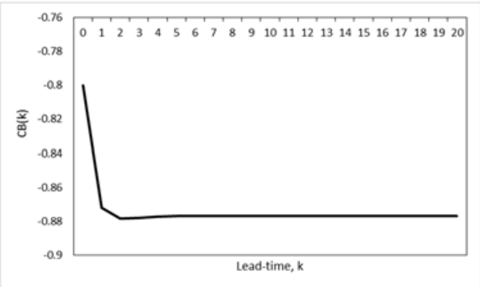
A: $\phi_1 = 0.9,\ \phi_2 = -0.2,\ \theta_1 = 0.5,\ \theta_2 = -0.06$
$\lambda_1^{\phi} = 0.4,\ \lambda_2^{\phi} = 0.5,\ \lambda_1^{\theta} = 0.2,\ \lambda_2^{\theta} = 0.3$



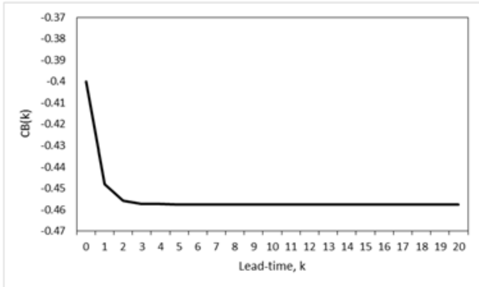
B: $\phi_1 = 0.5,\ \phi_2 = -0.05,\ \theta_1 = 0.3,\ \theta_2 = -0.02$
$\lambda_1^{\phi} = 0.138,\ \lambda_2^{\phi} = 0.361,\ \lambda_1^{\theta} = 0.1,\ \lambda_2^{\theta} = 0.3$



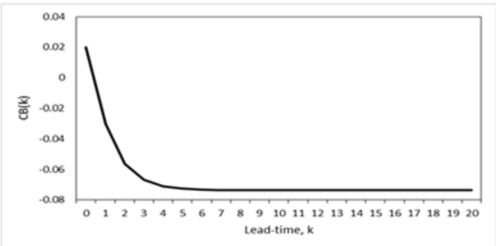
C: $\phi_1 = 0.5,\ \phi_2 = -0.06,\ \theta_1 = 0.9,\ \theta_2 = -0.2$
$\lambda_1^{\phi} = 0.2,\ \lambda_2^{\phi} = 0.3,\ \lambda_1^{\theta} = 0.227,\ \lambda_2^{\theta} = 0.877$



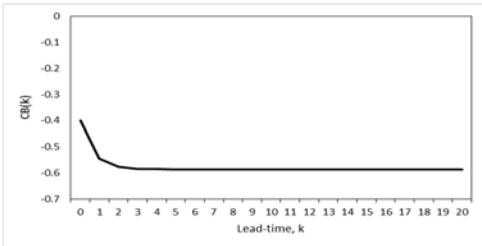
D: $\phi_1 = 0.3,\ \phi_2 = -0.02,\ \theta_1 = 0.5,\ \theta_2 = -0.05$
$\lambda_1^{\phi} = 0.1,\ \lambda_2^{\phi} = 0.2,\ \lambda_1^{\theta} = 0.138,\ \lambda_2^{\theta} = 0.361$



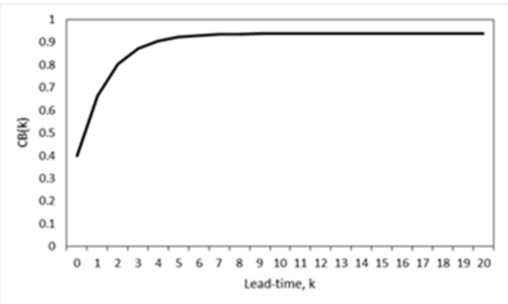
E1: $\phi_1 = 0.51,\ \phi_2 = -0.05,\ \theta_1 = 0.5,\ \theta_2 = -0.02,$
$p_1 > 0, \lambda_1^{\phi} = 0.132, \lambda_2^{\phi} = 0.377, \lambda_1^{\theta} = 0.043, \lambda_2^{\theta} = 0.456$



