A slightly depressing jump model: Intraday volatility pattern simulation

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Abstract

Hawkes Processes have been finding more applications in diverse areas of science, engineering and quantitative finance. In multi-frequency finance various phenomena have been observed, such as shocks, crashes, volatility clustering, turbulent flows and contagion. Hawkes processes have been proposed to model those challenging phenomena appearing across asset prices in various exchanges. The original Hawkes process is an intensity-based model for series of events with path dependence and self-exciting or mutual-exciting mechanisms. This paper introduces a slightly depressing process to model the reverse phenomenon of self-exciting mechanisms. Such a process models the decline in the intensity of jumps observed in market regimes. The proposed birth-immigration-death process captures the decline in jump intensity observed at the start of a daily trading regime while the classical immigration-birth process models an increase in jump intensity toward the close of daily trading. Each of these processes can be expressed as a special case of a simple bivariate Hawkes process.

Keywords: Hawkes process; jump detection; birth-death-immigration; financial series; intraday simulation

JEL classification: G15, G17, C4, C5

1 Introduction

Hawkes processes, the intensity-based stochastic processes, attracted the concentrated attention of finance researchers lately. The original processes were proposed in 1971 in studying path evolution of stochastic processes through a self-exciting or mutually exciting mechanism. In contrast to the commonly seen point processes in the finance literature, such as Poisson processes, Hawkes processes provide the possibility of studying the entire path showing how an observable variable evolves. This family of processes capture the process history and thereby allow the intensity of the path's potential development to be conditionally predicted based on such history. This type of intensity-based model includes the self-exciting processes introduced by Hawkes (1971a) and mutually-exciting processes Hawkes (1971b), Hawkes (1972), and Hawkes and Oakes (1974).

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The applications of Hawkes processes to financial economics, quantitative finance, and financial engineering transcend scale and frequency from macro-prudential slowly-moving scale to high-frequency finance Bacry et al. (2015) for a review of Hawkes theory and financial applications. These include, for example, applications to insurance, credit risk, risk exposure and value-at-risk calculations. Contagion effects or spillovers occur in asset pricing, volatility and many other important observable financial variables. Hawkes processes are proposed to model macro-prudential market conditions and cross-asset excitation creating bubbles or crashes. In the latter case, contagious fire-selling creates a downward spiral effect on market prices causing a crash. Such crashes have frequently been observed to be short-lived within a trading day but often leading to market instability associated with large financial losses. These observations naturally lead us to examine the current macro-prudential financial market conditions post the recent 2008 liquidity crisis and 2010 Eurozone sovereign debt crisis: contagion and clustering effects become newly emerged key issues in finance. Hawkes process type of intensity models have great capacity to incorporate such features and have been much used in the finance literature since about 2005, while learning particularly from early applications in seismology. Researchers have built various applications utilizing Hawkes processes in identifying earthquakes and aftershocks, which carries the contagion characters; see Hawkes and Adamopoulos (1973) and Ogata (1981), Ogata (1988) and Ogata (1998). More recently. Hawkes processes have found applications in many areas of Science and social processes too numerous to discuss here.

So far Hawkes processes feature the self exciting side of the process as a way of describing the transmission of excitation forward in time. From empirical observations, the opposite can happen, i.e. the process can exhibit frequent jumps and taper down in intensity to noise oscillations, which are not identified as jumps. In that sense the process undergoes a reverse self-excitation, which may be termed a *self-depressing process*. For example, in modeling daily jumps in the price process of many stocks, it is known that a dominant pattern appears to have large jumps in the morning undergoing a depressing process through local successive jumps of gradually diminishing magnitudes and less intensity until settling toward noise-like oscillations during mid-day trading. This is followed by a self-exciting process before the close of trading. In multi-dimensional asset price interactions, the model for Hawkes processes with mutually-exciting jumps describes the contagion in jumps across tightly coupled assets. Conversely, the same transmission mechanism facilitates mutually-depressing jumps. This means that reduced intensity of jumps may, in the absence of idiosyncratic news about the asset, lead to reduced intensity in tightly coupled assets. In section 4 we introduce a bivariate Hawkes process with two types of event: type 1 events are the observed jumps and the occurrence of type 2 events reduces the intensity of type 1 events, causing less of them to occur (a depressing effect). This process can also be described in terms of a classic birth-death-immigration process.

The literature has richly documented studies on spillover effects in explaining such economic phenomena. However, almost all these studies applied dynamic models such as the GARCH family of models which, if they include jumps, are usually developed on the basis of Poisson processes. Over recent years, it has become apparent that traditional diffusion models are not adequate to describe the sudden changes in behaviour that increasingly tend to occur in financial time series, possibly due in part to the rise in high frequency trading. Furthermore, Poisson-based models lack the property of memory: this causes lack of reflection of path evolution of the time series and its potential effect on forecasting. Especially, these models are mostly mean-reverting and therefore unlikely to capture inhibitory and extreme events we now often see. One great example is the mini-flash crash on May 06, 2010 and, more importantly, one more often sees such crashes documented even within a trading day. For instance, on January 27, 2014, within the first quarter of an hour of trading (9:30-9:45) Nanex spotted a mini crash on a similar scale to the famous 2010 crash; and 179 crashes were recorded between January 2012 to January 27, 2014. Apple's stock was reported with a price plunge and rebound before the close of January 25, 2013, which sent the stock down 1.7% and shrunk the market value of Apple nearly to $7 billion, over 1 million shares changing hands.

It is not difficult, in particular in finance applications, to see the importance of such features. For example, when studying trading behaviour, it is more natural to think that mid-quotes change and the trade arrival of a stock are interconnected and how such interconnection works in the past would have impact on trading behaviour in the future both individually and mutually (Bowsher (2007) and Large (2007)).

In section 3 we describe a model for daily log-returns but incorporating information on jumps observed in intra-day data recorded at intervals of one or two minutes. The aim is to understand more about how jumps and diffusion, considering both returns and volatility, interact with Hawkes processes. At the same time, we develop a model that is more informative than daily data only but simpler to use than a full high frequency approach. Such a model might appeal to commercial users. In this paper we are not studying microstructure or the high frequency phenomenon, but the lower frequency data such as one-minute data to identify intra-day jumps on a moderate timescale and use them to improve decision-making capacity. For that reason we keep the models and methods of analysis simple with applications on both sides of risk and portfolio optimization.

2 Literature on financial jumps

The literature on financial jumps may be viewed as addressing two main issues: one is concerned with modeling jumps through stochastic processes; the other aims to identify jumps, to develop criteria for jump detection and then to capture the dynamics of jumps through a process. For example, modeling financial jumps with Lévy processes has been proposed incrementally since Mandelbrot introduced it in 1963, from Poisson processes to compound Poisson processes to more general Lévy processes, see Cont and Tankov (2004). Those models are not generally concerned with the mechanism with which jumps have occurred but with postulating better models that can capture more statistical aspects of jumps present in a given price process. Such models include Merton's equilibrium model (1976), which assumes continuous log-price updates to be driven by two components: a standard Brownian motion diffusion process and a continuous counting process of occurrence of jumps, although occurring not so often. Mostly, the jumps are assumed to occur as a Poisson process with perhaps only a handful of events per year. The jump sizes are often supposed to follow a Normal distribution. However, this seems extremely unrealistic as jumps must be large and have effects on the underlying process by definition. Further, it is natural to expect that the discontinuity in the price updating process can be both positive and negative with the probability density of the size distribution being near zero at the origin. Undoubtedly, standard jump diffusion models have difficulties in accurately capturing all these features. Another model of jumps is given by Kou (2002) wherein jump sizes are asymmetrically distributed with a memoryless property and jumps that are modeled with a component of a Lévy process superposed on a diffusion process. This line of research is important in pricing and hedging as it aims to account for the documented shortcomings of the Black-Scholes model. It would be even more so if we could incorporate jump features that typical diffusion jump modelling could not capture. Aït-Sahalia et al. (2010) develop a Hawkes jump-diffusion mutually-exciting process with mean-reverting intensity dynamics and implement a Generalized Method of Moments (GMM)-based estimation procedure for data fitting. Again, this method could not sufficiently reflect the features, such as non-Gaussian properties, which are present in the real data.

On the other hand, in jump identification and detection, there is no universal definition of jumps but it is commonly agreed that jumps refer to discontinuities in the price process causing prices to sharply move away from the mean level. Some scholars studying market microstructure consider that, in the high frequency trading realm, every trade constitutes a jump, however small. In contrast, the classical ARCH/GARCH models suggest that there are no jumps and the price process is a complete diffusion process. A number of different methods have been proposed to detect jumps. Andersen and Bolleslev (1998) suggest to utilize realized volatility (RV) to capture the quadratic variation in the log-price process. Andersen et al. (2001) and Barndorff-Nielsen (2002) point out that in the absence of jumps the realized volatility estimate represents the integrated volatility. This measure of calculating realized volatility has been further improved by Barndorff-Nielsen and Shephard (2004) and Barndorff-Nielsen and Shephard (2006) in defining bi-power variation (BV), which measures volatility through observing the product of absolute values of log-returns in consecutive intervals. Andersen et al. (2007) study the difference between the bi-power variation and realized volatility in order to identify significant jumps (e.g. jumps with large size). Andersen et al. (2010) further develop a Z estimator to identify significant intra-day jumps. This Z score is a combination of realized tri-power quarticity, RV and BV measures.

Apart from the commonly used Andersen method, there are other jump detection methods found in the literature. For example, Huang and Tauchen (2005) develop a relative jump (RJ) measure to approximate jumps as the difference between the realized and bi-power variation proportional to the realized volatility. Aït-Sahalia and Jacod (2009) look at the log-return differences of an asset to the power of 4, comparing the results from two different sampling frequencies in order to detect jumps. Lee and Mykland (2008), instead of focusing on realised volatility, propose to observe the magnitude of individual returns relative to a measure of local volatility so as to establish the significance of jump dynamics. In the context of high frequency jump detection, jumps are referred to also as rare events wherein relatively sharp movements in price occur with light volume as in Bozdog et al. (2015). Recently, Buckle et al. (2017) suggest a new method of running jumps (BCH method) particularly to detect flash-crash type of jumps, utilising an approach that includes a local volatility measure. It seems likely that use of this local volatility measure also further improves the Lee and Mykland (2008) method as it avoids masking effects, whereby a jump can be missed if the volatility measure with which it is compared is inflated by a number of large returns occurring in neighbouring intervals.

3 Methodology

Given the advantages of Hawkes processes in modelling contagion and flexibility of incorporating various underlying distributions, stochastic modelling utilizing self- or cross-excitation become natural for applications in finance. Up to date, many applications have been well documented including limit order book structures, high frequency trading, liquidity spillovers, financial jumps, market sentiment etc. When studying the intra-day structure of price behaviour, the literature and market observations suggest that not only extreme events such as jumps or order flow flooding occur more regularly, but also they tend to occur in a concentrated short period. Therefore, our thoughts not only dwell in the aspect of utilizing Hawkes processes to detect and monitor such individual market events and related price movements, but also to gain some reflection of the market-wide behaviour and even the market structure. Therefore we propose the following two scientific setups, and Hawkes processes are embedded underneath to tie the knots of both: 1) we propose a local volatility measure based jump detection methods that would allow for occurrence of jumps in neighbouring or consecutive time intervals so that it reflects the idea of Hawkes processes advance stochastic modelling of contagion; and 2) we suggest a birth-death-immigration process to proxy a novel, simple but realistic structure of trading. The events of birth-death-immigration take different interpretations in a financial context. For example, one possible interpretation is summarized in the following correspondence for one-asset trading participants: birth corresponds to a buy-initiated order, death for a sell-initiated order while immigration corresponds to new conditional in-flux or out-flux of trading agents over a time horizon. Investigating various interpretations for the model takes us outside the intended objectives of this paper. In Section 3, a local volatility measure based on jump detection methods with the ability to handle successive jumps is proposed. In section 4, we set out details of the birth-death-immigration process. Further, we provide the simulated examples with the model fitting in Section 5.

3.1 From Diffusion to Jumps

Diffusion models in finance have a long history starting from the simplest geometric Brownian motion models. Louis Bachelier wrote his thesis on the theory of speculations in 1900 (Bachelier (1900)) and his seminal work remained essentially unknown to the finance literature until Paul A. Samuelson rediscovered it, see Davis and Etheridge (2006). Bachelier's work introduced Brownian motion to finance applications in the context of option theory and marked the beginning of financial mathematics. In 1973, the Black-Scholes and Merton model of diffusion gave a formula for option pricing. Further developments in jump-diffusion models were mainly made in conjunction with improving option pricing models. It has been realized that pure diffusion models should not be expected to properly model jumps because diffusion processes have continuous paths. Thus attempts were focused on extending the diffusion model in two directions: one is to add a jump term of Poisson type, like the Merton model in Merton (1976), hence obtaining jump-diffusion models; another approach models volatility itself as a separate but coupled process, hence getting a stochastic volatility class of models like the popular Heston stochastic volatility model. One may further include jumps in volatility itself as a stochastic process scaling the diffusion term in the return process. It is not easy to include all contributions in this paper; see Cont and Tankov (2004).

As an example, if we take the affine GARCH(1,1) model of Heston and Nandi (2000); see Christoffersen et al. (2008), we write

$$r_t = r + \lambda_z h_t + \sqrt{h_t} z_t; \tag{1}$$

$$h_{t+1} = \omega + b h_t + a(z_t - c\sqrt{h_t})^2, \tag{2}$$

where $r_t, t \in [0, T]$ is a sequence of daily log-returns; r the risk-free rate; h_t is a sequence of daily volatility; z_t is a series of i.i.d. random variables distributed as $N(0, 1)$. The other symbols are parameters that will usually be held constant over a year, perhaps longer. λ_z represents a volatility premium for the return. a determines the kurtosis of the distribution of log-returns. The c parameter results in an asymmetric influence of shocks: large negative z_t raises the volatility more than large positive z_t and controls the skewness of the log-return distribution. If $c = 0$, we have the GARCH(1,1) model.