E2: $\phi_1 = 0.3,\ \phi_2 = -0.02,\ \theta_1 = 0.5,\ \theta_2 = -0.01$
$p_1 < 0,\ \lambda_1^{\phi} = 0.1,\ \lambda_2^{\phi} = 0.2,\ \lambda_1^{\theta} = 0.021,\ \lambda_2^{\theta} = 0.479$



F1: $\phi_1 = 0.5,\ \phi_2 = -0.01,\ \theta_1 = 0.3,\ \theta_2 = -0.02$
$p_1 > 0,\ \lambda_1^{\phi} = 0.021,\ \lambda_2^{\phi} = 0.479,\ \lambda_1^{\theta} = 0.1,\ \lambda_2^{\theta} = 0.2$



F2: $\phi_1 = 0.5,\ \phi_2 = -0.02,\ \theta_1 = 0.51,\ \theta_2 = -0.05,$
$p_1 < 0,\ \lambda_1^{\phi} = 0.043,\ \lambda_2^{\phi} = 0.456,\ \lambda_1^{\theta} = 0.132,\ \lambda_2^{\theta} = 0.377$

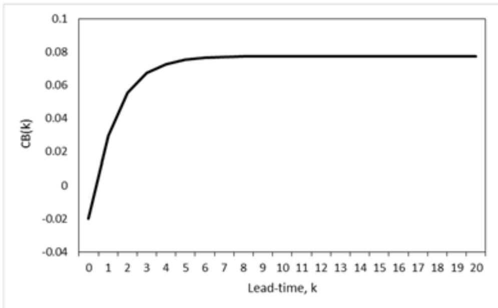


Table 1. Example CB(k) responses for ARMA(2,2) demands

Then, at least theoretically, positive and negative impulse responses are present, and increasing bullwhip over the lead-time is not possible.

5. Concluding Remarks

We have introduced a new bullwhip metric, $CB(k)$, useful when large order and demand variances are present; that is, when (near) non-stationary demand exists. Theorem 1 showed the positivity of the order impulse response determines the essential character of $CB(k)$ over the lead-time. We did this by studying the eigenvalues $0<\left\{\lambda_j^{\phi},\lambda_j^{\theta}\right\}<1$ of the demand process rather than AR and MA parameters directly. This proved to be efficient as only the order of the eigenvalues determines a lead-time/bullwhip relationship, not the specific value of the eigenvalues or the demand parameters. We found three different sets of eigenvalue orderings exist: increasing bullwhip effect over the lead-time, no bullwhip effect over the lead-time, and a bullwhip/lead-time relationship that depends on specific values of eigenvalues. Theorem 2 identifies a class of orderings for which the demand process behaves as a low pass filter that is sufficient to describe when the bullwhip is an increasing function of the lead-time. Within this class, the "strength" of the low pass filter directly influences the strength of $CB(k)$. We illustrated our results by studying the ARMA(2,2) demand process; higher order ARMA processes can be studied. Departing from the set of orderings satisfying Theorem 2 and its inverse, the number of sign changes in the r_i's complicates the analysis of $CB(k)$. The case of one change in sign is relatively easy to deal with. Using an extra property, when two sign changes are present, results can also be obtained; however, when more than two changes of sign occur, difficulties arise.

11. References

Alwan, L.C., Liu, J.J., and Yao, D-Q., 2003. Stochastic characterization of upstream demand processes in a supply chain. *IIE Transactions*, 35, 207-219.

Box, G.E.P., Jenkins, G.M., and Reinsel, G.C., 1994. *Time series analysis, forecasting, and control*. Third Edition, Holden-Day, San Francisco.

Brown, R.G., 1963. *Smoothing, Forecasting, and Prediction of Discrete Time Series*. Dover Publications, Inc. Mineola, New York.