A general class of jump-diffusion models can be obtained by modifying the above two equations:

$$r_t = r + \lambda_z h_t + \lambda_y v_t + \sqrt{h_t} w_t = r + \lambda_z h_t + \lambda_y v_t + \sqrt{h_t} z_t + \sum_{i=1}^{n_t} x_{t,i}; \tag{3}$$

$$h_{t+1} = \omega + b h_t + a(w_t - c\sqrt{h_t})^2, \tag{4}$$

where $w_t = z_t + y_t/\sqrt{h_t}$ and $y_t = \sum_{i=1}^{n_t} x_{t,i}$ is the sum of n_t jumps in the log-return process on day t ($y_t = 0$ if $n_t = 0$). Here v_t represents some kind of jump volatility. Note that the jumps in the log-return process on day t also contribute to the volatility on the following day through equation (4).

There is a flexible class of models based on Hawkes processes that are capable of describing intraday clustering of jumps and correlation between activity on neighbouring days. This offers many choices, including well-known Poisson and auto-regressive Poisson models as special cases. So the task is to narrow down the choices with a view to getting a simple class of models that give a good fit to most of the data. So far this description applies to a single asset but we are interested in a multivariate analysis. This could be a collection of indexes from different regions (S&P, FTSE, Hang-Seng); an underlying asset and a number of derivatives based on it; a single asset traded in multiple markets; shares in a number of companies whose business is thought to be related in some way. Correlations between series would be introduced by (i) mutually exciting jumps and (ii) there would be a set of equations (3) and (4) for each series but the z variables in these equations would probably show inter-series correlations.

3.2 Algorithmic jump identification

Before attempting to fit any of these equations it will be necessary to identify the number and size of jumps each day. We utilize the BCH detection method introduced by Buckle, Chen & Hawkes (2017). Different from the much-used 'RV-BV' detection method (see Barndorff-Nielsen and Shephard (2004), Barndorff-Nielsen and Shephard (2006); Andersen et al. (2010), which uses 'realized volatility' and 'bi-power variation', the BCH method examines all intraday returns scaled by a jump-robust measure of intraday volatility for each trading day.

The basic idea is to estimate realized volatility using the median of absolute returns of consecutive intraday intervals. The choice of the number of intervals has to be sufficient so that some level of omission of large neighbouring jumps in the volatility measure can be assured to prevent them from masking the occurrence of jumps that we are seeking to detect.

Therefore, the BCH method suggests to use a sum of rolling medians of nine measurements as an estimate of volatility on day t based on a series of M equally-spaced log-returns on that day. It is denoted by $Med9RV_{t,M}$ and defined as follows:

$$Med9RV_{t,M} = 1.82184 \frac{M}{M-8} \sum_{i=5}^{M-4} \left(med|r_{t,i-4}|...|r_{t,i}|...|r_{t,i+4}| \right)^2. \tag{5}$$

The scale factor 1.82184 is chosen so that the estimator is unbiased in the event that the returns are i.i.d. $N(0, \sigma^2)$.This is similar to the $Med3RV$ method suggested by Andersen et al. (2012). However, the use of 9 intervals is clearly more effective to tackle the issue of multiple jumps occurring in successive intervals: it can eliminate the effect of 4 neighbouring large returns, whereas the median of 3 can only eliminate the effect of 1 large neighbouring return. BCH provides plenty of empirical evidence at 2-minute intraday level to show that their $Med9RV$ captures jumps more effectively, especially for the flash crash type of jumps like the one we experienced on May 06, 2010: in contrast the RV-BV method fails to identify any jumps in S&P 500 index on that day. If going into ultra-high frequency data, we believe this method could be even more efficient as the occurrence of jumps and jump clusters should be more persistent.

Further based on the $Med9RV$, the returns can be scaled and written as:

$$r^*_{t,i} = \frac{r_{t,i} - med(r_t)}{\sqrt{Med9RV_{t,M}/M}}, \tag{6}$$

where $med(r_t)$ is the median return of day t. Call these 'scaled jump returns'. At the same time it is useful to think of this scaled return as the ex-post realized *local Sharpe ratio*. Indeed, consider the $med(r_t)$ as the local return and the BCH scaled jump returns to $\sqrt{Med9RV_{t,M}/M}$ as the local volatility. The ex-post Sharpe ratio follows Sharpe (1994) wherein the benchmark portfolio is the S&P500 while the local ratio is "benchmarked" to its own local median return and approximate realized standard deviation; thus the term *relative jump* makes sense both in time and return.

A further scale factor, f_i, can be introduced in order to allow for the classic U-shape of daily volatility. This would also reduce the number of jumps found near the beginning and the end of every trading day. Then, the scaled jump return will become

$$r^*_{t,i} = \frac{r_{t,i} - med(r_t)}{f_i \sqrt{Med9RV_{t,M}/M}}, \tag{7}$$

where $f_i = A\alpha^i + B\beta^{M+1-i} + C$ for suitable choice of A, B, C.

The authors suggest that the method can be adjusted to adapt to the detection purpose better. For instance, if the objective is to capture large jumps, which usually are considered to be more likely associated with large potential financial loss (or gain), a significant jump threshold, η, can be set up to suit this purpose: say defining J_η indicating the scaled jump returns with absolute values greater than η.

Further, one can scan through all individual jump returns and replace those with small absolute values, e.g. less than 1, by 0 and call remaining jumps 'non-trivial jumps'. This has two purposes: 1) to smooth out the small variations between big jumps; and 2) to identify large changes that evolve over several intervals. One can accumulate a run of positive non-trivial jumps to form a 'positive running jump'; similarly, a run of negative non-trivial jumps can be accumulated to form a negative running jump.These running jumps can be treated as a block for further studies, for example identifying those with absolute value larger than some critical value. Hence, the robustness of detection can be enhanced as no large changes can be missed.

The jump process: The jump identification process will depend upon some control parameter (find more or less jumps) which will, of course, affect the model fitting. For a full development of the jump-diffusion model, which is needed for prediction, we need to specify a model for the occurrence of jumps. Here we have quite a bit of flexibility, and pinning this down to a more specific form is a major part of the problem. Here is a general framework.

Intraday behaviour of jumps: We consider a set of K return series and aim to model the multivariate jump process by a mutually exciting Hawkes process with the intensity for events in the ith series within day t given by

$$\lambda_{i:t}(u) = v_{i:t} + \sum_{j=1}^{K} \sum_{s_{j:t:r} < u} \gamma_{ij}(u - s_{j:t:r}), \tag{8}$$

where $\gamma_{ij}(w)$ is the contribution that an event in series j makes to the intensity of events in series i at time w later (the cross-excitation effect); $s_{j:t:r}$ is the time of the rth event in series j on day t; $v_{i:t}$ is the base (Poisson) intensity of events in series i on day t. For simplicity, we suppose that the intensity does not depend on the size of jumps. This gives a general specification of intraday jump occurrence. Simple exponential exciting kernels,

$$\gamma_{ij}(w) = \alpha_{ij} e^{-\beta_{ij} w}, w > 0, \tag{9}$$

often suffice: it may be that further simplifications, such as many of the beta coefficient being equal might be good. However, in many finance applications it is claimed that some longer-term effects are also needed. This may be done by including a second exponential term in the kernel or using a power-law kernel: see discussion in Bacry et al (2015; section 3.1).

Rollover effects between days: The occurrence of jumps on one day may also have an effect on the probability of events on the following day. This could be introduced through the base Poisson intensity, say

$$v_{i:t} = v_i + \sum_{j=1}^{K} \phi_{ij}(n_{j:t-1}) + \kappa_i(h_{i:t}). \tag{10}$$

This depends, in a rather vaguely specified way, on the numbers of jumps $n_{j:t-1}$ in the various series on the previous day and the volatility $h_{i:t}$ on the same day. So lots of things to research there in deciding on what those ϕ, κ functions look like. Details of such a model-fitting process will be published elsewhere.

4 Birth-Death-Immigration

Applications of self-exciting Hawkes processes are well documented. What could be a new Hawkes process that models an opposing event triggering a dampening of intensities? Alternatively, what is a slightly depressing intensity process? As an example, it is observed, that the opening hour of a market undergoes a self-depressing intensity regime while the end of day witnesses a self-exciting regime demonstrated by relative jumps. For systemic events, a trigger event can cause a self-exciting chain of events while a regulatory measure aims at reversing the self-excitation to self-depressing intensity. In this section, a self-depressing intensity process is introduced.

4.1 Distribution of number of jumps

In this section we begin by considering a special case of a univariate self-exciting Hawkes process in which the exciting kernel is a constant, $\gamma(w) = \alpha$. Then if there have been $n(u)$ events in interval $(0, u)$ the intensity at time u is $\lambda(u) = v + n(u)\alpha$. This is equivalent to a classic immigration-birth process with immigration rate v and birth rate α. Clearly, the intensity of such a process increases towards the end of a day. The probability generating function of the immigration-birth process is given in many standard texts, for example Jones and Smith (2009), as

$$G_N(z,t) = E[z^{N(t)}] = e^{-vt}\left[1 - z\left(1 - e^{-\alpha t}\right)\right]^{-v/\alpha}. \tag{11}$$

This is a negative binomial distribution. We can obtain the probabilities by expanding in powers of z and picking the coefficient of z^r: thus

$$P(N(t)=r) = e^{-vt}v(v+\alpha)(v+2\alpha)\dots(v+(r-1)\alpha)\big(1-e^{-\alpha t}\big)^r / (\alpha^r r!)\,,\ r=1,2,\dots$$
$$P(N(t)=0) = e^{-vt}.$$

The above Hawkes process special case raises the intensity towards the end of the day. In contrast to this, we could use a model for the start of the day with a certain amount of inhibition that reduces the intensity as time passes. One natural choice is an immigration-death process, which would reduce the intensity, but with some local excitement. Therefore, we propose to have a classic immigration-birth-death process with immigration rate v per unit time; death rate β per live individual per unit time; birth rate α per live individual per unit time and initial number of live individuals $N(0)$. Now suppose

- **Every birth and every immigration corresponds to a jump**: this increases the live population by 1, and therefore increases the jump intensity by α (the self-exciting bit). At the same time it increases the death rate by β.
- We do not actually see any deaths, but when they occur the underlying population decreases by 1 and so the jump intensity will **decrease** by α and the death rate will decrease by β.
- If the initial population size, $N(0)$, is reasonably large and the death rate larger than the birth rate then we expect to see several jumps early on then decreasing with time and settling down to a steady process.

The probability distribution of the population size at time t is well known. However, we are interested in the distribution of the number of jumps (births and immigrations) in time interval $(0,t)$. To set up some notation, we let the total number of jumps in $(0,t)$ be

$$X(t) = \sum_{r=1}^{N(0)} x_r(t) + \vec{x}(t),$$

where $x_r(t)$ is the number of jumps due to the rth initial individual and $\vec{x}(t)$ denotes the number of jumps due to all immigrants and their descendants. Similarly, let the total population size at time t be

$$N(t) = \sum_{r=1}^{N(0)} n_r(t) + \vec{n}(t).$$

Let $G_x(z,t)$ be the probability generating function (p.g.f.) for the number of births in interval $(0,t)$ arising from a birth-death process, starting from 1 initial individual; let $G_{\vec{x}}(z,t)$ be the p.g.f. for the number of jumps resulting from all immigrants and their descendants in $(0,t)$. It is clear from the model that these various variables are independent, so that the distribution of the total number of jumps, $X(t)$, in $(0,t)$, starting with population size $N(0)$, is given by

$$G_X(z,t) = G_x(z,t)^{N(0)} G_{\vec{x}}(z,t). \tag{12}$$

Simple birth-death component

We seek to derive the first term in this result, starting from one initial individual, by obtaining the backward equation, conditioning on what happens in $(0,\delta t)$. Thus

$$G_x(z,t) = \alpha\delta t z\{G_x(z,t-\delta t)\}^2 + \beta\delta t + [1-(\alpha+\beta)\delta t]G_x(z,t-\delta t).$$

The three terms correspond to the following events: a birth (which is responsible for the term z) followed by a birth-death process with two initial individuals and, of course, just counting the births; a death, then nothing more; neither birth nor death followed by the original process, but over time interval of duration $t - \delta t$ instead of t. Letting $\delta t \to 0$, we get the Partial Differential Equation (PDE):

$$\frac{\partial G_x(z,t)}{\partial t} = \alpha z\{G_x(z,t)\}^2 - (\alpha+\beta)G_x(z,t) + \beta.$$

This can be solved subject to the initial condition $G_x(z,0) = 1$, since there can be no jumps in the initial interval of size zero. The solution can be written as

$$G_x(z,t) = \frac{\phi_2(1-\phi_1) - \phi_1(1-\phi_2)e^{-ct}}{(1-\phi_1) - (1-\phi_2)e^{-ct}}, \tag{13}$$

where c, ϕ_1, ϕ_2 are functions of z.

$$c(z) = \sqrt{(\alpha+\beta)^2 - 4\alpha\beta z} \tag{14}$$

and

$$\phi_1(z) = (\alpha+\beta+c)/(2\alpha z);\ \phi_2(z) = (\alpha+\beta-c)/(2\alpha z). \tag{15}$$

Immigration-birth-death process

The second term in equation (12) is the p.g.f. for the number of immigrants and all their descendants that are born in the interval $(0,t)$. This satisfies the backward equation

$$G_{\vec{x}}(z,t) = (v\delta t z G_x(z,t-\delta t) + (1 - v\delta t))G_{\vec{x}}(z,t-\delta t),$$

and hence

$$\frac{\partial G_{\vec{x}}(z,t)}{\partial t} = v(zG_x(z,t) - 1)G_{\vec{x}}(z,t)$$

with initial condition $G_{\vec{x}}(z,0) = 1$. Simple integration yields the solution

$$G_{\vec{x}}(z,t) = exp\big\{\int_0^t v[zG_x(z,s) - 1]ds\big\}. \tag{16}$$

The integral can be evaluated to yield

$$G_{\vec{x}}(z,t) = \left(\frac{c}{\alpha z\left[1 - \phi_2 + (\phi_1 - 1)\, e^{ct}\right]}\right)^{v/\alpha} e^{-vt(1-\phi_1 z)}. \tag{17}$$

4.2 Some special cases

Notation-wise it is sometimes better to express explicitly the dependence of the functions on the parameters. Thus, for example, we can write $G_X(z,t:\alpha,\beta,v)$ instead of $G_X(z,t)$. We now consider some special cases, notably when some parameter is set to zero.