Chen, F., Ryan, J.K., and Simchi-Levi, D., 2000. The impact of exponential smoothing forecasts on the bullwhip effect. *Naval Research Logistics*, 47, 269-286.

Dejonckheere, J., Disney, S.M., Lambrecht, M.R., and Towill, D.R., 2003. Measuring and avoiding the bullwhip effect: A control theoretic approach. *European Journal of Operational Research*, 147 (3), 567-590.

Deziel, D.P., and Eilon, S., 1967. A linear production-inventory control rule. *The Production Engineer*, 46 (2), 93-104.

Duc, T.T.H., Luong, H.T., and Kim, Y-D., 2008. A measure of bullwhip effect in supply chains with a mixed autoregressive-moving average demand process. *European Journal of Operational Research*, 187 (1), 243-256.

Gaalman, G., 2006. Bullwhip reduction for the ARMA demand model: The proportional OUT policy, versus the full-state feedback policy. *Automatica*, 42 (8), 1283-1290.

Gaalman, G. and Disney, S.M., 2006. State space investigation of the bullwhip problem with ARMA(1,1) demand processes *International Journal of Production Economics*, 104 327–339 121 (2), 454-463.

Gaalman, G. and Disney S.M., 2007. The evolution of ARMA demand in supply chains: A state-space approach. *Annual Conference of the Operations Research Society, Edinburgh*, Scotland, 4th-6th September.

Gaalman, G., and Disney, S.M., 2012. Bullwhip behaviour in the Order-Up-To policy with ARIMA demand. *4th World Production and Operations Management Conference*, Amsterdam, 1st–5th July.

Gaalman, G., and Disney, S.M., 2009. On bullwhip in a family of order-up-to policies with ARMA(2,2) demand and arbitrary lead-times. *International Journal of Production Economics*, 121 (2), 454-463.

Gilbert, K., 2005. An ARIMA Supply Chain Model. *Management Science,* 51 (2), 305-310.

Kailath, T., 1980. *Linear Systems*. Prentice Hall, Englewood Cliffs.

Lee, H.L., So, K.C., and Tang, C.S., 2000. The value of information sharing in a two-level supply chain. *Management Science*. 46 (5), 626-643.

Li, Q., Disney, S.M., and Gaalman, G., 2014. Avoiding the bullwhip effect using damped trend forecasting and the order-up-to replenishment policy. *International Journal of Production Economics.* 149, 3-16.

Li, G., Yu, G., Wang, S., and Hong Yan, H., 2017. Bullwhip and anti-bullwhip effects in a supply chain. *International Journal of Production Research*, 55 (18), 5423-5434.

Luong, H.T., 2007. Measure of bullwhip effect in supply chains with autoregressive demand process. *European Journal of Operational Research*, 180 (3), 1086-1097.

Luong, H.T., and Phien, N.H., 2007. Measure of bullwhip effect in supply chains: The case of high order autoregressive demand process. *European Journal of Operational Research*, 183 (1), 197-209.

Tsypkin, Y.Z., 1964. *Sampling systems theory and its applications*. Vol. 1, Oxford: Pergamon Press.

Urban, T.L., 2005. A periodic-review model with serially-correlated, inventory-level-dependent demand. *International Journal of Production Economics*, 95 (3), 287-295.

Wang, X., and Disney, S.M., 2016. The bullwhip effect: progress, trends and directions. *European Journal of Operational Research*, 250 (3), 691-701.

Zhang, X., 2004. The impact of forecasting methods on the bullwhip effect. *International Journal of Production Economics,* 88 (1), 15-27.