- Case 1: $\alpha = 0$. There are no births so that there are no jumps arising from the initial individuals; the immigration-birth process is a pure immigration process, i.e. a Poisson process. Thus

  $$G_x(z,t:0,\beta,v) \equiv 1 \text{ and } G_{\vec{x}}(z,t:0,\beta,v) = e^{vt(z-1)}.$$

- Case 2: $\beta = 0$. From equations (14) and (15) we have

$$c = \alpha;\ \phi_1 = 1/z; \phi_2 = 0. \tag{18}$$

Substitution into equation (13) yields

$$G_x(z, t : \alpha, 0, v) = \frac{e^{-\alpha t}}{1 - z\left(1 - e^{-\alpha t}\right)},$$

a Geometric distribution. There are no deaths so, in the process starting from 1 initial individual, the jumps form a pure birth process (the initial individual is not included) and the above result is equivalent to that in Beaumont (1983). With no deaths; the birth-death-immigration process is just an immigration-birth process and substitution of the values in equation (18) into equation (17) yields the negative binomial distribution given in equation (11). Putting these two together gives us, from equation (12),

$$G_X(z, t : \alpha, 0, v) = \left\{\frac{e^{-\alpha t}}{1 - z\left(1 - e^{-\alpha t}\right)}\right\}^{N(0)+v/\alpha}. \tag{19}$$

- Case 3: $v = 0$. The jumps arising from a single initial individual are the births. The distribution is given by the p.g.f. $G_x(z, t : \alpha, \beta, 0)$ in equation (13) without simplification, unless $\alpha = 0$ or $\beta = 0$ and we get the special cases already discussed above. The immigration process is null (no immigrants and therefore no births either) so has no jumps and $G_{\vec{x}}(z, t : \alpha, \beta, 0) \equiv 1$.

4.3 Means and Variances

From the independence of the components shown in equation (12), it is clear that

$$E[X(t)] = N(0)E[x(t)] + E[\vec{x}(t)];\ var[X(t)] = N(0)var[x(t)] + var[\vec{x}(t)]. \tag{20}$$

These means and variances can be obtained in the usual way from the first two partial derivatives w.r.t. z at $z = 1$ of the probability generating functions derived in subsection 4.1. However, although this is straightforward in principle (and we did actually do it) the algebra is complicated by the fact that c, ϕ_1, ϕ_2 are functions of z. We present instead an alternative derivation using backward equations and double expectation, conditioning on the events that may happen in the first small interval $(0, \delta t)$.

4.3.1 Means

Let $m_x(t) = E(x(t)) = E[E(x(t)|\mathcal{F}_1(\delta t))]$. The events, probabilities and consequences are shown in Table 1.

Table1

event in $(0, \delta t)$	death	birth	nothing
probability	$\beta\delta t$	$\alpha\delta t$	$1-(\alpha+\beta)\delta t$
$n(\delta t)$	0	2	1
$x(\delta t)$	0	1	0
$E(x(t)\|\mathcal{F}_1(\delta t))$	0	$1 + 2m_x(t - \delta t)$	$m_x(t - \delta t)$
$Var(x(t)\|\mathcal{F}_1(\delta t))$	0	$2\sigma_x^2(t - \delta t)$	$\sigma_x^2(t - \delta t)$

Thus

$$m_x(t) = \beta\delta t \times 0 + \alpha\delta t(1 + 2m_x(t-\delta t)) + (1 - (\alpha+\beta)\delta t)m_x(t-\delta t),$$

which becomes in the limit as $\delta t \to 0$

$$\frac{\partial m_x(t)}{\partial t} = \alpha - (\beta - \alpha)m_x(t),$$

or, multiplying by integrating factor $e^{(\beta-\alpha)t}$,

$$\frac{\partial}{\partial t}\left(m_x(t)e^{(\beta-\alpha)t}\right) = \alpha e^{(\beta-\alpha)t}.$$

Therefore

$$m_x(t)e^{(\beta-\alpha)t} - m_x(0) = \int_0^t \alpha e^{(\beta-\alpha)t}dt = \frac{\alpha}{\beta-\alpha}\left[e^{(\beta-\alpha)t} - 1\right],$$

or, since $m_x(0) = 0$,

$$E(x(t)) = m_x(t) = \frac{\alpha}{\beta-\alpha}\left[1 - e^{-(\beta-\alpha)t}\right]. \tag{21}$$

A similar approach can be used for the mean of all events generated by immigrants, for which the relevant information is given in Table 2 below:

Table2

event in $(0,\delta t)$	immigrant arrives	nothing
probability	$v\delta t$	$1 - v\delta t$
$n(\delta t)$	1	0
$\vec{x}(\delta t)$	1	0
$E(\vec{x}(t)\|\mathcal{F}_2(\delta t))$	$1 + m_x(t-\delta t) + m_{\vec{x}}(t-\delta t)$	$m_{\vec{x}}(t-\delta t)$
$Var(\vec{x}(t)\|\mathcal{F}_2(\delta t))$	$\sigma_x^2(t-\delta t) + \sigma_{\vec{x}}^2(t-\delta t)$	$\sigma_{\vec{x}}^2(t-\delta t)$

$\mathcal{F}_2$ is the filtration describing the events of the immigration process. Then

$$m_{\vec{x}}(t) = v\delta t(1 + m_x(t-\delta t) + m_{\vec{x}}(t-\delta t)) + (1 - v\delta t)m_{\vec{x}}(t-\delta t),$$

since either there is an immigrant (probability $v\delta t$) or there is not (probability 1 -$v\delta t$). In the former case, one event has already happened and this now also generates further events starting from one individual, as discussed in the previous result above, while the immigration process proceeds as before. In the latter case we just have the continuation of the immigration process. Thus

$$\frac{\partial}{\partial t}m_{\vec{x}}(t) = v\left[1 + m_x(t)\right] \Rightarrow m_{\vec{x}}(t) = \int_0^t v\left[1 + m_x(s)\right]ds.$$

Substituting from (21) and completing a simple integration yields

$$E(\vec{x}(t)) = m_{\vec{x}}(t) = \frac{v}{\beta-\alpha}\left\{\beta t + \frac{\alpha}{\beta-\alpha}\left[e^{-(\beta-\alpha)t} - 1\right]\right\}. \tag{22}$$

The total expected number of observable events, $E((X(t))$, is then obtained by substituting these two results into (20).

4.3.2 Variances

We can proceed in a similar way to find variances using the general double expectation result for variances

variance = expectation of conditional variance + variance of conditional expectation.