Appendix. Eigenvalue Transformation

The state space representation of the demand (3), (4) is in observable canonical form. Iff the AR part eigenvalues are distinct, the matrix D_ϕ can be transformed into a diagonal form using the matrix U where the rows are m independent left eigenvectors via

$$D_\phi = U^{-1}\Lambda_\phi U \tag{18}$$

with Λ_ϕ the diagonal matrix $(\lambda_1^\phi \quad \lambda_2^\phi \quad \cdots \quad \lambda_m^\phi)$, U the Vandermonde matrix becomes

$$U = \begin{pmatrix} \left(\lambda_1^\phi\right)^{m-1} & \left(\lambda_1^\phi\right)^{m-2} & \cdots & 1 \\ \left(\lambda_2^\phi\right)^{m-1} & \left(\lambda_2^\phi\right)^{m-2} & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ \left(\lambda_m^\phi\right)^{m-1} & \left(\lambda_m^\phi\right)^{m-1} & \cdots & 1 \end{pmatrix} \tag{19}$$

and the inverse matrix U^{-1}

$$U^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -\phi_1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ -\phi_{m-1} & \cdots & -\phi_1 & 1 \end{pmatrix} \begin{pmatrix} s_1 & s_1 & \cdots & s_1 \\ s_2\lambda_1^\phi & s_2\lambda_2^\phi & \cdots & s_2\lambda_m^\phi \\ \vdots & \vdots & \cdots & \vdots \\ s_m\left(\lambda_1^\phi\right)^{m-1} & s_m\left(\lambda_2^\phi\right)^{m-1} & \cdots & s_m\left(\lambda_m^\phi\right)^{m-1} \end{pmatrix} \tag{20}$$

with

$$s_l = \frac{1}{\prod_{\substack{j=1 \\ j\neq l}}^{m}(\lambda_l - \lambda_j)}, \tag{21}$$

see Kailath (1980). The state space demand process in (3) and (4) then becomes

$$\upsilon_{t+1} = \Lambda_\phi \upsilon_t + G_\lambda \eta_t,\; z_{t+1} = M_\lambda \upsilon_{t+1} + \eta_{t+1},\; \upsilon_t = U y_t,\, M_\lambda = MU^{-1} = \begin{pmatrix} s_1 & s_2 & \cdots & s_m \end{pmatrix},$$

$$G_\lambda = UG = \begin{pmatrix} \sum_{j=1}^{m} \lambda_1^{m-j}(\phi_j - \theta_j) \\ \sum_{j=1}^{m} \lambda_2^{m-j}(\phi_j - \theta_j) \\ \vdots \\ \sum_{j=1}^{m} \lambda_m^{m-j}(\phi_j - \theta_j) \end{pmatrix} = \begin{pmatrix} \prod_{j=1}^{m} (\lambda_1^\phi - \lambda_j^\theta) \\ \prod_{j=1}^{m} (\lambda_2^\phi - \lambda_j^\theta) \\ \vdots \\ \prod_{j=1}^{m} (\lambda_m^\phi - \lambda_j^\theta) \end{pmatrix}. \tag{22}$$

The impulse response can then be rewritten as

$$p_{k+1} = M(D_\phi^k)G = MU^{-1}(\Lambda_\phi^k)UG \tag{23}$$

Substitution results in

$$p_{k+1} = \sum_{i=1}^{m} r_i^\phi (\lambda_i^\phi)^k,\; r_i = \left(\frac{\Pi_{l=1}^{m} \left(\lambda_i^\phi - \lambda_l^\theta \right)}{\Pi_{\substack{l=1 \\ l \neq i}}^{m} \left(\lambda_i^\phi - \lambda_l^\phi \right)} \right). \tag{24}$$

Appendix B. Proof of the Theorem 2.
First we note the z-domain pole-zero transfer function of p_{k+1}, is given by

$$G(z) = \frac{\prod_{j=1}^{m} (z - \lambda_j^\theta)}{\prod_{j=1}^{m} (z - \lambda_j^\phi)}. \tag{25}$$

Theorem 1 showed that the increasing monotonicity of bullwhip is equivalent to a positive impulse response of $G(z)$. When $\lambda_j^\phi > \lambda_j^\theta$ each $G_j(z) = \left(z - \lambda_j^\theta\right) / \left(z - \lambda_j^\phi\right)$ in $G(z) = \prod_{j=1}^{m} G_j(z)$ has a positive impulse response. The convolution property then shows that the product $G(z)$ also has a positive impulse response.