Variance of x(t): If, as before, we condition on what happens in the interval $(0, \delta t)$ we get

$$Var(x(t)) = E\left(Var(x(t)|\mathcal{F}_1(\delta t))\right) + Var\left(E(x(t)|\mathcal{F}_1(\delta t))\right). \tag{23}$$

Using the results in Table 1 we get, ignoring terms of order $o(\delta t)$,

$$\begin{aligned}E\left(Var(x(t)|\mathcal{F}_1(\delta t))\right) &= \beta\delta t \times 0 + \alpha\delta t \times 2\sigma_x^2(t-\delta t) + [1-(\alpha+\beta)\delta t] \times \sigma_x^2(t-\delta t)\\ &= \{1-\delta t(\beta-\alpha)\}\,\sigma_x^2(t-\delta t);\end{aligned}$$

$$\begin{aligned}&Var\left(E(x(t)|\mathcal{F}_1(\delta t))\right) = E\left(\left[E(x(t)|\mathcal{F}_1(\delta t))\right]^2\right) - \left(E(x(t)\right)^2\\ &= \beta\delta t \times 0 + \alpha\delta t(1+2m_x(t-\delta t))^2 + (1-(\alpha+\beta)\delta t)\,m_x^2(t-\delta t) - m_x^2(t)\\ &= \delta t\left\{\alpha(1+2m_x(t-\delta t))^2 - (\alpha+\beta)m_x^2(t-\delta t) - \frac{\delta m_x^2(t)}{\delta t}\right\}.\end{aligned}$$

Put these two expressions together; divide by δt and take the limit as it goes to zero and we get

$$\frac{\partial}{\partial t}\left(e^{(\beta-\alpha)t}\sigma_x^2\left(t\right)\right) = e^{(\beta-\alpha)t}\left\{\alpha + 4\alpha m_x(t) - (\beta-3\alpha)m_x^2(t) - \frac{\partial m_x^2(t)}{\partial t}\right\}.$$

Substitute for $m_x\left(t\right)$ from equation (21) and perform the integration and, with a little bit of algebraic manipulation, we get

$$\sigma_x^2(t) = \frac{\alpha(\alpha+\beta)}{(\beta-\alpha)^3}\left\{\left(\beta+\alpha e^{-(\beta-\alpha)t}\right)\left(1-e^{-(\beta-\alpha)t}\right)\right\} - \frac{4\alpha^2\beta}{(\beta-\alpha)^2}e^{-(\beta-\alpha)t}t. \tag{24}$$

Note that $\sigma_x^2(0) = 0$, as it should, and $\sigma_x^2(\infty) = \frac{\alpha\beta(\alpha+\beta)}{(\beta-\alpha)^3}$. Also when $\beta = 0$, we get the variance of a pure linear birth process: $\sigma_x^2(t) = e^{\alpha t}\left(e^{\alpha t}-1\right).$

Variance of $\vec{x}(t)$: We now use a similar method, with the information in Table 2, to obtain the variance over (0,t) of events due to immigrants and their descendants. Ignoring terms of order $o(\delta t)$, we get

$$\begin{aligned}E\left(Var(\vec{x}(t)|\mathcal{F}_2(\delta t))\right) &= v\delta t\left(\sigma_{\vec{x}}^2(t-\delta t) + \sigma_x^2(t-\delta t)\right) + (1-v\delta t)\sigma_{\vec{x}}^2(t-\delta t)\\ &= \sigma_{\vec{x}}^2(t-\delta t) + v\delta t\sigma_x^2(t-\delta t);\end{aligned}$$

$$\begin{aligned}Var\left(E(\vec{x}(t)|\mathcal{F}_2(\delta t)\right) &= E\left(\left[E(\vec{x}(t)|\mathcal{F}_2(\delta t)\right]^2\right) - m_{\vec{x}}^2(t)\\ &= v\delta t(1+m_x(t-\delta t)+m_{\vec{x}}(t-\delta t))^2 + (1-v\delta t)m_{\vec{x}}^2(t-\delta t) - m_{\vec{x}}^2(t).\end{aligned}$$

Add these two expressions; divide by δt and take the limit as it goes to zero and we get

$$\frac{\partial}{\partial t}\left(\sigma_{\vec{x}}^2(t)+m_{\vec{x}}^2(t)\right)=v\left\{\sigma_x^2(t)+\left(1+m_x(t)+2m_{\vec{x}}(t)\right)\left(1+m_x(t)\right)\right\}$$

so that

$$\sigma_{\vec{x}}^2(t)+m_{\vec{x}}^2(t)=\int_0^t v\left\{\sigma_x^2(t)+\left(1+m_x(t)+2m_{\vec{x}}(t)\right)\left(1+m_x(t)\right)\right\}dt.$$

Using the previous results for $m_x(t), m_{\vec{x}}(t), \sigma_x^2(t)$, in this equation; doing the integration and some algebraic manipulation, we get

$$\begin{aligned}\sigma_{\vec{x}}^2(t)=&\frac{\beta(\alpha^2+\beta^2)}{(\beta-\alpha)^3}vt+\frac{4\alpha^2\beta}{(\beta-\alpha)^3}vte^{-(\beta-\alpha)t}\\&+\frac{\alpha v}{(\beta-\alpha)^4}\left(\alpha^2-2\alpha\beta-3\beta^2\right)\left(1-e^{-(\beta-\alpha)t}\right)-\frac{v\alpha^3}{(\beta-\alpha)^4}\left(1-e^{-2(\beta-\alpha)t}\right).\end{aligned}\tag{25}$$

Note that $\sigma_{\vec{x}}^2(0)=0$, as it should. Also $\sigma_{\vec{x}}^2(\delta t)=\nu\delta t+o(\delta t)$, again it should. Also, when $\alpha=0$, then $\sigma_{\vec{x}}^2(t)=vt$: which is correct, as it is just the variance of the Poisson process of immigrants. When $\beta=0$ then $\sigma_{\vec{x}}^2(t)=\frac{v}{\alpha}e^{\alpha t}\left(e^{\alpha t}-1\right)$, which is the variance of a simple linear immigration-birth process.

4.4 A bivariate Hawkes process

We began this section with a simple special case of a univariate Hawkes process then introduced a classic birth-death-immigration process to model a jump process with activity near the beginning of a day. This is useful because we can use traditional methods to study its stochastic dynamics. We now show that this can also be described as a bivariate interacting Hawkes process.

Equation (8) gives a general form of what is known as a mutually-exciting Hawkes process: but not everything has to be exciting. Consider a bivariate process with type-1 events being our jumps, $N_1(t)$, which are births or immigrations according to our earlier description; type-2 events will be deaths, $N_2(t)$. Let $\gamma_{ij}(w)=0$ for $w\le 0$, for each $i,j=1$ or 2; and for $w>0$ let $\gamma_{11}(w)=\alpha, \gamma_{12}(w)=-\alpha, \gamma_{21}(w)=\beta, \gamma_{22}(w)=-\beta$. Take the Poisson base rates to be $\upsilon_1=\upsilon+\alpha N(0)$ and $\upsilon_2=\beta N(0)$. Then, for this process, equation (8) can be written as

$$\begin{pmatrix}\lambda_1(u)\\ \lambda_2(u)\end{pmatrix}=\begin{pmatrix}\upsilon+\alpha N(0)\\ \beta N(0)\end{pmatrix}+\int_{s=0}^{u}\begin{pmatrix}\alpha & -\alpha\\ \beta & -\beta\end{pmatrix}\begin{pmatrix}dN_1(s)\\ dN_2(s)\end{pmatrix}.\tag{26}$$

The population size at any time is $N(t)$. This increases or decreases by 1 every time there is an event, or stays the same: $dN(t)=dN_1(t)-dN_2(t)$. Note, in particular, that the intensity of the jump process is given by the first element of the above vector expression

$$\lambda_1(u)=\upsilon+\alpha N(0)+\int_{s=0}^{u}\alpha dN_1(s)-\int_{s=0}^{u}\alpha dN_2(s)=\upsilon+\alpha N(t).\tag{27}$$

We cannot call this a mutually- exciting process because some events, deaths, decrease the intensity of both jumps and further deaths. Of course, the intensities must remain non-negative. Usually this is guaranteed by taking all elements of the exciting matrix, Γ, to be non-negative: in this case it is guaranteed because the intensity of deaths goes to zero as soon as the population size becomes zero, with the process being re-started when an immigrant comes along.

Usually the kernel matrix, Γ, of a mutually-exciting Hawkes process is a function of time difference w whose integral (from 0 to infinity) is, in a sense, not too large so as maintain stability. Here we have a constant Γ but the integral is not important as we are dealing with a finite range of one day, taken to be of length 1.

Equation (27) may be compared with equation (1) of Ertekin et al. (2015), which can be written closer to our notation as

$$\lambda(t) = \lambda_0 \left[1 + g_1 \left(\int_{s=0}^{t} g_2(t-s)dN_1(s) \right) - g_3 \left(\int_{s=0}^{t} g_4(t-s)dN_2(s) \right) \right]. \tag{28}$$

This is what they call a Reactive Point Process (RPP) in which $N_1(t)$ is a series of serious events (fires, explosions, power failures) in the underground electrical grid of New York City and $N_2(t)$ is a series of inspection events. g_2 is a self-exciting kernel function and g_4 is a regulation function: serious events are expected to be less likely to occur after an inspection, so the intensity decreases. The model is non-linear because the *saturation functions* g_1, g_3, which start as the identity function then decrease away from the origin, prevent the excitation and regulation, respectively, from becoming too large if there are many events close together. The inspection events $N_2(t)$ are not considered as random, but are taken as given. Therefore, apart from the non-linearities, this is really just a self-exciting process with an exogenous function that controls the base rate of the process.

5 Simulation examples

The model was introduced with the expectation that it might exhibit behaviour that is usually seen near the start of a trading day. That is, jumps tend to occur early in the trading day and decrease to a fairly flat level. There is often some increased intensity towards the end of the day, which could be modelled by the same process with different parameter values.

A file containing R code to simulate the model and a file describing its use are available as supplementary material. Here we just illustrate a few results. Each graph extends over the time interval $(0, 1)$, which represents one trading day. The model parameters are quoted as a vector $(N(0), \alpha, \beta, v)$.

Figure 1 shows the jump intensities, which is $\frac{\partial E(X(t))}{\partial t}$, for models $(10, \alpha, 1, 0.1)$, for $\alpha = 0, 0.2, .., 1.2$. This gives just the kind of thing we were hoping for: the intensity starts high and drops steadily down as t increases for appropriate values of α..

Note that these are unconditional expected intensities: nice smooth curves. An example of the actual conditional intensity, for one simulation of the model with parameters (20,2,4,0.1), is shown in Figure 2. This is a step function, $\alpha N(t) + v$, that goes up and down as births, immigrations and deaths occur. The jump intensity trends downwards, finishing very flat with the very low immigration rate that holds when the number of individuals in the population has reduced to zero.

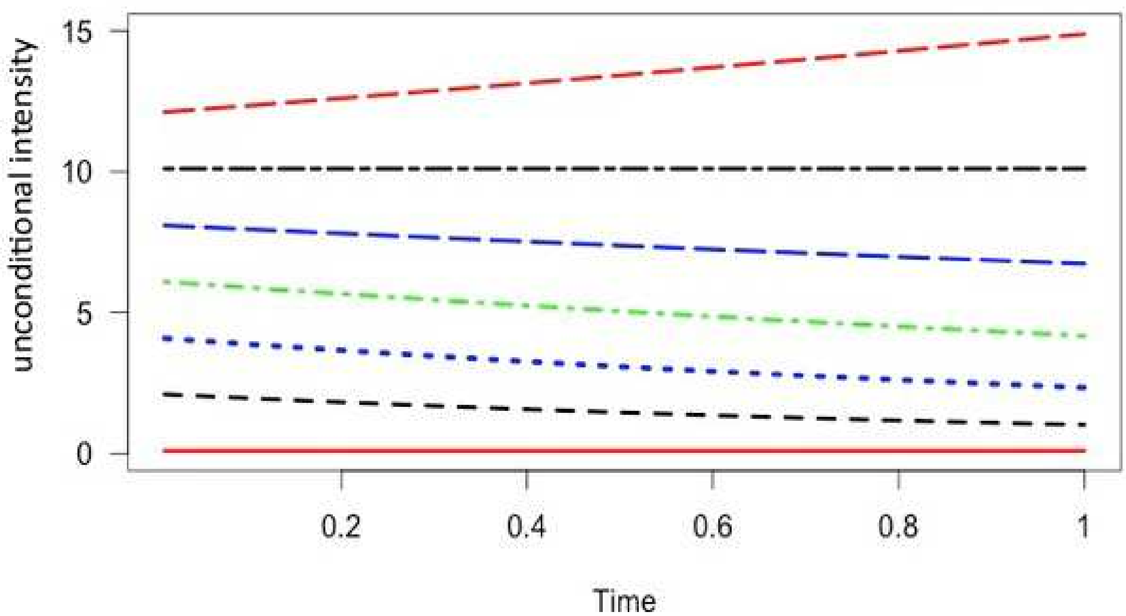

Figure 1: Expected jump intensities for models $(10, \alpha, 1, 0.1)$, $\alpha = 0, 0.2, 0.4, ..., 1.2$ with topmost curve corresponding to $\alpha = 1.2$.

We now simulate a 3-phase model in which the parameters change over the three time intervals (0, 0.4), (0.4, 0.7) and (0.7, 1). If our actual trading day were 9:30 to 16:00 these ranges would correspond to (9:30,12:06) (12:06,14:03) and (14:03, 16:00). We start with $N(0) = 25$ individuals and take parameters (α, β, v) to be $(2, 4, 2)$, $(1, 3, 2)$ and $(2, 0, 3)$, respectively, over those three time intervals. The starting population sizes at the start of the second and third intervals are the stochastic population sizes that were in force at the end of the previous interval. We perform one simulation that results in 23 jumps at times

$$0.031, 0.043, 0.072, 0.080, 0.112, 0.204, 0.216, 0.314, 0.317, 0.377, 0.379$$
$$0.410, 0.438, 0.479$$
$$0.702, 0.726, 0.772, 0.861, 0.879, 0.880, 0.895, 0.912, 0.992.$$

These are illustrated in Figure 3.

In Figure 4 we also plot the population size over the whole time range $(0, 1)$ for this simulation. The parameters have been chosen to produce the standard financial market pattern: there is an active period early on, then a quiet midday period followed by another active period during which time traders are trying to balance their books before the market closes.

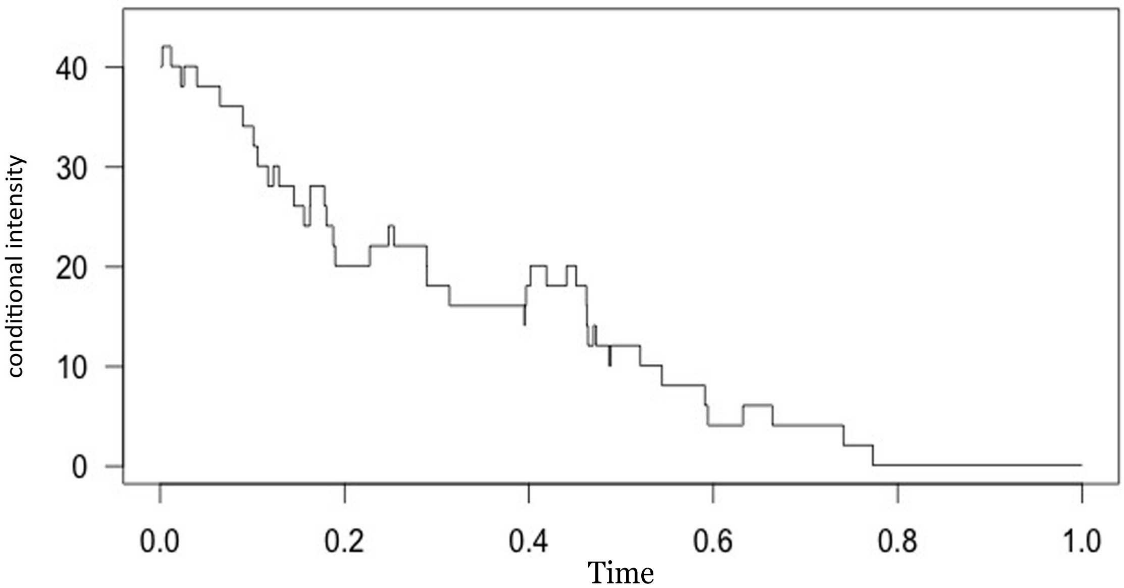

Figure 2: Conditional Intensity for a simulation of model = (20, 2, 4, 0.1)

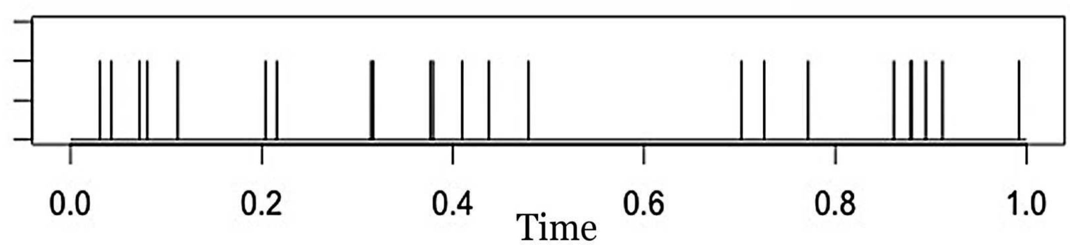

Figure 3: Jump times for 3-phase model

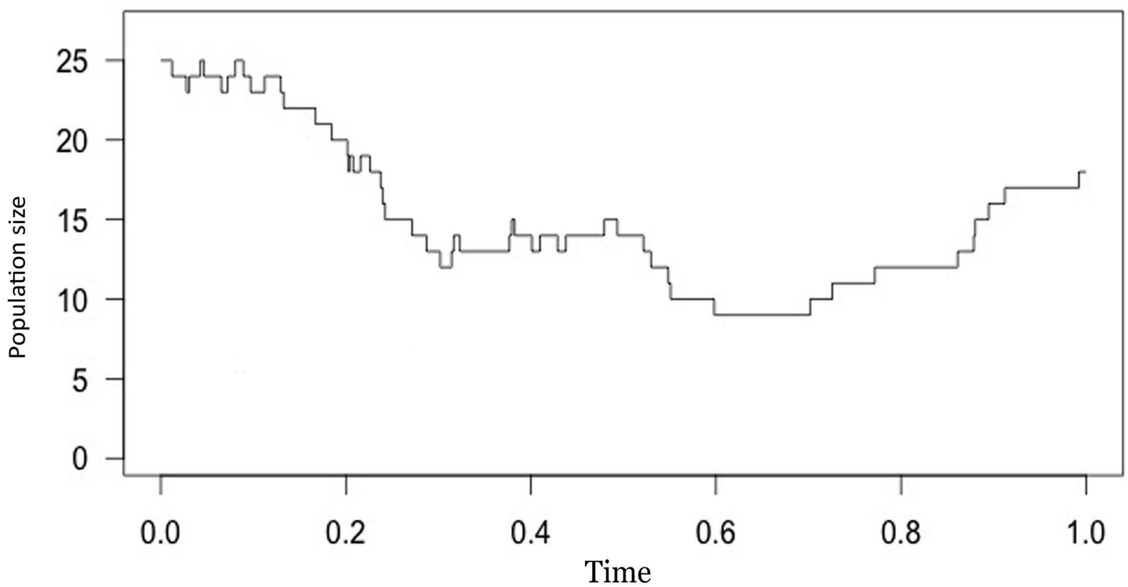

Figure 4: Variation of population size over the whole interval for simulated 3-phase model

5.1 Fitting the model

The simulations have confirmed that the model has the kind of properties that we were looking for. The next problem is to devise a method of estimating the model parameters from observed data, bearing in mind that we consider the parameters are likely to be different over different parts of the trading day. For example, we may want to fit the model over the first 90 minutes of a trading day and make a separate fit over the last 90 minutes.

We begin by considering what kind of data we might be observing. Note in particular that although the population size at each time forms an integral part of the model, we do not actually observe it: we merely observe the times at which visible events occur (defined as the immigrations and births of a BDI model). We do not observe deaths. We do not even observe event sizes, such as the size of a price jump or the volume of a trade. Of course, for any real series we might observe such things but, as they do not form part of the model, they will not help us to fit the model. If the events of interest to us are jumps in asset prices, they may be identified using methods discussed in section 3.2.

Because there are 4 parameters in our model (α, β, υ and the initial population size, $N(0)$, at the start of the day) we would need probably a minimum of about 20 events within the 90 minute period in order to get reasonable parameter estimates for a single day. This means that we need to be dealing with at least moderately high frequency data with a resolution of at least one minute, preferably a few seconds or better. If this is not feasible one might fit the model over several days, assuming that α, β, υ remain constant from day to day but allowing a different $N(0)$ each day.

Because there are hidden variables, deaths, in the model it does not seem to be possible to use maximum likelihood to estimate the parameters, even using EM (there are so many places that deaths could occur). However, it is possible to calibrate the model, making use of the means and variances derived in section 4.3. This will be reported in detail elsewhere.

6 Conclusion

Hawkes processes provide a wealth of models for financial applications with appropriate methodology. The authors posit that the question of finding a set of jumps can be approached as a design question of a set J to inform on practical objectives such as financial risk propagation, portfolio management, and algorithmic trading strategies. The design problem can be captured using a set of jumps J thought of as a function of the available sample data and its granularity, the available detection techniques, and the proposed model to calibrate. In many cases the data patterns inspire the process model that captures data patterns. As an example, in this paper, U-shaped daily patterns of price activities and associated jumps inspire the introduction of a new process termed self-depressing intensity process. While the classical self-exciting process can capture end-of-day trading patterns, the opening hour of trading patterns suggest a decrease in jump intensity towards a stable mid-day trading pattern unless emerging news disturb the patterns. The self-depressing process of birth-death-immigration (BDI) type will prove to be a useful Hawkes model in various application domains. Furthermore, the extension to multivariate Hawkes process will be implemented for multi-asset jump analysis. In future research we show how the BDI parameters can be calibrated for given financial patterns.

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